BR8135 2

INIS-mf--6666

Weyl tensors for asymmetric complex curvatures

Colber G. Oliveira

Departamento de Física, Universidade de Brasília, Brasil

Scheduled to appear in July/August 1981 - Journal of Mathematical Physics (USA)

Abstract

Considering a second rank Hermitian field tensor and a general Hermitian connection ~~we construct~~ the associated complex curvature tensor. The Weyl tensor that corresponds to this complex curvature is determined. The formalism is applied to the Weyl unitary field theory and to the Moffat gravitational theory.

1-Introduction

Recently the study of complex geometry in relativity is a subject of increasing interest(1). In the definition of algebraically special spaces the null tetrad formalism of Newman-Penrose is generalized to a complex null tetrad(2). The curvature and Weyl tensors become complex objects and their algebraic properties have been studied(3). Presently we consider the problem of determination of the Weyl tensors associated to a complex curvature derivable from a general Hermitian connection. Along with the complex affinity we consider a Hermitian field tensor $g_{\mu\nu}$ which generalizes the usual symmetric metric tensor of general relativity. No relation is imposed, a priori, between the connection and the field tensor. Each particular relationship between these quantities is specific of each particular theory, as for instance general relativity, the Weyl theory, or complex formalisms such as the Moffat gravitational theory. It is shown that the general curvature is a fourth rank tensor skew symmetric in the last pair of indices, which contains several complex Ricci tensors.

The method used for the determination of the complex Weyl tensor is a decomposition process which separates the curvature into components with a well determined symmetry. This method has the advantage of providing a direct way for the determination of the spinors associated to the components of the complex curvature and of the complex Weyl tensors.

It is the purpose of this paper to apply these results to some unitary field theories of interest, such as the Weyl theory and the Moffat theory. However, the general geometric results may also be applied as an extension of the use of complex geometries in "complex relativity", since presently the vierbeins are completely general complex quantities that generate the Hermitian field tensor. The formula giving the field tensor in terms of the vierbeins is the same formula used in the Moffat theory.

Although the complex vierbeins are implicitly contained in the definition of the Hermitian metric field, they are not used in the present paper since all results apply directly to the connection and curvature as function of the Hermitian field tensor. As it is well known, the use of vierbeins is necessary for the determination of the spinors associated to the curvature and Weyl tensors, and as we have said before the present formalism is easily translated in terms of complex vierbeins and two-component spinors.

In the second section we define the affine curvature tensor constructed with a Hermitian connection, and decompose this complex tensor into components with determined symmetry. The Ricci tensors and scalars of curvature are obtained by contractions with the Hermitian field tensor. In the third section we derive the complex Weyl tensors associated to each part of the previous decomposition. Finally, in the fourth section we apply the formalism to some unitary field theories of interest.

2 - The connection and the curvature

A general asymmetric connection may give rise to different forms of defining the parallel transport of vectors. For a contravariant vector field we may use the two possible definitions:

$$\delta A^{\mu} = - \Gamma^{\mu}_{\alpha\beta} A^{\alpha} dx^{\beta} \quad , \quad \delta A^{\mu} = - \Gamma^{\mu}_{\beta\alpha} A^{\alpha} dx^{\beta} \tag{2-1}$$

In order to distinguish these two forms of definition of the variation in the coordinates of the vector. A under in infinitesimal parallel transport, we use the notation suggested by Einstein[4]: we call the first of the equations (2-1) by $\delta \overset{+}{A}{}^{\mu}$ and the second one by $\delta \overset{-}{A}{}^{\mu}$. Accordingly, two possible definitions of covariant derivatives follow as

$$dx^{\nu} \overset{+}{A}{}^{\mu}_{;\nu} = dA^{\mu} - \delta \overset{+}{A}{}^{\mu} \quad , \quad dx^{\nu} \overset{-}{A}{}^{\mu}_{;\nu} = dA^{\mu} - \delta \overset{-}{A}{}^{\mu}$$

explicitly one has

$$\overset{+}{A}{}^{\mu}_{;\nu} = A^{\mu}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} A^{\alpha} \tag{2-2}$$

$$\overset{-}{A}{}^{\mu}_{;\nu} = A^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\alpha} A^{\alpha} \tag{2-3}$$

The expressions for $\delta A_{\underset{+}{\mu}}$, $\delta A_{\underset{-}{\mu}}$ are directly derived from the definitions (2.1) by imposing the condition that the lenght of the vector A is unchanged under parallel transport. Consequently, covariant derivatives of the classes + and - may be written for an arbitrary tensor field.

The asymmetric connection may be decomposed into symmetric and antisymmetric parts. In this paper we take the antisymmetric part of the connection as a purely imaginary third rank tensor. With this choice the connection $\Gamma^{\mu}_{\alpha\beta}$ becomes Hermitian with respect to the covariant indices : $\Gamma^{*\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$. Complex tensor fields where used by Einstein(4) in his complex nonsymmetric unitary theory. Recently this theory has been reviewed and generalized by Moffat(5). In these theories the metric tensor is a second order Hermitian tensor. This complex property of the metric allows for a simple expression of $g_{\mu\nu}$ in terms of a field of complex vierbeins(6). Thus, one of the advantages of the complex formulation is the direct possibility of translating the theory in terms of two-component spinors, a property which is not shared by the real formulation which works with an asymmetric real tensor $g_{\mu\nu}$.

The affine curvature tensor may be introduced by the commutator

$$A^{\overset{\sigma}{+}}{}_{;\underset{+}{\mu};\nu} - A^{\overset{\sigma}{+}}{}_{;\nu;\underset{+}{\mu}} = R^{\sigma}{}_{\lambda\mu\nu} + 2\ \Gamma^{\lambda}_{[\mu\nu]}\ A^{\overset{\sigma}{+}}{}_{;\lambda}$$

and has the value

$$R^{\sigma}_{\lambda\ \mu\nu}(\Gamma) = \partial_{\nu}\ \Gamma^{\sigma}_{\lambda\mu} - \partial_{\mu}\ \Gamma^{\sigma}_{\lambda\nu} + \Gamma^{\rho}{}_{\lambda\mu}\ \Gamma^{\sigma}_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu}\ \Gamma^{\sigma}_{\rho\mu} \qquad (2\text{-}4)$$

This same curvature tensor may be derived by calculating the variation in the components of a vector field around an infinitesimal closed loop, and is also called, by this reason, as the curvature of rotation. The general curvature tensor $R^{\sigma}{}_{\lambda\mu\nu}$ is presently a complex fourth rank tensor antisymmetric in the indices μ,ν. Accordingly, we may write

$$R^{\sigma}_{\lambda\mu\nu}(\Gamma) = T^{\sigma}_{\lambda\mu\nu} + i\ S^{\sigma}_{\lambda\mu\nu} \qquad (2\text{-}5)$$

$$T^{\sigma}{}_{\lambda\mu\nu} = -\ T^{\sigma}{}_{\lambda\nu\mu}\ ,\quad S^{\sigma}{}_{\lambda\mu\nu} = -\ S^{\sigma}{}_{\lambda\nu\mu}$$

the explicit expression of these two tensors is

$$T^{\sigma}{}_{\lambda\mu\nu} = G^{\sigma}{}_{\lambda\mu\nu} + \Gamma^{\sigma}_{[\alpha\nu]}\,\Gamma^{\alpha}_{[\lambda\mu]} - \Gamma^{\sigma}_{[\alpha\mu]}\,\Gamma^{\alpha}_{[\lambda\nu]} +$$

$$+ 2\,\Gamma^{\alpha}_{[\nu\mu]}\,\Gamma^{\sigma}_{[\lambda\alpha]} \tag{2-6}$$

$$S^{\sigma}{}_{\lambda\mu\nu} = \Gamma^{\overset{\sigma}{+}}_{[\underset{+}{\lambda}\ \underset{+}{\mu}];\nu} - \Gamma^{\overset{\sigma}{+}}_{[\underset{+}{\lambda}\ \underset{+}{\nu}];\mu} \tag{2-7}$$

where $G^{\sigma}{}_{\lambda\mu\nu}$ is affine curvature tensor counstructed with the symmetric part of the affinity $\Gamma^{\alpha}{}_{\mu\nu}$. We mention that here the quantity $\Gamma^{\alpha}{}_{(\mu\nu)}$ is not necessarily equal to the Christoffel symbols. The formulas (2-6) and (2-7) show explicitly the covariance property of the decomposition (2-5). There is a priori no relationship between the connection $\Gamma^{\alpha}{}_{\mu\nu}$ and the Hermitian tensor field $g_{\mu\nu}$. This relationship will be characteristic of each particular theory considered, as for instance general relativity, a semi-metric theory such as the unitary Weyl theory or the asymmetric unitary theory suggested by Moffat. Thus, we keep the formalism in a general form, but we need the Hermitian tensor field for lowering the contravariant index of curvature, since a discussion of the symmetry properties of the curvature will be necessary.

The conventions for lowering and raising indices with a Hermitian metric are well known :

$$A_{\alpha} = g_{\alpha\beta}\,A^{\beta} \quad , \quad A^{\alpha} = g^{\alpha\beta}A_{\beta} \quad ,$$

$$g^{\lambda\alpha}\,g_{\alpha\beta} = g^{\alpha\lambda}\,g_{\beta\alpha} = \delta^{\lambda}{}_{\beta} \quad .$$

Thus,

$$R_{\alpha\lambda\mu\nu}(\Gamma) = g_{\alpha\sigma}\,R^{\sigma}{}_{\lambda\mu\nu} = F_{\alpha\lambda\mu\nu} + i\,H_{\alpha\lambda\mu\nu} \tag{2-8}$$

In the next section we will determine the Weyl tensor associated to the general curvature tensor $R_{\alpha\lambda\mu\nu}$. For the determination of this tensor we will use the method of decomposition of the curvature in a sum of factors with a determined symmetry. The following notation will be used :

$$S_{\mu\nu} = S_{\underline{\mu\nu}} + S_{\underset{\vee}{\mu\nu}}$$

$$S^{*}_{\underline{\mu\nu}} = S_{\underline{\nu\mu}} \quad , \quad S^{*}_{\underset{\vee}{\mu\nu}} = - S_{\underset{\vee}{\nu\mu}}$$

$S_{\mu\nu}$ being any arbitrary second rank complex tensor, and $S_{\underline{\mu\nu}}$, $S_{\underset{\vee}{\mu\nu}}$ denote its Hermitian and Anti-Hermitian parts. This decomposition may be extended for a fourth rank complex tensor according to

$$R_{\mu\nu\rho\sigma} = R_{\underline{\mu\nu}\,\underline{\rho\sigma}} + R_{\underline{\mu\nu}\,\underset{\vee}{\rho\sigma}} + R_{\underset{\vee}{\mu\nu}\,\underline{\rho\sigma}} + R_{\underset{\vee}{\mu\nu}\,\underset{\vee}{\rho\sigma}} \tag{2-9}$$

where, for instance,

$$R_{\underline{\mu\nu}\,\underline{\rho\sigma}} = \frac{1}{4} \left(R_{\mu\nu\rho\sigma} + R^{*}_{\mu\nu\sigma\rho} + R^{*}_{\nu\mu\rho\sigma} + R_{\nu\mu\sigma\rho}\right),$$

with similar expressions for the remaining components in (2-9). From (2-8) we have for the several terms in (2-9)

(*)

$$R_{\underline{\mu\nu}\,\underline{\rho\sigma}} = i\, H_{[\mu\nu]\,[\rho\sigma]} \tag{2-10}$$

$$R_{\underset{\vee}{\mu\nu}\,\underline{\rho\sigma}} = i\, H_{(\mu\nu)\,[\rho\sigma]} \tag{2-11}$$

$$R_{\underline{\mu\nu}\,\underset{\vee}{\rho\sigma}} = F_{(\mu\nu)\,[\rho\sigma]} \tag{2-12}$$

(*) where $H_{[\mu\nu]\,[\rho\sigma]} = \frac{1}{2} (H_{\mu\nu\rho\sigma} - H_{\nu\mu\rho\sigma})$. Recall that $H_{\mu\nu\rho\sigma}$ is anti-symmetric on (ρ, σ) as consequence of (2-5).

$$R_{\underset{\vee}{\mu\nu}\,\underset{\vee}{\rho\sigma}} = F_{[\mu\nu][\rho\sigma]} \tag{2-13}$$

It is of interest to introduce the quantities

$$B_{\mu\nu\rho\sigma} = \frac{1}{2}\left(F_{[\mu\nu][\rho\sigma]} + F_{[\rho\sigma][\mu\nu]}\right) \tag{2-14}$$

$$E_{\mu\nu\rho\sigma} = \frac{1}{2}\left(F_{[\mu\nu][\rho\sigma]} - F_{[\rho\sigma][\mu\nu]}\right) \tag{2-15}$$

$$J_{\mu\nu\rho\sigma} = \frac{1}{2}\left(F_{(\mu\nu)[\rho\sigma]} + F_{(\rho\sigma)[\mu\nu]}\right) \tag{2-16}$$

$$I_{\mu\nu\rho\sigma} = \frac{1}{2}\left(F_{(\mu\nu)[\rho\sigma]} - F_{(\rho\sigma)[\mu\nu]}\right) \tag{2-17}$$

$$L_{\mu\nu\rho\sigma} = \frac{1}{2}\left(H_{[\mu\nu][\rho\sigma]} + H_{[\rho\sigma][\mu\nu]}\right) \tag{2-18}$$

$$M_{\mu\nu\rho\sigma} = \frac{1}{2}\left(H_{[\mu\nu][\rho\sigma]} - H_{[\rho\sigma][\mu\nu]}\right) \tag{2-19}$$

$$K_{\mu\nu\rho\sigma} = \frac{1}{2}\left(H_{(\mu\nu)[\rho\sigma]} + H_{(\rho\sigma)[\mu\nu]}\right) \tag{2-20}$$

$$U_{\mu\nu\rho\sigma} = \frac{1}{2}\left(H_{(\mu\nu)[\rho\sigma]} - H_{(\rho\sigma)[\mu\nu]}\right) \tag{2-21}$$

Accordingly, one gets

$$R_{\underline{\mu\nu}\,\underline{\rho\sigma}} = L_{\mu\nu\rho\sigma} + M_{\mu\nu\rho\sigma} \tag{2-22}$$

$$R_{\underset{\vee}{\mu\nu}\,\underline{\rho\sigma}} = K_{\mu\nu\rho\sigma} + U_{\mu\nu\rho\sigma} \tag{2-23}$$

$$R_{\underline{\mu\nu}\,\underset{\vee}{\rho\sigma}} = J_{\mu\nu\rho\sigma} + I_{\mu\nu\rho\sigma} \tag{2-24}$$

$$R_{\underset{\vee}{\mu\nu}\,\underset{\vee}{\rho\sigma}} = B_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma} \tag{2-25}$$

The tensors B, E, J and I satisfy the symmetry properties

$$B_{\mu\nu\rho\sigma} = -B_{\nu\mu\rho\sigma} = -B_{\mu\nu\sigma\rho} = B_{\rho\sigma\mu\nu} \tag{2-26}$$

$$E_{\mu\nu\rho\sigma} = - E_{\nu\mu\rho\sigma} = - E_{\mu\nu\sigma\rho} = - E_{\rho\sigma\mu\nu} \tag{2-27}$$

$$J_{\mu\nu\rho\sigma} = J_{\rho\sigma\mu\nu} \quad , \quad I_{\mu\nu\rho\sigma} = - I_{\rho\sigma\mu\nu} \tag{2-28}$$

The tensors L, M, K and U satisfy the same sequence of symmetries. The curvature $R_{\mu\nu\rho\sigma}$ has 192 independent components. Accordingly to our decomposition this total number of components is separated into the tensor B with 21 components, E with 15, J + I with 60 and the imaginary parts L, M and K + U with the same number of independent elements. In sequence we write down the Ricci tensors that correspond to these eight elements which compose the curvature. All contractions are carried out using the complex metric $g_{\mu\nu}$. For the components $B_{\mu\nu\rho\sigma}$ we have from (2-26)

$$B_{\nu\sigma} = g^{\rho\nu} B_{\mu\nu\rho\sigma} = B^{*}_{\sigma\nu} \tag{2-29}$$

Thus, the Ricci tensor of the curvature $B_{\mu\nu\rho\sigma}$ is a Hermitian second rank tensor. In what follows all contractions are taken as in equation (2-29). For the remaining components we find

$$E_{\nu\sigma} = - E^{*}_{\sigma\nu} \, , \quad J_{\nu\sigma} = J^{*}_{\sigma\nu} \, , \quad I_{\nu\sigma} = - I^{*}_{\sigma\nu} \, ,$$

$$L_{\nu\sigma} = - L^{*}_{\sigma\nu} \, , \quad M_{\nu\sigma} = M^{*}_{\sigma\nu} \, , \quad K_{\nu\sigma} = - K^{*}_{\sigma\nu} \, , \; U_{\nu\sigma} = U^{*}_{\sigma\nu} \, .$$

A further contraction generate the scalars of curvature. They satisfy the conditions

$$B = B^{*} \, , \quad E = - E^{*} \, , \quad J = J^{*} \, , \quad I = - I^{*} \, , \; L = - L^{*} \, ,$$

$$M = M^{*} \, , \quad K = - K^{*} \, , \quad U = U^{*}$$

With these results we can determine the several elements that compose the Weyl tensor associated to the general complex curvature.

3 - The Weyl tensor of the curvature $R_{\mu\nu\rho\sigma}$

In the determination of the Weyl tensor of the curvature $R_{\mu\nu\rho\sigma}$ it is of interest to introduce the four-index quantity $g_{\mu\nu\rho\sigma}$ given by

$$g_{\mu\nu\rho\sigma} = g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho} \tag{3-1}$$

since $g_{\mu\nu} = g_{\underline{\mu\nu}}$ we have

$$g_{\mu\nu\rho\sigma} = -g_{\mu\nu\sigma\rho} = -g_{\nu\mu\rho\sigma} = g^{*}_{\rho\sigma\mu\nu} \tag{3-2}$$

The Weyl tensor will be composed of four elements $\overset{(i)}{W}_{\mu\nu\rho\sigma}$ $(i=1...4)$, each one of these elements correspond to some combination of the several terms in the equations (2-22) through (2-25). First we select the components B and M and define the tensors

$$\Lambda_{\mu\nu\rho\sigma} = B_{\mu\nu\rho\sigma} + M_{\mu\nu\rho\sigma} = \Lambda^{*}_{\rho\sigma\mu\nu}$$

$$P_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\nu\lambda\sigma}\, a^{\lambda}_{\cdot\rho} - g_{\mu\nu\lambda\rho}\, a^{\lambda}_{\cdot\sigma}\right) = P^{*}_{\rho\sigma\mu\nu}$$

where

$$a^{\lambda}_{\cdot\rho} = \Lambda^{\lambda}_{\cdot\rho} - \frac{\Lambda}{4}\,\delta^{\lambda}_{\rho}\ , \quad \Lambda = B + M = \Lambda^{*}$$

The Weyl tensor "of the class (1)" is given by

$$^{(1)}W_{\mu\nu\rho\sigma} = \Lambda_{\mu\nu\rho\sigma} - P_{\mu\nu\rho\sigma} - \frac{\Lambda}{12}\, g_{\mu\nu\rho\sigma} \tag{3-3}$$

and has the same symmetries of the tensor $\Lambda_{\mu\nu\rho\sigma}$. In addition, it satisfies

$$^{(1)}W_{\nu\sigma} = {}^{(1)}W^{\mu}_{\ \nu\mu\sigma} = 0$$

Following with this process we select the components E and L and introduce the tensors

$$Q_{\mu\nu\rho\sigma} = E_{\mu\nu\rho\sigma} + L_{\mu\nu\rho\sigma} = - Q^{*}_{\rho\sigma\mu\nu}$$

$$T_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\nu\lambda\sigma} \, q^{\lambda}_{\cdot\rho} - g_{\mu\nu\lambda\rho} \, q^{\lambda}_{\cdot\sigma}) = - T^{*}_{\rho\sigma\mu\nu}$$

with

$$q^{\lambda}_{\cdot\rho} = Q^{\lambda}_{\cdot\rho} - \frac{Q}{4} \delta^{\lambda}_{\ \rho} \, , \; Q = E + L = - Q^{*}$$

The Weyl tensor $^{(2)}W$ is given by the expression

$$^{(2)}W_{\mu\nu\rho\sigma} = Q_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma} - \frac{Q}{12} g_{\mu\nu\rho\sigma} \tag{3-4}$$

all symmetries presented by $Q_{\mu\nu\rho\sigma}$, are equally satisfied by $^{(2)}W_{\mu\nu\rho\sigma}$, and we have the conditions

$$^{(2)}W_{\nu\sigma} = 0$$

By a similar choice we finally take the tensors (J, U) and (I, K) and form the quantities

$$\Omega_{\mu\nu\rho\sigma} = J_{\mu\nu\rho\sigma} + U_{\mu\nu\rho\sigma} = \Omega^{*}_{\rho\sigma\mu\nu}$$

$$\Lambda_{\mu\nu\rho\sigma} = I_{\mu\nu\rho\sigma} + K_{\mu\nu\rho\sigma} = - \Lambda^{*}_{\rho\sigma\mu\nu}$$

$$\Delta_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\nu\lambda\sigma} \, \omega^{\lambda}_{\cdot\rho} - g_{\mu\nu\lambda\rho} \, \omega^{\lambda}_{\cdot\sigma}) = \Delta^{*}_{\rho\sigma\mu\nu}$$

$$\Phi_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\nu\lambda\sigma} \, \phi^{\lambda}_{\cdot\rho} - g_{\mu\nu\lambda\rho} \, \phi^{\lambda}_{\cdot\sigma}) = - \Phi^{*}_{\rho\sigma\mu\nu}$$

where

$$\omega^{\lambda}_{\cdot\rho} = \Omega^{\lambda}_{\ \rho} - \frac{\Omega}{4} \delta^{\lambda}_{\ \rho} \, , \quad \phi^{\lambda}_{\ \rho} = \Lambda^{\lambda}_{\cdot\rho} - \frac{\Lambda}{4} \delta^{\lambda}_{\ \rho}$$

These elements generate the Weyl tensors "of the classes (3) and (4)" according to

$$ {}^{(3)}W_{\mu\nu\rho\sigma} = \Omega_{\mu\nu\rho\sigma} - \Delta_{\mu\nu\rho\sigma} - \frac{\Omega}{12}\, g_{\mu\nu\rho\sigma} \qquad (3\text{-}5) $$

$$ {}^{(4)}W_{\mu\nu\rho\sigma} = \Lambda_{\mu\nu\rho\sigma} - \Phi_{\mu\nu\rho\sigma} - \frac{\Lambda}{12}\, g_{\mu\nu\rho\sigma} \qquad (3\text{-}6) $$

${}^{(3)}W$ and ${}^{(4)}W$ have the same symmetry of the tensors Ω and Λ, and satisfy

$$ {}^{(3)}W_{\nu\sigma} = {}^{(4)}W_{\nu\sigma} = 0 $$

Thus, the Weyl tensor associated to the general, complex, curvature is given by the sum of the four elements ${}^{(i)}W$:

$$ W_{\mu\nu\rho\sigma} = \sum_{i=1}^{4} {}^{(i)}W_{\mu\nu\rho\sigma} $$

and has the form

$$ W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left[g_{\mu\nu\lambda\sigma}\,(R^{\lambda}_{\cdot\,\rho} - \frac{R}{4}\,\delta^{\lambda}{}_{\rho}) - g_{\mu\nu\lambda\sigma}\,(R^{\lambda}_{\cdot\,\sigma} - \frac{R}{4}\,\delta^{\lambda}{}_{\sigma})\right] $$
$$ - \frac{1}{12}\, R\, g_{\mu\nu\rho\sigma}\, , \qquad (3\text{-}7) $$

where

$$ R^{\lambda}_{\cdot\,\rho} = g^{\lambda\mu}\, g^{\beta\alpha}\, R_{\alpha\mu\beta\rho} $$

The complex Weyl tensor $W_{\mu\nu\rho\sigma}$ has the symmetry property $W_{\mu\nu\rho\sigma} = - W_{\mu\nu\sigma\rho}$, which is the only symmetry property presented by the curvature tensor $R_{\mu\nu\rho\sigma}$. A formula like (3-7) would have been written without the necessity of going through the process of decomposition used in the section 2 and in this section. However, we have used this process of decomposition of the curvature $R_{\mu\nu\rho\sigma}$ since it allows directly for the determination of the two-components spinors that correspond to the several parts of the curvature tensor and of the Weyl tensors ${}^{(i)}W$. The formula (3-7) along with the symmetry property of the Weyl tensor W do not contain sufficient information for the determination of these curvature spinors.

4 - Applications of the formalism

In the previous sections we have considered a general, complex, curvature associated to a general Hermitian connetion $\Gamma^{\mu}{}_{\nu\lambda}$, and have used a Hermitian field tensor $\underset{\sim}{g}_{\mu\nu}$. No relation between the connection and the field tensor was used. Presently we particularize this general formalism for two cases of interest : the asymmetric, complex, field theory of Moffat and the semi-metric, real, unitary theory proposed by Weyl[7].

(i) Moffat's theory

In this theory the metric and the connection are Hermitian objects which satisfy the field equations (in absence of fermion sources)

$$g_{\mu\nu;\lambda} - g_{\alpha\nu}\,\Gamma^{\alpha}_{\cdot\,\mu\lambda} - g_{\mu\alpha}\,\Gamma^{\alpha}_{\cdot\,\lambda\nu} = 0 \tag{4-1}$$

$$\underset{\sim}{g}^{[\mu\nu]}{}_{,\nu} = 0 \tag{4-2}$$

$$* \; R_{(\mu\nu)}(\Gamma) = 0 \tag{4-3}$$

$$* \; R_{[\mu\nu]}(\Gamma) = -\frac{2}{3}\, w_{[\mu,\nu]} \tag{4-4}$$

The quantity w_{μ} is a vector gauge field constructed as the vector of torsion of an affine connection $W^{\alpha}_{\cdot\,\mu\lambda}$ that is related to the $\Gamma^{\alpha}_{\cdot\,\mu\lambda}$ by

$$W^{\alpha}{}_{\mu\lambda} = \Gamma^{\alpha}{}_{\mu\lambda} - \frac{2}{3}\,\delta^{\alpha}{}_{\mu}\, W_{\lambda}$$

with

$$W_{\lambda} = W^{\alpha}{}_{[\lambda\alpha]} = i w_{\lambda}$$

Consequently the vector of torsion of the Hermitian connection vanishes: $\Gamma_\mu = \Gamma^\alpha{}_{[\lambda\alpha]} = 0$. The remaining quantities in (4-2), (4-3) and (4-4) are given by

$$\underset{\sim}{g}^{[\mu\nu]} = \sqrt{-g}\; g^{[\mu\nu]} \quad , \; g = |g_{\mu\nu}|$$

$$* R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\Gamma) + \frac{4 \Pi G}{k^2 c^4} \; \tau_{\mu\nu}$$

$$\tau_{\mu\nu} = - i \; (g_{[\mu\nu]} + g^{[\lambda\beta]} \, g_{\beta\nu} \, g_{\mu\lambda} + \frac{1}{2} \, g^{[\beta\lambda]} \, g_{\beta\lambda} \, g_{\mu\nu}) = \tau^*_{\nu\mu}$$

We recall that the Hermitian field tensor may be written as $g_{\mu\nu} = g_{(\mu\nu)} + i\, g_{[\mu\nu]}$. G is the gravitational constant, and $k = iK$ where K has the dimension $L^{1/2}\, M^{-1/2}\, T$. The Ricci tensor $R_{\mu\nu}(\Gamma)$ is Hermitian, and consequently has the same form as the $g_{\mu\nu}$ written above. The same conclusion holds for the tensor $*R_{\mu\nu}(\Gamma)$. Finally, the following identification is made :

$$\Lambda = \frac{Kc^4}{12\, \pi G} \, w_\mu$$

The vector Λ_μ in the Einstein-Maxwell limit of the theory (which is obtained for $K \to 0$) generates the field strenght $F_{\mu\nu}$ (this is obtained from the equation (4-4)), the Maxwell equations follow from the equations (4-2) in this limit.

Going back to our general curvature tensor given by (2-5), (2-6) and (2-7), we can write

$$R^{\sigma}_{\cdot\,\lambda\mu\nu}(\Gamma) = Y^{\sigma}_{\cdot\,\lambda\mu\nu} + i\, V^{\sigma}_{\cdot\,\lambda\mu\nu}$$

with

$$Y^{\sigma}_{\cdot\,\lambda\mu\nu} = G^{\sigma}_{\cdot\,\lambda\mu\nu} - \Gamma^\rho{}_{[\lambda\mu]}\, \Gamma^\sigma{}_{[\rho\nu]} + \Gamma^\rho{}_{[\lambda\nu]}\, \Gamma^\sigma{}_{[\rho\mu]}$$

$$V^{\sigma}_{\cdot\,\lambda\mu\nu} = \partial_\nu \Gamma^\sigma{}_{[\lambda\nu]} - \partial_\mu \Gamma^\sigma{}_{[\lambda\nu]} + \Gamma^\rho{}_{[\lambda\mu]}\, \Gamma^\sigma{}_{[\rho\nu]} + \Gamma^\sigma{}_{(\rho\nu)}\, \Gamma^\rho{}_{[\lambda\mu]}$$
$$- \Gamma^\rho{}_{(\lambda\nu)}\, \Gamma^\sigma{}_{[\rho\mu]} - \Gamma^\sigma{}_{(\rho\mu)}\, \Gamma^\rho{}_{[\lambda\nu]}$$

Accordingly, the complex Ricci tensor may be written as $R_{\lambda\nu}(\Gamma) = Y_{\lambda\nu}(\Gamma) + i\, V_{\lambda\nu}(\Gamma)$, imposing that the vector of torsion of the connection Γ vanishes, one obtains

$$Y_{\lambda\nu}(\Gamma) = G_{\lambda\nu} - \Gamma^{\rho}{}_{[\lambda\sigma]}\, \Gamma^{\sigma}{}_{[\rho\nu]} = Y_{\nu\lambda}(\Gamma) \quad (*)$$

$$V_{\lambda\nu}(\Gamma) = -\partial_{\sigma}\, \Gamma_{[\lambda\nu]} + \Gamma^{\rho}{}_{(\lambda\sigma)}\, \Gamma^{\sigma}{}_{[\rho\nu]} + \Gamma^{\sigma}{}_{(\rho\nu)}\, \Gamma^{\rho}{}_{[\lambda\sigma]} -$$

$$- \Gamma^{\sigma}{}_{(\rho\sigma)}\, \Gamma^{\rho}{}_{[\lambda\nu]} = - V_{\nu\lambda}(\Gamma)$$

Thus, the conditions $\Gamma_{\mu} = 0$ imply that $R_{\nu\lambda}(\Gamma)$ is Hermitian. Accordingly, in the application of our formalism to the Moffat theory we have to impose the conditions

$$E_{\lambda\nu} = I_{\lambda\nu} = L_{\lambda\nu} = K_{\lambda\nu} = 0$$

In this case the components of the Weyl tensor assume the form

(1) $W_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma} - P_{\mu\nu\rho\sigma} - \frac{A}{12}\, g_{\mu\nu\rho\sigma}$

(2) $W_{\mu\nu\rho\sigma} = Q_{\mu\nu\rho\sigma}$

(3) $W_{\mu\nu\rho\sigma} = \Omega_{\mu\nu\rho\sigma} - \Lambda_{\mu\nu\rho\sigma} - \frac{\Omega}{12}\, g_{\mu\nu\rho\sigma}$

(4) $W_{\mu\nu\rho\sigma} = \Lambda_{\mu\nu\rho\sigma}$

(*) We recall that for any metrical theory, such as the Einstein-Cartan theory, we have

$$\Gamma^{\rho}{}_{\mu\nu} = \left\{ {}^{\ \rho}_{\mu\ \nu} \right\} + g^{\rho\sigma}\, (\Gamma_{[\mu\sigma]\nu} + \Gamma_{[\nu\sigma]\mu} + \Gamma_{[\mu\nu]\sigma}) \,.$$

Then, in our case

$$\Gamma^{\rho}{}_{\rho\nu} = \left\{ {}^{\ \rho}_{\rho\ \nu} \right\} = \partial_{\nu} \log \sqrt{-g} = \Gamma^{\rho}{}_{(\rho\nu)}$$

This implies that $G_{\lambda\nu} = G_{\nu\lambda}$

The explicit value for the Weyl tensor is obtained from the field equations (4-3) and (4-4) and from the above equations:

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{4\pi G}{Kc^4}\Big[i\ (F_{\mu\rho}\, g_{\nu\sigma} - F_{\nu\rho}\, g_{\mu\sigma} - F_{\mu\sigma}\, g_{\nu\rho}$$

$$+ F_{\nu\sigma}\, g_{\mu\rho}) + \frac{1}{2K}\,(\tau_{\mu\rho}\, g_{\nu\sigma} - \tau_{\nu\rho}\, g_{\mu\sigma} - \tau_{\mu\sigma} g_{\nu\rho} + \tau_{\nu\sigma} g_{\nu\rho})\Big]$$

$$- \frac{1}{12}\left[\frac{4\pi G}{K^2 c^4}\, g^{(\rho\sigma)}\alpha_{(\rho\sigma)} - g^{[\rho\sigma]}\,\frac{4\pi G}{Kc^4}\,(2\, F_{\sigma\rho} + \frac{1}{K}\,\beta_{[\sigma\rho]})\right] g_{\mu\nu\rho\sigma} \tag{4-5}$$

where,

$$\tau_{\mu\nu} = \alpha_{(\mu\nu)} + i\ \beta_{[\mu\nu]}$$

$$F_{\mu\nu} = 2\ A_{[\mu,\nu]}$$

In the Einstein-Maxwell limit of the theory, we have

$$\frac{1}{K}\,\beta_{[\mu\nu]} \rightarrow -2\ F_{\mu\nu}, \quad -\frac{1}{K^2}\,\alpha_{(\mu\nu)} \rightarrow 2\ T_{\mu\nu}$$

where $T_{\mu\nu}$ is the Maxwell energy-momentum tensor. It is easy to verify that in this limit $W_{\mu\nu\rho\sigma} \rightarrow R_{\mu\nu\rho\sigma}$ in empty space(*) (no charges and currents). For the full theory, which is an unitary field theory the expression for the complex Weyl tensor is given by (4-5), since in this situation the sources terms cannot be distinguished from the other dynamical factors.

(ii) <u>Weyl's theory</u>

For the unitary field theory proposed by Weyl we have

(*) In this case $R_{\mu\nu\rho\sigma}$ is interpreted as the, real, Riemann-Christoffel tensor.

$$\Gamma^{\rho}{}_{\mu\nu} = \{{}_{\mu}{}^{\rho}{}_{\nu}\} + \frac{1}{2}\,(\delta^{\rho}{}_{\mu}\,\phi_{\nu} + \delta^{\rho}{}_{\nu}\,\phi_{\mu} - g_{\mu\nu}\,\phi^{\rho}) \tag{4-6}$$

$$\Gamma^{\rho}{}_{[\mu\nu]} = 0$$

where the metric $g_{(\mu\nu)}$ and the gauge vector field ϕ_{μ} are subjected to the gauge transformations

$$g'_{\mu\nu} = \lambda\, g_{\mu\nu} \quad , \quad \phi'_{\mu} = \phi_{\mu} - \partial_{\mu} \log \lambda \quad .$$

The affinity given by (4-6) is invariant under these transformations. Accordingly, the curvature tensor associated to the Weyl connection is also gauge invariant. In this case we have

$$L_{\mu\nu\rho\sigma} = M_{\mu\nu\rho\sigma} = K_{\mu\nu\rho\sigma} = U_{\mu\nu\rho\sigma} = 0$$

The only remaining components of the curvature are the quantities B, E, J and I. Thus, the components of the Weyl tensor have the form

$$(1)\quad W_{\mu\nu\rho\sigma} = B_{\mu\nu\rho\sigma} - P_{\mu\nu\rho\sigma} - \frac{B}{12}\, g_{\mu\nu\rho\sigma} \tag{4-7}$$

$$(2)\quad W_{\mu\nu\rho\sigma} = E_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma} - \frac{E}{12}\, g_{\mu\nu\rho\sigma} \tag{4-8}$$

$$(3)\quad W_{\mu\nu\rho\sigma} = J_{\mu\nu\rho\sigma} - \Delta_{\mu\nu\rho\sigma} - \frac{J}{12}\, g_{\mu\nu\rho\sigma} \tag{4-9}$$

$$(4)\quad W_{\mu\nu\rho\sigma} = I_{\mu\nu\rho\sigma} - \Phi_{\mu\nu\rho\sigma} - \frac{I}{12}\, g_{\mu\nu\rho\sigma} \tag{4-10}$$

A long but straightforward calculation gives

$$(1)\quad W_{\rho\mu\nu\sigma} = C_{\rho\mu\nu\sigma} + \frac{1}{2}\, g_{\rho[\sigma}\phi_{\nu]}\,\phi_{\mu} + \frac{1}{2}\, g_{\mu[\nu}\phi_{\sigma]}\,\phi_{\rho} +$$

$$+\, g_{\sigma[\mu}\phi_{\rho]}\,\phi_{\nu} + g_{\nu[\rho}\phi_{\mu]}\,\phi_{\sigma}$$

$$(2)\quad W_{\rho\mu\nu\sigma} = 0$$

(3) $W_{\rho\mu\nu\sigma} = \frac{1}{2} (g_{\rho\mu} \phi_{\nu\sigma} + g_{\nu\sigma} \phi_{\rho\mu})$

(4) $W_{\rho\mu\nu\sigma} = \frac{1}{2} (g_{\rho[\mu}\phi_{\nu]\sigma} + g_{\sigma[\rho}\phi_{\mu]\nu} + g_{\nu[\sigma}\phi_{\mu]\rho})$

where $C_{\rho\mu\nu\sigma}$ is the Weyl tensor for the Riemann-Christoffel curvature. $\phi_{\mu\nu}$ is the field tensor associated to the potentials ϕ_μ by the definition $\phi_{\mu\nu} = 2\ \phi_{[\mu,\nu]}$. In any application of the theory only gauge invariant quantities have physical significance. As was mentioned before the Weyl curvature tensor is a quantity of this type : $R'^{\mu}_{\cdot\nu\rho\sigma} = R^{\mu}_{\cdot\nu\rho\sigma}$. An inspection in the equations (4-7) through (4-10) shows that the several terms involved in these equations change under a gauge transformation by a multiplicative factor λ. Thus, we have for all components of the Weyl tensor : $^{(i)}W'_{\mu\nu\rho\sigma} = \lambda\ ^{(i)}W_{\mu\nu\rho\sigma}$. This implies that the physical components of the Weyl tensor are given by

$$^{(i)}\underset{\sim}{W}_{\mu\nu\rho\sigma} = \frac{^{(i)}W_{\mu\nu\rho\sigma}}{(-g)^{1/4}}$$

And the gauge invariant Weyl tensor associated to the curvature tensor $R^{\mu}_{\ \nu\rho\sigma}(\Gamma)$ is of the form

$$\underset{\sim}{W}_{\mu\nu\rho\sigma} = \sum_{i=1}^{4}\ ^{(i)}\underset{\sim}{W}_{\mu\nu\rho\sigma}$$

The expression for $\underset{\sim}{W}_{\mu\nu\rho\sigma}$ has a general form, since in the previous calculations we have not used any set of possible field equations. For each choice of field equations we can particularize the expression of the Ricci-Christoffel tensor in (4-7).

5 - Conclusion

We have seen how to determine the four complex components of the

Weyl tensor that correspond to a general complex non-Riemannian curvature. The method presently used for the calculation of the complex Weyl tensors may be directly applied for the determination of the four Weyl spinors that correspond to these tensors. In the determination of these spinors we have to work with a general complex set of vierbeins. Thus, the process of projection on the two-dimensional complex spin space has to be properly re-defined. After the determination of the components of the Weyl spinor we may obtain its Petrov classification, and determine the necessary conditions for the existence of radiative fields for general complex geometries. The knowledge of such conditions is clearly important for any application of this formalism, in particular, it is of direct interest for the case of the two unitary field theories considered in this paper.

Acknowledgement

I want to express my appreciation to D^{r} P. Letelier for some interesting discussions on the subject. I also want to thank Mr. Edmo D. Campos for the help in part of the calculations.

References

(1) E. T. Newman, J. Winicour - J. Math. Phys., 15, 426 (1974)
J. Math. Phys., 15, 113 (1974)

(2) C. W. Fette, A. Janis, E. T. Newman - J. Math. Phys., 17, nº 5. 660 (1975)

(3) J. Plebanski - J. Math. Phys., 16, nº 12, 2395 (1975)

(4) A. Einstein - Ann. Math., 46, 578 (1945)
Rev. Mod. Phys., 20, 35 (1948)
A. Einstein, G. Strauss - Ann. Math., 47, 731 (1946)

(5) J. W. Moffat - Phys. Rev. D15, 3520 (1977)

(6) D. W. Sciama - J. Math. Phys. 2, 472 (1961)

(7) H. Weyl - Z. Physik, 56, 330 (1929)