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EFFECTIVE THEORIES WITH BROKEN FLAVOUR SYMMETRY

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ABSTRACT

We generalise the work of Ovrut and Schnitzer on effective theories derived from a non Abelian Gauge Theory to include the physically interesting case of broken flavour symmetry. Our calculations are performed at the 1-loop level. We show that at an intermediate stage in the calculations two distinct renormalised gauge coupling constants appear, one describing gauge field coupling to heavy particles and the other describing coupling to light particles. Appropriately modified Slavnov-Taylor identities are shown to hold. We also consider a simple alternative to the Ovrut-Schnitzer rules for calculating with effective theories.

I. INTRODUCTION

The Appelquist-Corazzone decoupling theorem[1] states that heavy fields in a general renormalisable field theory decouple from processes involving only light fields, which can then be described in terms of an effective light field Lagrangian with effective fields and parameters.

Recently Ovrut and Schnitzer[2,3,15] and Weinberg[4] have shown how to calculate parameters of the effective theory in terms of those of the complete theory in particular models.

In the model of a non abelian gauge theory used by Ovrut and Schnitzer, all the fermions of the theory had the same mass and were regarded as the heavy particles, ghosts and gluons were the light particles.

In this paper we consider a non abelian gauge theory with broken flavour symmetry. Our model has both light and heavy fermions.

We are interested in obtaining expressions for the parameters of one effective light field Lagrangian in terms of the parameters of a second effective light field Lagrangian, when both are decoupled forms of the same complete theory. Our principle motivation is the application of such results to the area of effective weak non leptonic (penguin generating) Hamiltonians[5,6], but of course our results will have a broader applicability.

The calculations in this paper are based upon the 'earlier' methods[2, 3] of Ovrut and Schnitzer, they were completed before the availability of the extended methods[15].

Our calculations are presented at the one loop level where both methods are equally valid and easy to apply, and in fact give identical results. In the earlier method one considers an intermediate renormalisation scheme, known as heavy field renormalisation. (Decoupling performed at this level yields an unrenormalised effective field theory). In this paper we demonstrate the existence of two heavy field renormalised gauge coupling constants and derive (perturbatively) appropriately modified S-T type identities for this scheme.

A single gauge coupling constant is restored when we perform either light field renormalisation (so as to obtain the complete theory renormalisation scheme) or decoupling renormalisation (so as to obtain a bare effective light field Lagrangian).

This point was originally overlooked by Ovrut and Schnitzer[2].

The extended methods of Ovrut and Schnitzer[15] in fact now avoid recourse to partial renormalisation and present a more attractive approach to multiloop decoupling calculations, needless to say the difficulties encountered in earlier methods are now overcome.

In the final section to this work we consider a simpler alternative to the Ovrut-Schnitzer[12] rules for calculating with effective Lagrangians.

We shall begin by Specifying precisely the meaning of the various renormalisation schemes and notations employed.

II. THE EFFECTIVE LAGRANGIAN

A. The interconnecting renormalisation schemes

Figure I outlines the various renormalisation schemes we can consider in moving from the bare theory involving light and heavy fields, to the renormalised effective light field theory which is our goal. In all cases we use dimensional regularisation in $n = 4 + \epsilon$ dimensions, and renormalise using the minimal subtraction scheme.

Fields and parameters of the bare theory will be indicated by using the subscript o (eg σ_o). The complete theory renormalisation carries us to the renormalised complete theory, the fields and parameters of which are unadorned (eg σ). The complete theory renormalisation constants will be denoted by Z, so that

$$\sigma_o = Z\sigma \tag{2.1}$$

In Appendix A the complete theory renormalisation scheme is outlined (at the 1-loop level) for a non-abelian gauge theory with broken flavour symmetry which we take as our model.

It is possible to proceed from the bare theory to the renormalised complete theory in two steps. We firstly renormalise by removing all ultra-violet divergences due to loops with at least one heavy field internal line. This scheme will be called heavy field renormalisation and takes us to the heavy field renormalised theory, whose fields and parameters are denoted by carets (eg $\hat{\mathcal{O}}$). The associated renormalisation constants will be denoted by $\hat{Z}$, so

$$\mathcal{O}_0 = \hat{Z}\hat{\mathcal{O}}. \qquad (2.2)$$

Heavy field renormalisation is described in detail in section B.

The second step which takes us to the renormalised complete theory is to remove ultra violet divergences generated by loops which contain no heavy fields. For example, in our specific model loops at this stage contain only gluons, ghosts and light fermions. We refer to this step as light field renormalisation. Clearly the associated renormalisation constants are

$Z/\hat{Z} = Z^{(L)}$; and

$$\hat{\mathcal{O}} = Z^{(L)}\mathcal{O} \qquad (2.3)$$

The bare theory, the heavy field renormalised theory and the renormalised complete theory still contain heavy fields. After we have performed at least heavy field renormalisation we can eliminate the heavy fields to obtain effective bare light field theories. We will refer to this elimination as decoupling renormalisation, and describe it in detail in section C.

If we perform decoupling renormalisation on the heavy field renormalised theory we construct an effective theory which is apparently bare in that it still contains ultra violet divergences. We call this theory the effective light field theory, and label the fields and parameters of this theory with the subscript 'eo' (eg σ_{eo}). The renormalisation constants of decoupling renormalisation will be denoted by S, so that

$$\hat{\sigma} = S\sigma_{eo} \tag{2.4}$$

Our final goal, the renormalised effective light field theory can be reached by applying decoupling renormalisation to the renormalised complete theory, or by applying light field renormalisation to the effective light field theory.

Fields and parameters of the renormalised effective light field theory will be denoted by the subscript "e", eg σ_e.

The corresponding renormalisation constants are defined by

$$\sigma = S\,\sigma_e \tag{2.5}$$

$$\sigma_{eo} = Z^{(L)}\,\sigma_e \tag{2.6}$$

We have employed the same symbols here as in (2.3) and (2.4). In otherwords these symbols refer to the functional form. The distinction between the two cases is by the implied variable dependences. This is best illustrated later when the heavy field scheme has been discussed.

All of these interlocking renormalisations are summarised in figure I.

B. The Heavy field Renormalisation Scheme of our Model

In the complete theory (described in more detail in Appendix A) there are p fermion flavours. The n heaviest of these will be defined as the heavy fields of our model. Infinities due to loops containing at least one heavy fermion are absorbed by introducing Heavy Field Renormalisation.

The bare Lagrangian of the complete theory is given, in a covariant gauge, by

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A^a_{o\nu} - \partial_\nu A^a_{o\mu}\right)^2 - g_o C^{abc}\left(\partial^\mu A^{a\nu}_o\right) A^b_{o\mu} A^c_{o\nu}$$

$$- \frac{1}{4} g_o^2 C^{abc} C^{ade} A^b_{o\mu} A^c_{o\nu} A^{d\mu}_o A^{e\nu}_o - \frac{1}{2\alpha_o}\left(\partial^\mu A^a_{o\mu}\right)^2$$

$$- \xi^a_o \partial^2 \eta^a_o - g_o C^{abc}\left(\partial^\mu \xi^a_o\right) \eta^b_o A^c_{o\mu}$$

$$+ \sum_{j=1}^{p} \left\{ \bar{\psi}^j_{oA}\, i \not{\partial} \psi^j_{oA} - m_{oj} \bar{\psi}^j_{oA} \psi^j_{oA} + g_o T^a_{AB} \bar{\psi}^j_{oA} \gamma^\mu \psi^j_{oB} A^a_{o\mu} \right\} \qquad (2.7)$$

Here a, A, B are colour indices, ξ and η are ghost fields, j is a flavour index and we have broken the flavour symmetry of the theory by taking different bare masses m_{oj} for the fermions. For convenience increasing j corresponds to increasing m_j.

When the complete renormalisation is carried out on this theory, the renormalised theory is gauge invariant. This is expressed by the appearance of one renormalised coupling constant g, and the Slavnov-Taylor identities which relate the renormalisation coefficients. The complete theory renormalised Lagrangian is

$$\mathcal{L} = -\frac{1}{4} Z_3 \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - Z_1 g\, C^{abc}\left(\partial^\mu A^{a\nu}\right) A^b_\mu A^c_\nu$$

$$- \frac{1}{4} Z_1^2 Z_3^{-1} g^2 C^{abc} C^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2$$

$$- Z'_3\, \xi^a \partial^2 \eta^a - Z'_1\, g\, C^{abc}\left(\partial^\mu \xi^a\right) \eta^b A^c_\mu$$

$$+ \sum_{j=1}^{p} \left\{ Z^{(j)}_{2f} \bar{\psi}^j_A\, i \not{\partial}\, \psi^j_A - Z^{(j)}_M Z^{(j)}_{2f}\, m_j\, \bar{\psi}^j_A \psi^j_A \right.$$

$$\left. + Z^{(j)}_{1f}\, g\, T^a_{AB} \bar{\psi}^j_A \gamma^\mu \psi^j_B A^a_\mu \right\} \qquad (2.8)$$

and the Slavnov-Taylor relations between the Z's are

$$Z_3^{1/2} Z_g = Z_1/Z_3 = Z'_1/Z'_3 = Z^{(j)}_{1f}/Z^{(j)}_{2f} \qquad (2.9)$$

where $g_o = Z_g\, g$. (2.10)

Were we to write down the heavy fermion renormalised Lagrangian simply by putting carets on the fields and parameters in eqn (2.8) we discover an inconsistency, in that $\hat{Z}_g$ depends on which three (or four) point function is used to compute it. Associated with this is the breakdown of the Slavnov-Taylor identities in the form of eqn (2.9). To be specific $\hat{Z}^{(j)}_{1f}/\hat{Z}^{(j)}_{2f}$ depends on whether the fermion j is light or heavy.

Indeed the same breakdown of manifest gauge invariance at the level of the heavy field scheme occurs for the Ovrut-Schnitzer model in which all of the fermions are heavy. $\hat{Z}_g$ computed from the ghost-gluon vertex differs from $\hat{Z}_g$ computed from the fermion-gluon vertex, and $\hat{Z}_1'/\hat{Z}_3' \neq \hat{Z}_{1f}/\hat{Z}_{2f}$. Perhaps because they concentrated on the light particle sector of the theory Ovrut and Schnitzer did not comment on this apparent pathology of the theory.

Physically what is happening is clear if we note that the irreducible light fermion-gluon vertex function contains no heavy fermion loops at the one loop level, while the irreducible heavy fermion-gluon vertex function does have heavy fermion loops. Thus one would expect different heavy field renormalised coupling constants at these vertices, we should not be surprised that the assumption that these constants are the same leads to contradictions. Calculation shows that there are two renormalised coupling constants at this level, in terms of which the heavy field renormalised Lagrangian reads,

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}\hat{Z}_3 (\partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu)^2 - \hat{Z}_1 \, \hat{g}_2 \, C^{abc} (\partial^\mu \hat{A}^{a\nu}) \hat{A}^b_\mu \hat{A}^c_\nu \\
& - \frac{1}{4}\hat{Z}_4 \, \hat{g}_2^2 \, C^{abc} C^{ade} \hat{A}^b_\mu \hat{A}^c_\nu \hat{A}^{d\mu} \hat{A}^{e\nu} - \frac{1}{2\hat{\alpha}} (\partial^\mu \hat{A}^a_\mu)^2 \\
& - \hat{Z}_3' \hat{\xi}^a \partial^2 \hat{\eta}^a - \hat{Z}_1' \hat{g}_2 C^{abc} (\partial^\mu \hat{\xi}^a) \hat{\eta}^b \hat{A}^c_\mu \\
& + \sum_{j=1}^{p} \left\{ \hat{Z}^{(j)}_{2f} \hat{\bar{\psi}}^j_A \, i \not\partial \, \hat{\psi}^j_A - \hat{Z}^{(j)}_m \hat{Z}^{(j)}_{2f} \hat{m}_j \hat{\bar{\psi}}^j_A \hat{\psi}^j_A \right\} \\
& + \sum_{j=1}^{p-n} \hat{Z}^{(j)}_{1f} \hat{g}_2 T^a_{AB} \hat{\bar{\psi}}^j_A \gamma^\mu \hat{\psi}^j_B \hat{A}^a_\mu
\end{aligned}$$

$$+ \sum_{J=p-n+1}^{p} \hat{Z}_{1f}^{(j)} \hat{g}_1 T^a_{AB} \hat{\bar{\psi}}^j_A \gamma^\mu \hat{\psi}^j_B \hat{A}^a_\mu \qquad (2.11)$$

To avoid prejudice we introduced an additional renormalisation coefficient $\hat{Z}_4$ for the four gluon coupling, and we computed all of the $\hat{Z}$'s directly making no assumptions about the Slavnov-Taylor identitics. We have introduced two coupling constants $\hat{g}_1$ and $\hat{g}_2$, we define $\hat{Z}_{g_1}$ and $\hat{Z}_{g_2}$ by

$$g_o = \hat{Z}_{g_1} \hat{g}_1 = \hat{Z}_{g_2} \hat{g}_2 \qquad (2.12)$$

We utilise the minimal subtraction renormalisation scheme and dimensional regularisation in $n = 4 + \varepsilon$ dimensions to find at the one loop level

$$\hat{Z}_3 = 1 + \frac{\hat{g}^2}{16\pi^2} \left[\frac{8}{3} T(F)_h \right] \frac{1}{\varepsilon} + O(\hat{g}^4) \qquad (2.13a)$$

$$\hat{Z}_3' = 1 + O(\hat{g}^4) \qquad (2.13b)$$

$$\hat{Z}_{2f}^{(j)} = 1 + O(\hat{g}^4) \quad \text{for } j \le p-n \qquad (2.13c)$$

$$= 1 + \frac{\hat{g}^2}{16\pi^2} \left[2\hat{\alpha} C_2(F) \right] \frac{1}{\varepsilon} + O(\hat{g}^4) \quad \text{for } j > p-n \qquad (2.13d)$$

$$\hat{Z}_m^{(j)} = 1 + O(\hat{g}^4) \quad \text{for } j \le p-n \qquad (2.13e)$$

$$= 1 + \frac{\hat{g}^2}{16\pi^2} \left[6\, C_2(F) \right] \frac{1}{\varepsilon} + O(\hat{g}^4) \quad \text{for } j > p-n \qquad (2.13f)$$

$$\hat{Z}_{1f}^{(j)} = 1 + O(\hat{g}^4) \quad \text{for } j \le p-n \qquad (2.13g)$$

$$= 1 + \frac{\hat{g}^2}{16\pi^2} \left[2\hat{\alpha} C_2(F) + \left(\frac{3}{2} + \frac{\hat{\alpha}}{2} \right) C_2(A) \right] \frac{1}{\varepsilon} + O(\hat{g}^4) \quad \text{for } j > p-n \qquad (2.13h)$$

$$\hat{Z}_1 = 1 + \frac{\hat{g}^2}{16\pi^2}\left[\frac{8}{3}T(F)_h\right]\frac{1}{\varepsilon} + O(\hat{g}^4) \qquad (2.13j)$$

$$\hat{Z}_1' = 1 + O(\hat{g}^4) \qquad (2.13k)$$

$$\hat{Z}_4 = 1 + \frac{\hat{g}^2}{16\pi^2}\left[\frac{8}{3}T(F)_h\right]\frac{1}{\varepsilon} + O(\hat{g}^4) \qquad (2.13l)$$

$$\hat{Z}_{g_1} = 1 + \frac{\hat{g}^2}{16\pi^2}\left[\left(\frac{3}{2} + \frac{\hat{\alpha}}{2}\right)C_2(A) - \frac{4}{3}T(F)_h\right]\frac{1}{\varepsilon} + O(\hat{g}^4) \qquad (2.13m)$$

$$\hat{Z}_{g_2} = 1 + \frac{\hat{g}^2}{16\pi^2}\left[-\frac{4}{3}T(F)_h\right]\frac{1}{\varepsilon} + O(\hat{g}^4) \qquad (2.13n)$$

The group invariants are defined in the appendix (eqn A5) and we have witten

$$T(F)_h = n\,T(F)' \qquad (2.14)$$

recalling that there are n heavy fermions.

In eqn (2.13) we have written $\hat{g}$ for the renormalised coupling constant since when $O(\hat{g}^4)$ terms are ignored it does not matter whether we use $\hat{g}_1$ or $\hat{g}_2$. We will adopt this notation below.

Thus by explicit calculation at the one loop level we have found the Slavnov-Taylor type identities for $\hat{g}_2$

$$\hat{Z}_{g_2}\hat{Z}_3^{1/2} = \hat{Z}_1/\hat{Z}_3 = \hat{Z}_1'/\hat{Z}_3' = \hat{Z}_{1f}^{(j)}/\hat{Z}_{2f}^{(j)} \quad \text{for } j \le p-n \qquad (2.15)$$

we also find $\hat{Z}_4 = \hat{Z}_{g_2}^2\hat{Z}_3^2 = \hat{Z}_1^2/\hat{Z}_3$, (2.16)

and the Slavnov Taylor identities for $\hat{g}_1$

$$\hat{Z}_{g_1}\hat{Z}_3^{1/2} = \hat{Z}_{1f}^{(j)}/\hat{Z}_{2f}^{(j)} \quad \text{for } j > p-n\,. \qquad (2.17)$$

It should be emphasised that $\hat{g}_1$ enters only the heavy fermion coupling in the Lagragian (2.1i), and that all of the other couplings are determined by $\hat{g}_2$. We do not therefore anticipate any problems from the appearance of two coupling constants when we take the next step and eliminate the heavy fermions from the theory.

C. Decoupling Renormalisation and Effective Parameters

At this stage of the calculation we have a heavy field renormalised theory, but it still contains the heavy fields. Now we wish to eliminate the heavy fields from the theory. According to the decoupling theorem the effect of heavy fields on low energy processes i.e. those with only light particles in the initial and final states, can be absorbed into renormalisations of the parameters of the theory. This we refer to as decoupling renormalisation. After this transformation, the heavy field renormalised Lagragian becomes a bare effective light field Lagragian.

From our intentions it is clear that we will want to consider amplitudes in the renormalised complete theory in the limit where $-q^2$, a typical space like external 4 momentum, is small compared to m^2_{p-n+1} the square of the mass of the lightest heavy fermion. We introduce the symbol $A \overset{c}{=} B$ to represent the statement "A is equal to B in the limit that $|q^2| \ll m^2_{p-n+1}$". To apply this conditional equivalence to a function $F(q^2)$ we expand this function in terms of $x = |q^2|/m^2_{p-n+1}$ near $x = 0$, so $\hat{F}(q^2) = F' + O(|q^2|/m^2_{p-n+1})$.

Note that F' could have logarithmic q^2 dependances

The finite decoupling renormalisation constants, S are defined in a standard fashion by,

$$\left.\begin{aligned}
\hat{A}^a_\mu &= S_3^{\frac{1}{2}} A^a_{eo_\mu} \\
\hat{\psi}^{(j)}_A &\quad S^{(j)^{\frac{1}{2}}}_{2f} \psi^{(j)}_{eoA} \qquad j \leq p-n \\
\hat{\eta}^a &= S_3'^{\frac{1}{2}} \eta^a_{eo} \\
\hat{m}^{(j)} &= S^{(j)}_m m^{(j)}_{eo} \\
\hat{g}_2 &= S_g g_{eo}
\end{aligned}\right\} \qquad (2.18)$$

At the one loop level they may be calculated by first calculating heavy field renormalised propagators and vertices and equating the results with the corresponding vertices and propogators, calculated at zero loop order, using a bare effective light field lagrangian. This is amply illustrated in what follows.

The gluon propagator $\hat{D}^{ab}_{\mu\nu}(q)$ in the heavy fermion renormalised theory[16] is related to the corresponding vacuum polarisationscalar $\hat{\Pi}(q^2)$ by

$$\hat{D}^{ab}_{\mu\nu}(q) = \frac{1}{1 + \hat{\Pi}(q^2)} \; \frac{\left[-g_{\mu\nu} + (1 - \hat{\alpha}(1 + \hat{\Pi}(q^2))\; q_\mu q_\nu/q^2 \right] \; \delta^{ab}}{q^2 + i\varepsilon} \qquad (2.19)$$

We want to cast $\hat{D}^{ab}_{\mu\nu}$ into the form of a free gluon propagator, at least up to a constant. This constant looks like a renormalisation constant, but because we have removed heavy fermion induced ultra violet divergences it is finite. We see that

$$\hat{D}^{ab}_{\mu\nu}(q) \;\stackrel{c}{=}\; S_3 \; \frac{\left[-g_{\mu\nu} + (1 - \alpha_{eo})q_\mu q_\nu/q^2 \right] \; \delta^{ab}}{q^2 + i\varepsilon} \qquad (2.20)$$

where $$S_3 = \frac{1}{1 + \hat{\Pi}'} \qquad (2.21)$$

$$\alpha_{eo} = S_3^{-1}\,\alpha = (1 + \hat{\Pi}')\,\alpha$$

and $$\hat{\Pi}' \equiv \hat{\Pi}(q^2) \quad . \qquad (2.23)$$

At the one loop level $\hat{\Pi}'$ has no q^2 dependence, we find

$$\hat{\Pi}' = \frac{-\hat{g}^2}{16\pi^2} \left[\frac{4}{3}\, T(F)' \sum_{i=p-n+1}^{p} \ln\left(\frac{m_i^2}{\hat{\mu}^2}\right) \right] \qquad (2.24)$$

where $\hat{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$, γ_E being Euler's constant and μ the subtraction point. The expression for $\hat{\Pi}(q^2)$ can be obtained from $\Pi(q^2)$ of the complete theory as given in Appendix B, by noting that we are considering only heavy fermion loop contributions to the gluon propogator and these must be proportional to $T(F)'$.

Then from (2.21)

$$S_3 = 1 + \frac{\hat{g}^2}{16\pi^2} \left[\frac{4}{3}\, T(F)' \sum_{i=p-n+1}^{p} \ln\left(\frac{m_i^2}{\hat{\mu}^2}\right) \right] \qquad (2.25)$$

In a similar way we can write for the heavy field renormalised ghost propogator

$$\hat{D}_G^{ab}(q) = -S_3'\,\frac{\delta^{ab}}{q^2 + i\epsilon} \qquad (2.26)$$

S_3' is given in terms of the ghost vacuum polarisation scalar $\hat{\Pi}_G(q^2)$.

$$\hat{\Pi}_G' \equiv \hat{\Pi}_G\,(q^2) \qquad (2.27a)$$

$$\hat{\Pi}_G' = 0 \qquad \text{at the one loop level} \qquad (2.27b)$$

$$S_3' = (1 + \hat{\Pi}'_G)^{-1} \qquad (2.27c)$$

$$S_3' = 1 \qquad \text{at the one loop level} \qquad (2.27d)$$

To discuss the fermion propagator for the ith light fermion (i.e. $i \le p-n$)

$$\hat{S}_F^{(i)AB}(q) = \frac{\delta^{AB}}{\not{q} - \hat{m}_i - \hat{\Sigma}^{(i)}(\not{q}) + i\varepsilon} \qquad (2.28)$$

we write the self energy as

$$\hat{\Sigma}^{(i)}(\not{q}) = \hat{\Sigma}_1^{(i)}(q^2) + (\not{q} - \hat{m}_i)\,\hat{\Sigma}_2^{(i)}(q^2) \qquad (2.29)$$

Then the propagator can be written as

$$\hat{S}_F^{(i)AB}(q) = \frac{\delta^{AB}}{1 - \hat{\Sigma}_2^{(i)}(q^2)} \bullet \frac{1}{\not{p} - \hat{m}_i - [\hat{\Sigma}_1^{(i)}(q^2)/\{1-\hat{\Sigma}_2^{(i)}(q^2)\}] + i\varepsilon} \qquad (2.30)$$

Thus we can write $\hat{S}_F^{(i)AB}(q)$ in the form of a free fermion propagator

$$\hat{S}_F^{(i)AB}(q) = S_{2f}^{(i)} \frac{\delta^{AB}}{\not{p} - m_{eo}^{(i)} + i\varepsilon} \qquad (2.31)$$

where $$S_{2f}^{(i)} = \frac{1}{1 - \hat{\Sigma}_2^{(i)'}} \qquad (2.32)$$

and $$m_{eo}^{(i)} = S_m^{(i)^{-1}} \hat{m}_i \qquad (2.33)$$

with $$S_m^{(i)} = \frac{1 - \hat{\Sigma}_2^{(i)'}}{1 - \hat{\Sigma}_2^{(i)'} + (\hat{\Sigma}_1^{(i)'}/\hat{m}_i)} \qquad (2.34)$$

where $$\hat{\Sigma}_2^{(i)'} \in \hat{\Sigma}_2^{(i)}(q^2) \qquad (2.35a)$$

$$\hat{\Sigma}_1^{(i)'} \in \hat{\Sigma}_1^{(i)}(q^2) \qquad (2.35b)$$

In the one loop approximation $\hat{\sum}_1^{(i)'} = \hat{\sum}_2^{(i)'} = 0$, so that

$$S_{2f}^{(i)} = S_m^{(i)} = 1 \qquad i \leq p-n \tag{2.36}$$

We define the decoupling renormalisation constant S_g for the coupling constant $\hat{g}_2$ by

$$g_{eo} = S_g^{-1}\, \hat{g}_2 \tag{2.37}$$

where g_{eo} is the bare effective coupling constant. g_{eo} and S_g can be calculated from a number of different vertices. We begin with the 3 gluon vertex, the one particle irreducible amplitude for which in the heavy field renormalised theory we will write as $\hat{\Gamma}^{(3,0)}$.

At the symmetry point ($p^2 = q^2 = r^2$) in the one loop approximation

$$\hat{\Gamma}^{(3,0)}(p,q,r) = \hat{\Gamma}_1(q^2)\, C^{abc}\left\{g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-r)_\mu + g_{\lambda\mu}(r-p)_\nu\right\} \tag{2.38}$$

Defining $\hat{\Gamma}_1{}' \equiv \hat{\Gamma}_1(q^2)$ (2.39a)

then perturbatively $\hat{\Gamma}_1' = \hat{g}_2 - \frac{\hat{g}^3}{16\pi^2}\left[\frac{4}{3}T(F)'\sum_{i=p-n+1}^{p} \ell n\left(\frac{\hat{m}_i^2}{\hat{\mu}^2}\right)\right]$ (2.39b)

according to the one loop decoupling prescription we set

$$\hat{\Gamma}_1' = S_3^{-3/2}\, g_{eo}$$

hence $S_g = (1+\hat{\Pi}')^{3/2}\left(\frac{\hat{g}_2}{\hat{\Gamma}_1'}\right)$ (2.40a)

thus at the one loop level

$$S_g = 1 + \frac{\hat{g}^2}{16\pi^2}\left[-\frac{2}{3}T(F)'\sum_{i=p-n+1}^{p} \ell n\left(\frac{\hat{m}_i^2}{\hat{\mu}^2}\right)\right] \tag{2.40b}$$

so that

$$g_{eo} = \hat{g}_2 + \frac{\hat{g}^3}{16\pi^2}\left[\frac{2}{3}T(F)'\sum_{i=p-n+1}^{p} \ell n\left(\frac{\hat{m}_i^2}{\hat{\mu}^2}\right)\right] \tag{2.41}$$

is the coupling constant of the bare effective light field theory.

For consistency we should verify that the same S_g and hence the same g_{eo} are obtained from the other irreducible vertex functions - the light fermion-gluon, ghost-gluon and four gluon vertices.

For the light fermion-gluon vertex the one particle irreducible vertex function is $\hat{\Gamma}^{(1,2)}(q)$, where

$$\hat{\Gamma}^{(1,2)}(q) = i\gamma^{\mu}T^{a}_{AB}\,\hat{\Gamma}_2(q^2) \qquad (2.42)$$

and according to the one loop decoupling prescription

$$\hat{\Gamma}_2(q^2) \underset{\sim}{c} \hat{\Gamma}'_2 = S_3^{-1/2}\,S_{2f}^{-1}\,g_{eo} \qquad (2.43)$$

whence
$$S_g = S_3^{-1/2}\,S_{2f}^{-1}\,\frac{\hat{g}_2}{\hat{\Gamma}'_2} \qquad (2.44)$$

As $\hat{\Gamma}_2(q^2) = \hat{g}_2$ at the one loop level.

It can readily be verified that this leads to the same S_g as before.

The one particle irreducible vertex function for the ghost gluon vertex is

$$\hat{\Gamma}^{(1,0,2)} = C^{abc}\,q_{\mu}\,\hat{\Gamma}_3(q^2)\ , \qquad (2.45)$$

noting that the one loop decoupling prescription implies

$$\hat{\Gamma}_3(q^2) \underset{\sim}{c} \hat{\Gamma}'_3 = S_3^{-\frac{1}{2}}\,\tilde{S}_3^{-1}\,g_{eo}\ ,$$

then
$$S_g = S_3^{-\frac{1}{2}}\,\tilde{S}_3^{-1}\,\hat{g}_2/\hat{\Gamma}'_3 \qquad (2.46)$$

In this case $\hat{\Gamma}'_3 = \hat{g}_2$ at the one loop level and we obtain, once more, the same result for S_g.

At this point we note that three of the four possible vertices give unique values for S_g and g_{eo}. We have no suspicion that the four gluon vertex would not give th se same values, but we have not explicitly performed this additional calculation to verify our suspicions.

It is clear that the effective light field Lagrangian, expressed in terms of the bare effective quantities derived above.

$$
\begin{aligned}
\mathcal{L}_e = & -\frac{1}{4}\left(\partial_\mu A^a_{eo\nu} - \partial_\nu A^a_{eo\mu}\right)^2 - g_{eo}\, C^{abc}\left(\partial^\mu A^{a\nu}_{eo}\right) A^b_{eo\mu} A^c_{eo\nu} \\
& -\frac{1}{4} g^2_{eo}\, C^{abc} C^{adf} A^b_{eo\mu} A^c_{eo\nu} A^{d\mu}_{eo} A^{f\nu}_{eo} - \frac{1}{2\alpha_{eo}}\left(\partial_\mu A^{a\mu}_{eo}\right)^2 \\
& -\xi^a_{eo}\, \partial^2 \eta^a_{eo} - g_{eo}\, C^{abc}\left(\partial_\mu \xi^a_{eo}\right) \eta^b_{eo} A^{c\mu}_{eo} \\
& + \sum_{j=1}^{p-n}\left\{ \bar{\psi}^{(j)}_{eoA}\left[\left(i\partial\!\!\!/ - m^{(j)}_{eo}\right)\delta_{AB} + g_{eo}\, T^a_{AB}\, \gamma_\mu A^{a\mu}_{eo}\right]\psi^{(j)}_{eoB}\right\} \qquad (2.47)
\end{aligned}
$$

reproduces the above vertex functions and propagators to lowest order (i.e. in the zero loop calculation). That decoupling has taken place is equivalent to the statement that $\mathcal{L}_e$ and $\mathcal{L}$ give identical results for processes involving only light fields in external lines, in the limit $|q^2| \ll m^2_{p-n+1}$. Whether or not the method indeed leads to a decoupled effective theory is most easily investigated in terms of the light field renormalised version of (2.47) and this is done in the next section.

D. The Renormalised Effective Theory

It is clear that to zero loop order the Lagrangian of equation (2.47) reproduces the results of the heavy fermion renormalised theory for processes which involve only low q^2 light particles in the initial and final states. Since in doing one loop (and higher order) calculations light particle propagators are integrated over q^2 it is by no means obvious that higher order calculations with (2.47) reproduce higher order calculations of its parent theory.

In this section we show that at the one loop level this is the case by showing that the one loop decoupling prescription leads to renormalisation group functions of the effective light field parameters which are consistent with (2.47) being correct.

Logically we should now construct the renormalised effective light field theory from $\mathcal{L}_e$ in (2.47) in the usual way. However, since we want to use the known renormalisation group functions of the renormalised complete theory we proceed in a slightly round about manner. We first note that the renormalised effective light field theory can be obtained from the complete theory by applying the decoupling renormalisation.

In the preceding section we saw how to apply one loop decoupling to heavy field renormalised theory to obtain the bare effective light field theory.

$$\hat{\mathcal{O}} = S(\hat{\alpha}, \hat{g}_1, \hat{g}_2, \hat{m}_j)\mathcal{O}_{eo} \qquad (2.48)$$

The formal manipulations involved in constructing the renormalised effective light field theory from the renormalised complete theory are identical with those described in section C, only the interpretation of the parameters differ. We may therefore write

$$\mathcal{O} = S(\alpha, g, g, m_j)\,\mathcal{O}_e \qquad \text{[illegible]}$$

where the decoupling renormalisation constants S have the same functional form in (2.48) and (2.49).

In particular, from eqn (2.41) we can write

$$g_e = g + \frac{g^3}{16\pi^2}\left[\frac{2}{3}\,T(F)' \sum_{i=p-n+1}^{p} \ln\left(\frac{m_i^2}{\hat{\mu}^2}\right)\right] \qquad (2.50)$$

From this we can construct the β function of the effective light field theory

$$\beta_e(g_e) = \mu\frac{dg_e}{d\mu} \qquad (2.51)$$

$$= \left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \delta(g,\alpha)\frac{\partial}{\partial\alpha} + \sum_{i=1}^{p}\gamma_{m_i}(g,\alpha)\,m_i\frac{\partial}{\partial m_i}\right) g_e$$

Using the renormalisation group functions of the complete theory (as presented in Appendix A, eqn (A7)) we find

$$\beta_e(g_e) = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}C_2(A) + \frac{4}{3}T(F)_\ell\right] + O(g^5)$$

$$= \frac{g_e^3}{16\pi^2}\left[-\frac{11}{3}C_2(A) + \frac{4}{3}T(F)_\ell\right] + O(g_e^5)$$

$$\text{and}\quad T(F)_\ell = (p-n)\,T(F)' \qquad (2.52)$$

This is precisely the β function we would calculate from a Lagrangian of the form of (2.47).

In a similar way we compute the δ and γ_{m_j} functions of the effective theory.

Using (2.21), (2.22), (2.25) and heading (2.48) & (2.49) we find

$$\delta_e\,(g_e,\alpha_e) = \mu \frac{d}{d\mu}\alpha_e$$

$$= -\,2\alpha_e \left\{ \frac{g_e^{\;2}}{16\pi^2} \left[\left(\frac{\alpha_e}{2} - \frac{13}{6}\right) C_2(A) + \frac{4}{3}\, T(F)_{\ell} \right] \right\}$$

and using (2.33), (2.36), (2.48) & (2.49) we find

$$\gamma_{m_i e}\,(g_e,\alpha_e) = \frac{\mu}{m_e^{(i)}} \frac{dm_e^{(i)}}{d\mu} \qquad (i \leq p-n)$$

$$= \frac{g_e^{\;2}}{16\pi^2} \left[-6\, C_2(F) \right]$$

Both of these functions would follow from a one loop calculation with eqn (2.47).

We therefore conclude that, at least as far as one loop calculations are concerned, the Lagrangian $\mathcal{L}_e$ of eqn (2.47) does represent the effective Lagrangian of the theory. The heavy fermions have been eliminated from the fields appearing in the Lagrangian. They do, however, enter by determining the effective light field coupling constants gauge fixing parameter and mass parameters in terms of those of the complete theory.

III. EXPRESSIONS FOR EFFECTIVE PARAMETERS IN OUR MODEL

A. Basic Results

In this section we write down the relations which exist between two sets of effective parameters, derived from the same complete theory. This is done at the one loop level.

We need these expressions if we are to apply the decoupling theorem sequentially, removing heavy fermions one at a time. Such sequential removal is absolutely necessary in certain applications (see Gilman & Wise[5]).

We introduce a notation more convienient to this section [illegible] let α, g & $m^{(i)}$, $i \le p$, denote the renormalised complete theory parameters. α_k, g_k and $m_k^{(i)}$, $i \le k$, will denote the renormalised effective light field parameters for an effective theory with k light fermions.

We can utilise the results of the previous section to obtain our required relations. From (2.22), (2.33) and (2.37) employing (2.[illegible]) with 1-loop results (2.25), (2.36) and (2.40), then the effective parameters in the k quark theory, g_k, α_k, and $m_k^{(i)}$ are given in terms of the complete theory parameters by

$$g_\ell = g\left(1 + \frac{g^2}{16\pi^2}\,\frac{2}{3}\,T(F)'\sum_{i=\ell+1}^{p}\ln\frac{m^{(i)2}}{\mu^2}\right) \tag{3.1}$$

$$\alpha_\ell = \alpha\left(1 + \frac{g^2}{16\pi^2}\left(-\frac{4}{3}\right)T(F)'\sum_{i=\ell+1}^{p}\ln\frac{m^{(i)2}}{\hat{\mu}^2}\right) \tag{3.2}$$

$$m_\ell^{(i)} = m^{(i)} \qquad i \le \ell \tag{3.3}$$

We can invert eqns (3.1) and (3.2) to obtain (replacing ℓ by k)

$$g = g_k\left[1 + \frac{g_k^2}{16\pi^2}\left(-\frac{2}{3}\right)T(F)'\sum_{i=k+1}^{p}\ln\left(\frac{m^{(i)2}}{\hat{\mu}^2}\right)\right] \tag{3.4}$$

$$\alpha = \alpha_k\left[1 + \frac{g_k^2}{16\pi^2}\,\frac{4}{3}\,T(F)'\sum_{i=k+1}^{p}\ln\left(\frac{m^{(i)2}}{\hat{\mu}^2}\right)\right] \tag{3.5}$$

$$m^{(i)} = m_k^{(i)} \qquad i \le k \tag{3.6}$$

[Note that the masses in 'ℓn' terms correspond to heavy fermions]

We can now obtain the relationships between α_ℓ, g_ℓ, $m_\ell^{(j)}$ and α_k, g_k, $m_k^{(j)}$. They are

$$g_\ell = g_k \left[1 + \frac{g_k^2}{16\pi^2} \left(-\frac{2}{3}\right) T(F)' \sum_{i=k+1}^{\widetilde{\ell+1}} \ell n \left(\frac{m^{(i)2}}{\hat{\mu}^2}\right) \right] \tag{3.7}$$

$$\alpha_\ell = \alpha_k \left[1 + \frac{g_k^2}{16\pi^2} \left(\frac{4}{3}\right) T(F)' \sum_{i=k+1}^{\widetilde{\ell+1}} \ell n \left(\frac{m^{(i)2}}{\hat{\mu}^2}\right) \right] \tag{3.8}$$

$$m_\ell^{(i)} = m_k^{(i)} \tag{3.9}$$

where we have used the notation $\sum_{i=a}^{\tilde{b}} f_i = \sum_{i=a}^{p} f_i - \sum_{i=b}^{p} f_i$ (3.10)

We defer to section C a discussion of the conditions under which (3.7) - (3.10) are valid, and now consider how (3.7) - (3.9) may be generalised to the running effective parameters.

B. Extensions to running effective parameters

To avoid problems with the running gauge parameter we will assume here that we are working in the Landau gauge. So we shall discuss running fermion mass and the running coupling constant.

We consider a renormalisation group equation in terms of the parameters of an effective theory with k fermion flavours. A subset of the auxillary equations will read,

$$\frac{d\mu}{\mu} = \frac{dg_k}{\beta_k(g_k)} = \frac{dm_k^{(i)}}{\gamma_m(g_k)_k \, m_k^{(i)}} \tag{3.11}$$

We introduce the renormalisation group invariants, Λ_k and $\tilde{\Lambda}_k^{(i)}$ (i=1, ...k).

$$\Lambda_k = \mu \exp - \int^{g_k} \frac{dx}{\beta_k(x)} \tag{3.12}$$

$$\tilde{\Lambda}_k^{(i)} = m_k^{(i)} \exp - \int^{g_k} \frac{\gamma_m(x)_k}{\beta_k(x)} dx$$

clearly,

$$\frac{d}{d\mu} \Lambda_k = \frac{d}{d\mu} \tilde{\Lambda}_k^{(i)} = 0 \qquad (3.14)$$

We term (g_k, $m_k^{(i)}$; i=1, ... k) the physical point. The scaling solution of the renormalisation group equations is a particular case of the general solution of the equations. It relates Green's functions at the physical point to other points in the g_k - $m_k^{(i)}$ space which lie on a curve parameterised by a variable Q, and are defined by,

$$\mathrm{Ln}(Q/\Lambda_k) = \int^{\bar{g}_k(Q)} \frac{dx}{\beta_k(x)} \qquad (3.15)$$

$$\bar{m}_k^{(i)}(Q) = \tilde{\Lambda}_k^{(i)} \exp \int^{\bar{g}_k(Q)} \frac{\gamma_m(x)_k}{\beta_k(x)} dx \qquad (3.16)$$

(3.15) defines the running coupling constant $\bar{g}_k(Q)$ and (3.16) defines the running fermion masses in terms of the running coupling constant.

For k>ℓ we can generalise eqns (3.7) and (3.8) to

$$g_\ell(\mu) = S_{\ell k}^{(g)} (\mu,\ g_k(\mu),\ m^{(i)}\ (\mu);\ \ i=\ell+1,\ldots k) g_k\ (\mu) \qquad (3.17)$$

$$m_\ell^{(i)}(\mu) = S_{\ell k}^{(m)} (\mu, g_k(\mu),\ m^{(i)}\ (\mu);\ \ i=\ell+1,\ldots,k)\ m_k^{(i)}\ (\mu) \qquad (3.18)$$

i.e. S calculated to all orders and the implicit μ dependance is made clear. Comparing (3.12) & (3.15) we see that $g_k(\mu)$ and $\bar{g}_k(Q)$ are just the same function with differently named variables. Hence from (3.13) & (3.16) it also follows that $\bar{m}_k^{(i)}(Q)$ and $m_k^{(i)}(\mu)$ are the same functions with differently named variables.

One immediately concludes that (3.17) & (3.18) hold true for running parameters. i.e.

$$\bar{g}_{\ell}(Q) = S^{(g)}_{\ell k}\ (Q,\bar{g}_k(Q),\ \bar{m}^{(i)}\ (Q);\quad i=\ell+1,\ldots .k)\ \bar{g}_{k(Q)} \tag{3.19}$$

$$\bar{m}^{(i)}_k(Q) \;=\; S^{(m)}_{\ell k}(Q,\bar{g}_k(Q),\ \bar{m}^{(i)}(Q);\quad i=\ell+1,\ldots .k)\ \bar{m}^{(i)}_k(Q) \tag{3.20}$$

To some readers this point may seem rather trivial, but there do exist non-standard definitions of the running coupling constant [5] which are both μ and Q dependent and the analogue of (3.19) is not true.

Using the leading asymptotic expansions for $\bar{g}_n(Q)$ when Q is 'large' reveals that (3.19) & (3.20) contain the implicit relations between the sets $\left\{\Lambda_n, \mathcal{X}^{(i)}_n\right\}$ for differing n.

C. A simpler alternative to the Ovrut-Schnitzer's 'Rules for calculating with an effective Lagrangian'.

Ovrut & Schnitzer [12] in a recent paper set out rules for calculating with an effective Lagrangian. Recall their model is one without flavour symmetry breaking, m_{heavy} denotes the mass of the heavy fermions.

For convenience a brief resumé of their rules are given below.

Rule 1: The subtraction point choice, μ, is not arbitrary. It must be chosen to be μ_o where $\mu_o = 0\ (m_{heavy})$. This ensures that the effective parameters are well defined.

Rule 2: Typical external momenta, q, of effective light field Green's functions are not arbitrary. They must satisfy the following conditions.

(1) $|q|$ must be sufficiently small so that one can ignore terms of order $|q^2|/m^2_{heavy}$.

(2) $|q|$ must, at the same time be sufficiently large so that terms such as $\mathrm{Ln}\ |q^2|/_{\mu_o{}^2}$ do not invalidate perturbation theory. Such values of q will be denoted by q_o.

Rule 3: Effective Green's functions must be evaluated at μ_o and q_o given in rules 1 & 2. Green's functions at very much smaller $|q|$ are obtainable by scaling down the latter from $|q_o|$ to $|q|$ using the renormalisation group of the effective theory. Effective parameters evaluated at μ_o act as boundary conditions to the renormalisation group.

These rules certainly describe a valid procedure, but we feel they are unnecessarily complicated, a simpler approach is outlined below.

The problem is, how do we perturbatively evaluate an effective Green's function using an effective Lagrangian?

Denote the effective Green's function by $G(\mu^2, \bar{g}_k(\mu), \bar{\alpha}_k(\mu), \bar{m}_k^{(i)}(\mu) ; q^2)$. By the very nature of the effective Lagrangian it only applies to 'low energy' processes (relative to the energy scale where the next flavour can be excited). q^2 is thus a typical external momentum and is in this 'low energy' region. We wish to evaluate G perturbatively, this is a straight forward calculation. We want this perturbation series to make sense. All heavy fermion mass scales are implicit in $\bar{g}_k(\mu)$, $\bar{\alpha}_k(\mu)$ and $\bar{m}_k^{(i)}(\mu)$ they will not occur explicitly in the perturbation series for G. For this reason we can adopt the conventional approach to subtraction point choice, i.e. choose $\mu^2 = 0(q^2)$. Of course we must now face the problem of what is meant by $\bar{g}_k(\mu)$, $\bar{\alpha}_k(\mu)$ and $\bar{m}_k^{(i)}(\mu)$ at such 'small' μ^2 because it is clear from (3.1) - (3.3) that we cannot use those perturbation series to define them (i.e. give them numerical values).

The solution to this question is simple. The renormalisation group of the effective theory is used to relate the running effective parameters at a larger mass scale Q [17]. The value of Q is chosen so that the perturbation series for $S_{kp}^{(g)}(Q)$, $S_{kp}^{(m)}(Q)$ and $S_{kp}^{(\alpha)}(Q)$ defined in (19) and (20) do have small loop terms and can be used.

We suggest the following approach to accomplish this last step. It is clear that one loop terms in (3.19) and (3.20) (also see 3.4 - 3.6) will be small for a choice of Q, denoted Q_k given by,

$$Q_k^2 = (4\pi e^{-\gamma_E})^{-1} \; \bar{m}_p^{(k+1)}(Q) \; \bar{m}_p^{(p)}(Q_k) \tag{3.21}$$

k and p denoting the effective and complete theory number of flavours. (With 3.21) one obtains partial cancellation of +ve and -ve log terms in 3.4 and 3.5. Exact cancellation of the one loop terms follows if $4\pi e^{-\gamma_E} Q_k^2$ is chosen as the geometric mean of $\bar{m}_p^{(k+1)}(Q_k)^2$, $\bar{m}_p^{(k+2)}(Q_k)^2$, $\bar{m}_p^{(p)}(Q_k)^2$, but this is an unnecessary refinement.) We presume that R.G. invariants Λ_p, $\tilde{\Lambda}_p^{(i)}$ are known, so that $\bar{g}_p(Q)$, $\bar{m}_p^{(i)}(Q)$ and $\bar{\alpha}_p(Q)$ are known for Q above the onset of scaling. Q_k in (3.21) will in general be above the onset of scaling.

Thus using (3.19) and (3.20) one establishes $\bar{g}_k(Q_k)$ $\bar{m}_k^{(i)}(Q_k)$ and $\bar{\alpha}_k(Q_k)$. Using the asymptotic parameterisations of these running effective parameters, the above results are used to calculate the <u>effective</u> R.G. invariants Λ_k, $\tilde{\Lambda}_k^{(i)}$ (which are of course independent of the choice of Q_k). Having done this we are finished since running parameters $\bar{\alpha}_k(Q)$, $\bar{g}_k(Q)$ and $\bar{m}_k^{(i)}(Q)$ are now known for <u>all</u> Q above the onset of scaling. In particular $\bar{\alpha}_k(\mu)$, $\bar{g}_k(\mu)$ and $\bar{m}_k^{(i)}(\mu)$ can now be calculated.

One immediate criticism is that it is possible $\mu^2 = O(q^2)$ will not be above the onset of scaling. This problem really does not arise in practice as long considering processes above strange threshold. $\bar{g}_k^2(\mu)/4\pi$ will still be sufficiently small so that the perturbation series for the Green's function $G(\mu)$ does make sense.

The above approach is simple to put into practice and we do not have to face the unattractive task of scaling all the Green's functions.

In reality the presumption that 'complete theory' Λ_p and $\Upsilon_p^{(i)}$ must be known is not necessary. Using the above method one can calculate Λ_k, $\Upsilon_k^{(i)}$ using some other set of effective R.G. invariants Λ_ℓ, $\Upsilon_\ell^{(i)}$, along with flavour threshold estimates. Table I gives an example of such calculations.

IV. SUMMARY AND CONCLUSION

We have obtained three results in this paper:

(i) The model and calculations of Ovrut and Schnitzer have been generalised to a non abelian gauge theory with broken flavour symmetry, but only at the one loop level.

(ii) We have shown that two gauge coupling constants are required in the heavy field renormalised theory, and have demonstrated the extended Slavnov-Taylor identities appropriate to this case.

(iii) We have obtained relations between the effective parameters of one effective theory and those of another.

There are three obvious ways to attempt to proceed from this point. The first is to extend the above results to the two loop level. The major obstacles to such a calculation are the need for a two loop calculation of γ_m, the anamolous mass dimension, and of two loop constants in the complete theory renormalised functions such as $\Pi_G(q^2)$, etc. (Two loop log coefficients of renormalised Green's functions can be generated by use of the renormalisation group, see Ovrut & Schnitzer[2]). In fact we have only recently received references to the 2 loop γ_m calculation[13]. One needs to know the group invariant structure of two loop constants so as to assess whether or not a contribution comes from fermions. These would have a bearing on the 2 loop extensions of equations (3.7) - (3.9). The extended methods of Ovrut and Schnitzer[15] would have to be used.

Broken symmetry introduces a new problem to the calculations. At the 2 loop level $\hat{\sum}_1(q^2)$ and $\hat{\sum}_2(q^2)$ obtain non-trivial contributions. These would be deduced from the complete theory $\sum_1(q^2)$ and $\sum_2(q^2)$. Difficulties arise in the calculation of the latter because both heavy and light fermions are present simultaneously in 2-loop diagrams of the fermion propagator. One cannot therefore make use of the simplicity of the assumption,

$$|q^2| << m_i^2 \qquad i = 1, \ldots p$$

and still deduce correct $\hat{\sum}_1(q^2)$ and $\hat{\sum}_2(q^2)$. As a result a renormalisation group analysis generating the 2-loop coefficients becomes messy.

The second extension of this work which comes to mind is the formal demonstration of the gauge invariance of the heavy field renormalised Lagrangian. As it is simply a rearranged form of the complete theory Lagrangian it must be gauge invariant, in spite of the apprarance of two coupling constants. A formal demonstration of this would be an interesting exercise.

Finally this work can be extended by application, for example to the Gilman-Wise calculation. Work along these lines is in progress.

APPENDIX A

The Complete Theory M.S. Renormalisation Scheme of a NAGT with broken flavour symmetry

The bare Lagrangian of a general NAGT in a covariant gauge with p fermion flavours is given by,

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_{o_\nu} - \partial_\nu A^a_{o_\mu})^2 - g_o C^{abc}(\partial^\mu A^{a\nu}_o)A^b_{o_\mu} A^c_{o_\nu}$$

$$-\frac{1}{4} g_o^2 C^{abc} C^{ade} A^b_{o_\mu} A^c_{o_\nu} A^{d\mu}_o A^{e\nu}_o - \frac{1}{2\alpha_o}(\partial^\mu A^a_{o_\mu})^2$$

$$-\xi^a_o \partial^2 \eta^a_o - g_o C^{abc}(\partial^\mu \xi^a_o)\eta^b_o A^c_{o_\mu}$$

$$+\sum_{j=1}^{p}\left\{\bar\psi^j_{oA}\, i\partial\!\!\!/\, \psi^j_{oA} - m_{j_o}\bar\psi^j_{oA}\psi^j_{oA} + g_o T^a_{AB}\bar\psi^j_{oA}\gamma^\mu \psi^j_{oB} A^a_{o_\mu}\right\} \qquad \text{(A1)}$$

(ξ and η are ghost fields)

We define renormalisation constants as follows,

$$\left.\begin{aligned}
A^a_{o\mu} &= Z_3^{\frac{1}{2}} A^a_\mu \\
\eta^a_o &= Z_3'^{\frac{1}{2}}\eta^a \\
\psi^j_{Ao} &= Z_{2f}^{(i)\frac{1}{2}}\psi^j_A \\
\alpha_o &= Z_3\alpha \\
m_{jo} &= Z_m^{(i)} m_j \\
g_o &= Z_g\, g \\
g_o C^{abc}(\partial^\mu A^{a\nu}_o)A^b_{o_\mu}A^c_{o_\nu} &= Z_1\, g\, C^{abc}(\partial^\mu A^{a\nu})A^b_\mu A^c_\nu \\
g_o C^{abc}(\partial^\mu \xi^a_o)\eta^b_o A^c_{o_\mu} &= Z_1'\, g\, C^{abc}(\partial^\mu \xi^a)\eta^b A^c_\mu \\
g_o T^a_{AB}\bar\psi^j_{Ao}\gamma^\mu \psi^j_{Bo}A^a_{o_\mu} &= Z_{1f}^{(j)}\, g\, T^a_{AB}\bar\psi^j_A\gamma^\mu\psi^j_B A^a_\mu
\end{aligned}\right] \qquad \text{(A2)}$$

The flavour index j on the fermion related renormalisation constants is not really necessary for C.T. renormalisation but will be so for the H.F. renormalisation scheme, we introduce it here for uniformity of notation.

In the M.S. scheme we have confirmed the following 1-loop results. [Beware: results quoted for dimensional regularisation in $n = 4 + \varepsilon$, dimensions <u>not</u> $n = 4 - \varepsilon$, we do this to maintain an equivalence in notation to Ovrut and Schnitzer[2]].

$$
\left.
\begin{aligned}
Z_{2f}^{(j)} &= 1 + \frac{g^2}{16\pi^2}\left[\,2\alpha C_2(F)\,\right]\frac{1}{\varepsilon} \\
Z_3 &= 1 + \frac{g^2}{16\pi^2}\left[\,(\alpha - 13/3)\ C_2(A) + 8/3\ T(F)\,\right]\frac{1}{\varepsilon} \\
Z_3' &= 1 + \frac{g^2}{16\pi^2}\left[\,(\alpha/2 - 3/2)\ C_2(A)\,\right]\frac{1}{\varepsilon} \\
Z_g &= 1 + \frac{g^2}{16\pi^2}\left[\,11/3\ C_2(A) - 4/3\ T(F)\,\right]\frac{1}{\varepsilon} \\
Z_m^{(j)} &= 1 + \frac{g^2}{16\pi^2}\left[\,6\ C_2(F)\,\right]\frac{1}{\varepsilon} \\
Z_1 &= 1 + \frac{g^2}{16\pi^2}\left[\,(3/2\ \alpha - 17/6)\ C_2(A) + 8/3\ T(F)\,\right]\frac{1}{\varepsilon} \\
Z_{1f}^{(j)} &= 1 + \frac{g^2}{16\pi^2}\left[\,(3/2 + \alpha/2)\ C_2(A) + 2\alpha\ C_2(F)\,\right]\frac{1}{\varepsilon} \\
Z_1' &= 1\ \frac{g^2}{16\pi^2}\left[\,\alpha\ C_2(A)\,\right]\frac{1}{\varepsilon}
\end{aligned}
\right] \qquad (A3)
$$

The S-T identities

$$Z_3^{\frac{1}{2}} Z_g = Z_1/Z_3 = Z_1'/Z_3' = Z_{1f}^{(j)}/Z_{2f}^{(j)} \qquad \text{(A4)}$$

being used to generate Z_1, Z_1' & $Z_{1f}^{(j)}$. The group invariants are defined by:

$$T^a_{AB}\, T^a_{BC} = C_2(F)\, \delta_{AC} \; ; \; C_{abc}\, C_{acd} = C_2(A)\, \delta_{bd} \; ; \; Tr(T^a T^b) = T(F)'\, \delta^{ab}$$

$$\text{with } T(F) = PT(F)' \qquad \text{(A5)}$$

$\mathcal{L}$ written in terms of the renormalised fields and parameters looks like,

$$\begin{aligned}\mathcal{L} = &- 1/4\; Z_3(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - Z_1\, g\, C^{abc}\, (\partial^\mu A^{a\nu})\, A^b_\mu\, A^c_\nu \\ &- 1/4\; Z_1^2\, Z_3^{-1}\, g^2\, C^{abc}\, C^{ade}\, A^b_\mu\, A^c_\nu\, A^{d\mu}\, A^{e\nu} - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 \\ &- Z_3'\, \xi^a\, \partial^2 \eta^a - Z_1'\, g\, C^{abc}\, (\partial^\mu \xi^a)\, \eta^b\, A^c_\mu \\ &+ \sum_{j=1}^{p} \left\{ Z_{2f}^{(j)}\, \bar{\psi}^j_A\, i\not{\partial}\, \psi^j_A - Z_m^{(j)}\, Z_{2f}^{(j)}\, m_j\, \bar{\psi}^j_A\, \psi^j_A + Z_{1f}^{(j)}\, g\, T^a_{AB}\, \bar{\psi}^j_A\, \gamma^\mu\, \psi^j_B\, A^a_\mu \right\}\end{aligned} \qquad \text{(A6)}$$

with the standard definitions of the R.G. functions[9] we obtain the one loop results, $\beta(g) = + \frac{g^3}{16\pi^2}\left[-11/3\; C_2(A) + 4/3\; T(F)\right]$[8]

$$\left.\begin{aligned}\gamma_{m_j}(g,\alpha) &= \frac{g^2}{16\pi^2}\left[-6\; C_2(F)\right] \\ \gamma_f(g,\alpha) &= \frac{g^2}{16\pi^2}\left[\alpha\; C_2(F) \right] \\ \gamma_A(g,\alpha) &= \frac{g^2}{16\pi^2}\left[(\alpha/2 - 13/6)\; C_2(A) + 4/3\; T(F) \right] \\ \gamma_\xi(g,\alpha) &= \frac{g^2}{16\pi^2}\left[(\alpha/4 - 3/4)\; C_2(A) \right] \\ \delta(g,\alpha) &= -2\alpha\; \gamma_A(g,\alpha)\end{aligned}\right\} \qquad \text{(A7)}$$

APPENDIX B

THE COMPLETE THEORY M.S. RENORMALISED GLUON PROPAGATOR AND 3-GLUON VERTEX

The gluon progagator may be written in the form

$$D^{ab}_{\mu\nu}(q) = \delta^{ab}(1 + \pi(q^2))^{-1} \frac{(-g_{\mu\nu} + (1 - \alpha\,(1+\pi(q^2))\,q_\mu\,q_\nu/q^2)}{q^2 + i\varepsilon} \tag{B1}$$

For simplicity $\pi(q^2)$ is evaluated for the condition that

$$|q^2| << m_i^2 \qquad i = 1, \ldots\ldots, p \tag{B2}$$

The result in a model with broken flavour symmetry is easily obtained from the work of Ovrut and Schnitzer[2] with a few minor manipulations.
We have at the one loop level (same notation)

$$\pi(q^2) = \frac{-g^2}{16\pi^2}\left\{ A \sum_{i=1}^{p} \mathrm{Ln}\left(\frac{m^{(i)2}}{\hat{\mu}^2}\right) + D\,\mathrm{Ln}\left(\frac{-q^2}{\hat{\mu}^2}\right) - H \right\} + O\left(\frac{-q^2}{m^{(i)2}}\right) \tag{B3}$$

where

$$\left.\begin{aligned} A &= \frac{4}{3}\,T(F)' \\ D &= \left(\frac{\alpha}{2} - \frac{13}{6}\right) C_2(A) \\ H &= -\left(\frac{\alpha^2}{4} + \frac{\alpha}{2} + \frac{97}{36}\right) C_2(A) \end{aligned}\right] \tag{B4}$$

$T(F)'$ is defined in eqn (A5).

Further $\hat{\mu}^2 = 4\Pi e^{-\gamma_E}\,\mu^2$ (B5)

with μ the subtraction point and γ_E Euler's constant.

At the one loop level the 1PI amplitude $\Gamma^{(3,0)}(p,q,r)$ evaluated at the symmetry point $p^2 = q^2 = r^2$ can be written as

$$\Gamma^{(3,0)}(p,q,r) = \Gamma_1(q^2)\,\left[g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-r)_\mu + g_{\lambda\mu}(r-p)_\nu\,\right] \tag{B6}$$

$\Gamma_1(q^2)$ when evaluated under condition (B2) is given by,

$$\Gamma_1(q^2) = g - \frac{g^3}{16\Pi^2}\left\{ A \sum_{i=1}^{p} \mathrm{Ln}\left(\frac{m^{(i)2}}{\hat{\mu}^2}\right) + D\,\mathrm{Ln}\left(\frac{-q^2}{\hat{\mu}^2}\right) - H\right\} + O\left(\frac{-q^2}{m^{(i)2}}\right) \qquad \text{(B7)}$$

with

$$A = \frac{4}{3}\,T(F)'$$

$$D = \left(\frac{3\alpha}{4} - \frac{17}{12}\right) C_2(A) \qquad \text{(B8)}$$

$$H = \left([-\frac{3}{8} + \frac{23}{72}\,I] - \frac{3}{8}\,[1 + I]\alpha + \left[-\frac{5}{8} + \frac{I}{12}\right]\alpha^2 + \frac{\alpha^3}{24}\right) C_2(A)$$

where $I = -2\int_0^1 dx\,\frac{\mathrm{Ln}x}{x^2 - x + 1} = 2.3439072\ .....$ (B9)

(This is again taken from Ovrut and Schnitzer[2] with minor modifications) Although at the one level we do not need to know the complete theory renormalised fermion propagator, a useful reference in which it is fully evaluated with no approximations is de Rafael[10].

REFERENCES AND FOOT NOTES

1. T. Applequist & J. Carazzone Phys. Rev. D11, 2856 (1975).

2. B.A. Ovrut & H.J. Schnitzer Nucl. Phys. B179, 381 (1981).

3. B.A. Ovrut & H.J. Schnitzer Phys. Rev. D22, 2518, (1980); B.A. Ovrut & H.J. Schnitzer Phys. Lett. 100B, 403, (1981); B.A. Ovrut & H.J. Schnitzer Brandeis preprint (July 1980) 'Decoupling Theorems and effective field theories'.

4. S. Weinberg Phys. Lett 91B (1980) P.51.

5. F.J. Gilman & M.B. Wise Phys. Rev. D20, 2392 (1979).

6. M.A. Shifman, A.I. Vainshtein & V.I. Zakharov Nucl. Phys. B120, 316 (1977).

8. Strictly speaking

$\beta(g,\varepsilon) = \mu \frac{dg}{d\mu} = +g \frac{\varepsilon}{2} + \beta(g)$, because of Dimensional regularisation in $n = 4 + \varepsilon$ dimensions. It is $\beta(g)$ we list in (A7), the minor difference is always understood. The $g\frac{\varepsilon}{2}$ term does play a non-trivial role in evaluation of renormalisation group functions.

9. See for example, A. Peterman Phys. Rep. 53C, 157, (1979).

10. E. De Rafael in GIF, 10TH (1978) Vol. 2. Chpt. 6.

11. See footnote F2 in ref. (2).

12. B. Ovrut and H.J. Schnitzer, Brandeis preprint (Sept. 1980). 'Low energy & the threshold calculations using effective field theory.'

13. O. Nachtmann and W. Wetzel preprint HD-THEP-81-1 'The β function for effective quark masses' D.V. Nanopoulos and D.A. Ross Nucl. Phys. B157 (1979) P273. R. Tarrach, Marseille preprint (1980) 'The pole mass in perturbative Q.C.D.'

[The Nachtmann and Tarrach results agree, both disagreeing with Nanopoulos.]

14. C.T. Hill, 'Quark and Lepton masses from R.G. fixed points', FERMILAB-PUB-80/97-THY, Dec. 1980.

15. B.A. Ovrut and H.J. Schnitzer 'Gauge Theory and effective Lagrangians' Brandeis preprint (March 1981).

16. Note $\hat{D}^{ab}_{\mu\nu} \neq \langle T \{ \hat{A}^a_\mu \hat{A}^b_\nu \} \rangle_o$ • $\hat{D}^{ab}_{\mu\nu}$ refers to the sum of all Feynman diagrams contributing to the indicated Green's function and which contain at least one internal heavy line.

17. D.G. Unger and Y-P Yao 'Effective Parameters and the Renormalisation Group in Grand Unified Theories', Michigan preprint UM-HE 81-30 advocate a similar procedure.

TABLE I

n=	Λ_n^2 (GeV^2)	$\tilde{\Lambda}_n^{(n)2}$ (GeV^2)
3	.026	-
4	.023	0.862
5	.017	8.83
6	.0080	671
7	.0028	3190
8	.00057	32500

Λ_n^2 & $\hat{\Lambda}_n^{(n)2}$ examplifying the methods outlined in Section III C. Here results are specific to m.s. with $\Lambda^2_{4(m.s.)} = 0.023$ GeV^2 used as input along with threshold masses $m^{(4)} = 1.6$ GeV, $m^{(5)} = 4.7$ GeV $m^{(6)} = 35$ GeV, $m^{(7)} = 75$ GeV and $m^{(8)} = 230$ GeV.

FIGURE CAPTIONS

Figure 1: The interconnecting renormalisation schemes and their notations.

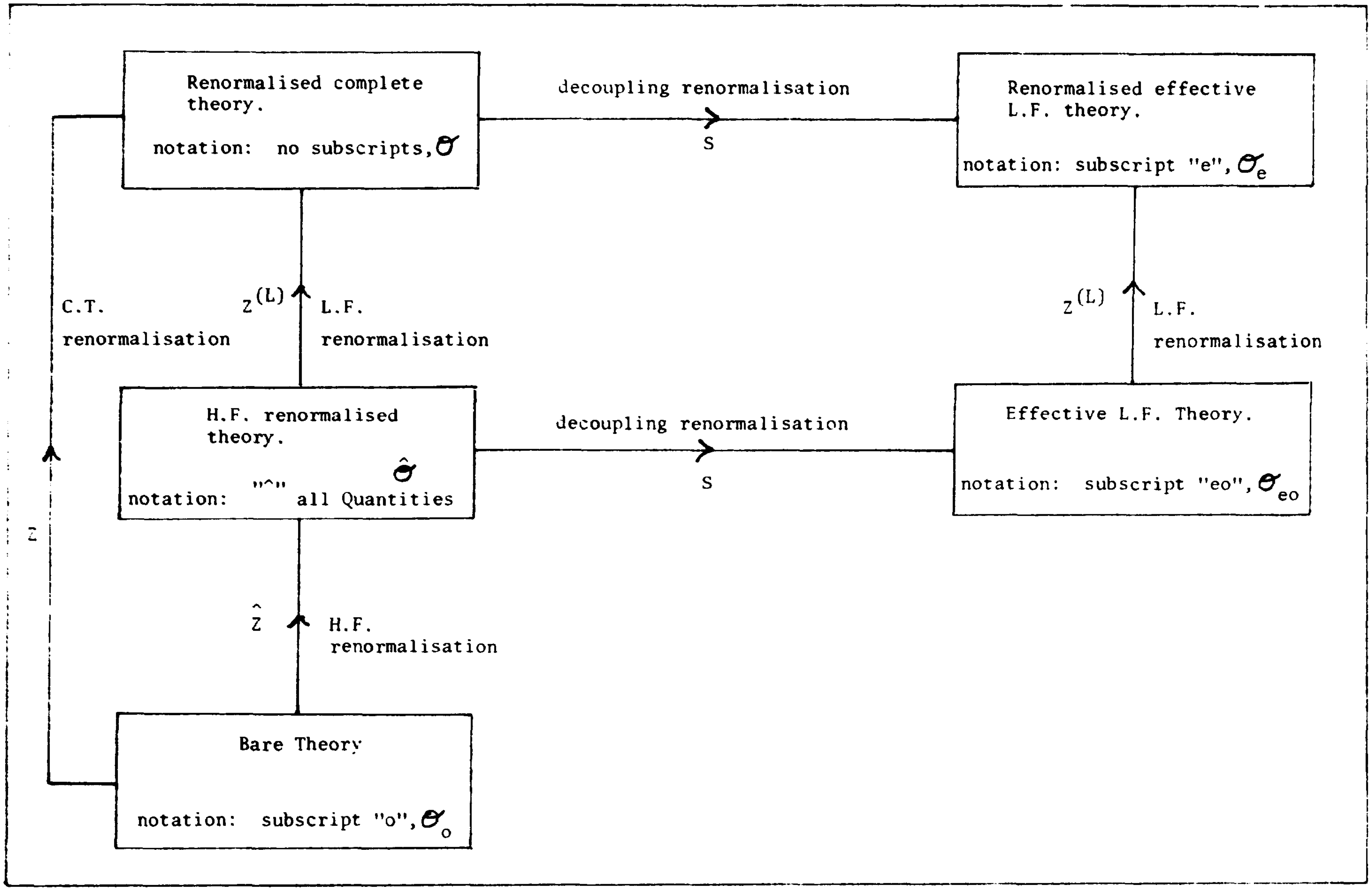
Renormalised complete theory.
notation: no subscripts, $\mathcal{O}$
decoupling renormalisation
S
Renormalised effective L.F. theory.
notation: subscript "e", $\mathcal{O}_e$
C.T. renormalisation
$Z^{(L)}$
L.F. renormalisation
$Z^{(L)}$
L.F. renormalisation
H.F. renormalised theory.
notation: "^" all Quantities $\hat{\mathcal{O}}$
decoupling renormalisation
S
Effective L.F. Theory.
notation: subscript "eo", $\mathcal{O}_{eo}$
Z
$\hat{Z}$
H.F. renormalisation
Bare Theory
notation: subscript "o", $\mathcal{O}_o$

FIG. 1