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FERMIONS AND BOSONS: A "SPINLESS" APPROACH*

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ABSTRACT. The fundamental difference between fermions and bosons is presented. The treatment used is based only on indistinguishability and its related implications on interference, with no mention to spin. Comparison between indistinguishable (fermions or bosons) and distinguishable identical particles are also made, yielding the enhancement (bosons) or inhibition (fermions) factors which determine the quantum distribution equations. (author).

RESUMO. É apresentada a diferença fundamental entre fermions e bosons. O tratamento utilizado se baseia apenas na indistinguibilidade e suas implicações relativas à interferência, sem mencionar o spin. São também feitas comparações entre partículas idênticas indistinguíveis (fermions ou bosons) e distinguíveis, fornecendo os fatores de amplificação (bosons) ou inibição (fermions) que determinam as equações das distribuições quânticas. (autor)

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I - INTRODUCTION

In a first course on Modern physics it is usual to introduce the concepts of fermions and bosons through their spin characteristics. This approach is not quite good, in our opinion, for two reasons: i) the concept of spin is a very abstract one, and difficult for young Physics or Engineering students to understand; and ii) the spin approach tends to hide the essential feature of the problem, that is, the interference phenomena which determines the behavior of fermions and bosons.

We present in next section a simple distinct approach based exactly on interference arguments, with no need of spin concepts.

II - INTERFERENCE APPROACH

We represent the eigenstates of a physical observable (say position, or momentum) of identical particles by the symbols α, β, γ, etc. In (one-dimensional) position representation, the eigenfunctions associated to those eigenstates are $\psi_\alpha(x)$, $\psi_\beta(x)$, $\psi_\gamma(x)$, etc. We assume that these identical particles interact weakly. There are two distinct cases: i) the particles are identical but distinguishable; and ii) the particles are identical and indistinguishable. Case i) occurs when the mean distance between two neighboring particles is large compared to their De Broglie Wavelength. Case ii) occurs otherwise, when there are interference effects between wave functions of individual particles.

The quantities P and Q defined in equations 1.a and 1.b give the probability of finding two identical distinguishable particles 1 and 2 in quantum states α and β respectively or β and α, particle 1 between positions x_1 and $x_1 + dx_1$, and particle 2 between x_2 and $x_2 + dx_2$.

$$P = |\psi_\alpha(x_1)\psi_\beta(x_2)|^2 dx_1\ dx_2 \qquad (1.a)$$

$$Q = |\psi_\alpha(x_2)\psi_\beta(x_1)|^2 dx_1\ dx_2 \qquad (1.b)$$

If the particles are indistinguishable, quantum theory does not allow us to reach so detailed information as is obtained through quantities P or Q separately. In particular, the word "respectively" in the previous paragraph is meaningless if we can not distinguish the particles. In this case, the amplitudes $\psi_\alpha(x_1)\psi_\beta(x_2)$ and $\psi_\alpha(x_2)\psi_\beta(x_1)$ correspond to indistinguishable events which must be treated as equiprobable, following the rules of quantum theory (1). In other words, these two amplitudes must have the same absolute value. This implies that they differ only by a phase factor as expressed in equation 2.

$$\psi_\alpha(x_2)\psi_\beta(x_1) = e^{i\theta}\ \psi_\alpha(x_1)\psi_\beta(x_2) \qquad (2)$$

Equation 2 can be interpreted as a rule: a permutation of two identical indistinguishable particles corresponds to multiply the associated amplitude by a phase factor $e^{i\theta}$. The interesting feature is that the phase difference θ is an observable quantity, and that it depends upon the nature of the particles we are dealing with (electron or photons, or neutrons, etc). The quantity R defined in equation 3 gives the probability of finding two identical <u>indistinguishable</u> particles 1 and 2 in quantum states α and β, one of them between positions x_1 and $x_1 + dx_1$, and the other between x_2 and $x_2 + dx_2$.

$$
\begin{aligned}
R &= |\psi_\alpha(x_1)\psi_\beta(x_2) + e^{i\theta}\psi_\alpha(x_2)\psi_\beta(x_1)|^2 \, dx_1 \, dx_2 = \\
&= \{ |\psi_\alpha(x_1)\psi_\beta(x_2)|^2 + |\psi_\alpha(x_2)\psi_\beta(x_1)|^2 + \\
&+ e^{i\theta}\psi^*_\alpha(x_1)\psi^*_\beta(x_2)\psi_\alpha(x_2)\psi_\beta(x_1) + \\
&+ e^{-i\theta}\psi^*_\alpha(x_2)\psi^*_\beta(x_1)\psi_\alpha(x_1)\psi_\beta(x_2) \} \, dx_1 \, dx_2 \qquad (3)
\end{aligned}
$$

Using the fact that the value of R must be invariant under permutation between particles (because they are indistinguishable), and from equation 3, it is easy to show that there are <u>only two</u> possible values for the phase difference θ [2]: 0 or π. The former corresponds to particles called <u>bosons</u>, and the latter to <u>fermions</u>. Electrons, protons and neutrons are examples of fermions, while photons, α-particles and phonons are bosons.

The physical difference between fermions and bosons becomes evident when we take α and β as the same eigenstate, that is $\beta = \alpha$ [3]. In this case, we note that the value of R vanishes for fermions, and is enhanced for bosons. Two fermions can not occupy the same quantum state, while two bosons present a greater tendency to occupy the same quantum state when compared to distinguishable particles.

Let us take an interval Δx, around some position x, and ask for the probability of finding two identical particles, in the same quantum state α, both in the same interval Δx. Equation 4 defines this probability $P_2(x,\Delta x)$ in the case of <u>distinguishable</u> particles, and equation 5 defines the correspondent probability $R_2(x,\Delta x)$ in the case of <u>indistinguishable</u> particles. Although there are only two allowable values for the phase factor $e^{i\theta}$, we maintain[4] the use of the symbol θ because it makes clear that we are dealing with interference phenomena.

$$P_2(x,\Delta x) = \iint_{\Delta x} dx_1\ dx_2 |\psi_\alpha(x_1)\psi_\alpha(x_2)|^2 = |\psi_\alpha(x)\psi_\alpha(x)|^2\ \Delta x^2 \tag{4}$$

$$R_2(x,\Delta x) = \frac{1}{2}\iint_{\Delta x} dx_1\ dx_2 |\psi_\alpha(x_1)\psi_\alpha(x_2) + e^{i\theta}\psi_\alpha(x_2)\psi_\alpha(x_1)|^2 =$$

$$= |\psi_\alpha(x)\psi_\alpha(x)|^2\Delta x^2\{1 + \cos\theta\} \tag{5}$$

The factor 1/2 in equation 5 is necessary in order to avoid double counting - the definition of the quantity R, above equation 3, does not distinguish which particle (1 or 2) is between x_1 and $x_1 + dx_1$ or between x_2 and $x_2 + dx_2$. In the case of fermions $R_2(x,\Delta x)$ vanishes, while in the case of bosons $R_2(x,\Delta x)$ is doubled when compared to $P_2(x,\Delta x)$ for two distinguishable particles. Equations 6.a and 6.b relate the case of distinguishable particles to the case of indistinguishable particles, through the factor $f_2(\theta)$ plotted in figure 1.

$$R_2(x,\Delta x) = P_2(x,\Delta x)f_2(\theta) \tag{6.a}$$

$$f_2(\theta) = 1 + \cos\theta \qquad (6.b)$$

Note that equation 6.b is formally the same which appears in the double slit interference problem. If $\theta = 0$ (bosons), there is constructive interference. If $\theta = \pi$ (fermions), the interference is destructive.

Equations 4 and 5 can be generalized for three particles 1, 2 and 3. The quantity $P_3(x,\Delta x)$, defined in equation 7, is the probability of finding three identical <u>distinguishable</u> particles 1, 2 and 3, in the same quantum state α, all in the same interval Δx. The quantity $R_3(x,\Delta x)$, defined in equation 8, is the analogous probability for the case of <u>indistinguishable</u> particles[5].

$$P_3(x,\Delta x) = \iiint_{\Delta x} dx_1\ dx_2\ dx_3 |\psi_\alpha(x_1)\psi_\alpha(x_2)\psi_\alpha(x_3)|^2 =$$

$$= |\psi_\alpha(x)\psi_\alpha(x)\psi_\alpha(x)|^2\ \Delta x^3 \qquad (7)$$

$$R_3(x,\Delta x) = \frac{1}{3!}\iiint_{\Delta x} dx_1\ dx_2\ dx_3 |\psi_\alpha(x_1)\psi_\alpha(x_2)\psi_\alpha(x_3) +$$

$$+ e^{i\theta}\psi_\alpha(x_2)\psi_\alpha(x_1)\psi_\alpha(x_3) + e^{2i\theta}\psi_\alpha(x_3)\psi_\alpha(x_1)\psi_\alpha(x_2) +$$

$$+ e^{3i\theta}\psi_\alpha(x_1)\psi_\alpha(x_3)\psi_\alpha(x_2) + e^{4i\theta}\psi_\alpha(x_2)\psi_\alpha(x_3)\psi_\alpha(x_1) +$$

$$+ e^{5i\theta}\psi_\alpha(x_3)\psi_\alpha(x_2)\psi_\alpha(x_1)|^2 =$$

$$= |\psi_\alpha(x)\psi_\alpha(x)\psi_\alpha(x)|^2\ \Delta x^3\ \{1 +$$

$$+ \frac{1}{3!}\sum_{n=1}^{5} (6-n)(e^{in\theta} + e^{-in\theta})\} \qquad (8)$$

The relation between the distinguishable and indistinguishable cases is made in equations 9.a and 9.b, through the factor $f_3(\theta)$ plotted in figure 2.

$$R_3(x,\Delta x) = P_3(x,\Delta x) f_3(\theta) \tag{9.a}$$

$$f_3(\theta) = 1 + \frac{1}{3!} \sum_{n=1}^{5} (6-n)(e^{in\theta} + e^{-in\theta}) \tag{9.b}$$

Equations 9.a and 9.b can be easily generalized for the case of N particles, resulting in equations 10.a and 10.b. The sum on equation 10.b was performed through geometrical series formulae, and can also be easily verified.

$$R_N(x,\Delta x) = P_N(x,\Delta x) f_N(\theta) \tag{10.a}$$

$$f_N(\theta) = 1 + \frac{1}{N!} \sum_{n=1}^{N!-1} (N!-n)(e^{in\theta} + e^{-in\theta}) =$$

$$= \frac{1}{N!} \frac{\operatorname{sen}^2(N!\theta/2)}{\operatorname{sen}^2(\theta/2)} \tag{10.b}$$

The interesting feature is that $f_N(0) = N!$ and that $f_N(\pi) = 0$. The former originates the so-called enhancement factor $1 + N$ for bosons, and the latter originates the inhibition factor $1 - N$ for fermions. Starting from those factors, one can[6] easily reach the Bose or Fermi distribution equations.

III - ACKNOWLEDGEMENTS

We are indebted to M. Bazin, S.L.A. de Queiroz, S.M. Oliveira and J.A. Swieca for helpful discussions.

REFERENCES

1 - R.P. Feynman, "The Feynman Lectures on Physics", vol III, Addison-Wesley (1965), Chapter 1.

2 - In one dimension it is possible to define a phase factor $e^{i\,\varepsilon(x_1-x_2)\theta}$, where $\varepsilon(x_1-x_2)$ is the signal function of x_1-x_2. Under this point of view, all real values are allowed for θ. In three dimensions, however, this device does not work. This argument has been given to us by Dr. J.A. Swieca.

3 - Strictly, we can not take $\beta = \alpha$ in equation 3, because, in this case, the linear combination of two amplitudes becomes unnecessary. We can take, however, close eigenstates $\beta \to \alpha$ of an observable such as position or momentum, which vary *continuously*.

4 - A similar treatment is used in the study of parastatistics

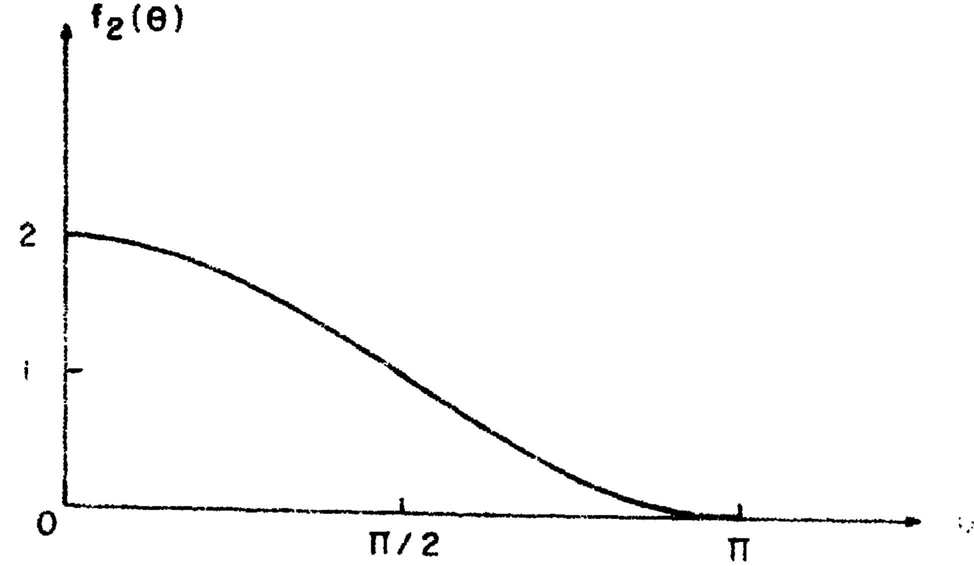
5 - In equation 8, starting from one possible amplitude $\psi_\alpha(x_1)\psi_\alpha(x_2)\psi_\alpha(x_3)$, we have built each of the other five possibilities by making just one permutation on the preceding one, and multiplying by a phase factor $e^{i\theta}$. It would be irrelevant if we had chosen a different order for making the permutation, because the phase factor $e^{i\theta}$ can take only two possible values: +1 or -1.

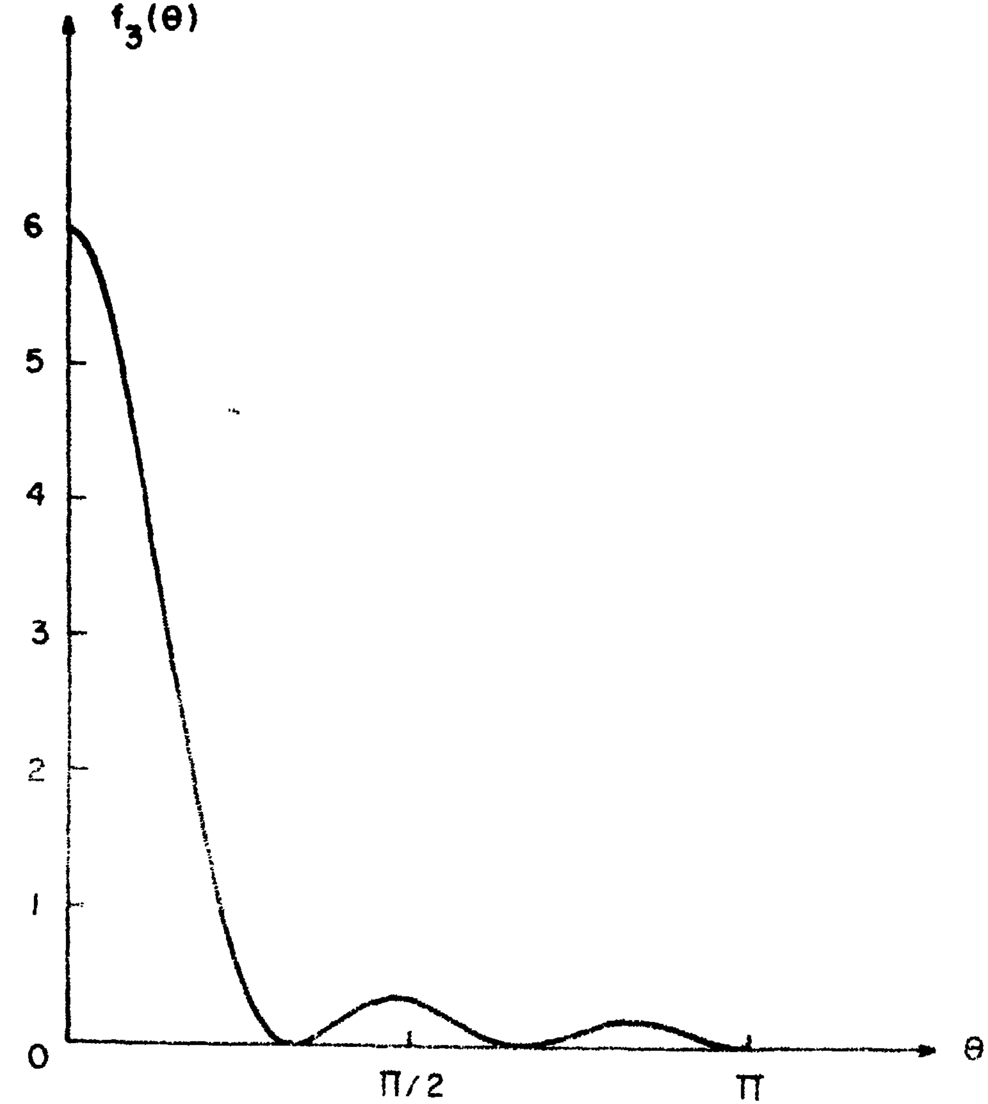
6 - See, for example, R. Eisberg and R. Resnick "Quantum Physics of Atoms, Molecules, Solids, Nuclei and Particles" John Wiley & Sons, Inc. (1974), chapter 11.

FIGURE CAPTIONS

Fig. 1 - Plot of $f_2(\theta)$.

Fig. 2 - Plot of $f_3(\theta)$.


f2(θ)
2
1
0
π/2
π


f3(θ)
6
5
4
3
2
1
0
Π/2
Π
θ