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# REMARKS ON LATTICE GAUGE MODELS+

H. Grosse

Institut für Theoretische Physik
Universität Wien

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Abstract

We report on a study of the phase structure of lattice gauge models where one takes as a gauge group a non-abelian discrete subgroup of SU(3). In addition we comment on a lattice action proposed recently by Manton and observe that it violates a positivity property.

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## I. Introduction

A complete study of a gauge field theory would consist first of a treatment of the classical theory, second of finding classical solutions to the field equations and third of an attempt to develop a quantization scheme. Since any study of the quantum field theoretical aspects requires the introduction of a cut-off, and introducing a cut-off destroys certain symmetries the theory has, one intends to find a procedure which allows to maintain as many properties as possible. The only ultraviolet cut-off procedure known to be consistent with gauge invariance which allows to stay in the appropriate space-time dimensions is Wilson's [1] method of formulating the theory on a lattice.

We intend to start with a few comments on the requirements one would like to have fulfilled by a quantum field theory [2], and discuss next ideas of testing confinement. In a second chapter we give the usual Wilson lattice approach fulfilling most of these requirements but being representation dependent. In order to overcome this disease Manton [3] proposed recently an alternative action which as we shall show violates Osterwalder-Schrader positivity [4] for any gauge group containing a U(1) subgroup. In a third part we comment on known phase-properties of U(1) and SU(2) gauge theories [5] and add properties for theories having a discrete non-abelian subgroup of SU(3) as gauge group [6].

In a classical treatment of gauge theories one starts with a gauge potential A and a field strength $F = dA + [A,A]$, both being Lie algebra valued forms. Clearly the relevant quantity is not A themselves, but the orbit of A under gauge transformations [A] which can be characterized by the holonomy group consisting of elements

$$g_c = P\,(\exp\ [i \int_c dz^\mu A_\mu(z)]) \qquad (1)$$

out of the gauge group G, where c denotes a nodeless closed curve (an element out of the loop space $\Omega$) and P means path ordering. Invariant functions on G can be characterized by the set $\{\chi_i(g_c)\,|\,c \in \Omega,\ \chi_i =$ irreducible character of G$\}$. In trying to quantize a gauge theory, this set plays an important role; n-point functions being gauge dependent are replaced by n-loop functions $S_n(W_{i_1}(c_1)\ldots W_{i_n}(c_n))$ where the Wilson-loop observables $W_i(c)$ should result from $\chi_i(g_c)$ by a suitable normal ordering [7]. If one would be able to construct a set of n-loop functions $S_n$ by some limiting procedure, the requirements one would like to check are:

a) local gauge invariance,
b) euclidean invariance implying Poincaré invariance in Minkowski space time,
c) Osterwalder-Schrader positivity, implying a positive definite scalar product and the spectral condition,
d) cluster properties in order to check whether one has a dynamical mass generation or not.

On a lattice one replaces b) by
b') lattice translation invariance, and adds in addition:
e) formally correct $a \to 0$ limit, where a being a lattice constant,
f) representation independence.

Next one would intend to study further properties the theory has, especially confinement properties, where one can imagine a number of suitable definitions for a SU(n) theory [7]:

a) Fields transforming nontrivial under $Z_n$ should not connect the vacuum to the physical one particle state.
b) Asymptotic states transform trivial under the center.
c) Physical states are bound states of two and three fermions.

In practice one clearly likes to simplify and one tries to get information about the effective potential, which two fermions will feel if one couples them to the gauge field, from the pure gauge theory themselves: Define [1]

$$V(L) = -\lim_{T\to\infty} \frac{1}{T} \ln |S_1(W(C))| \tag{2}$$

where C denotes a rectangular curve of extension L×T, then we speak of confinement in the sense of Wilson iff

$$V(L) \underset{L\to\infty}{=} \sigma\cdot L\cdot a^2 \tag{3}$$

with $\sigma > 0$ being the string tension.

## II. Formulation of Lattice Gauge Theories

Here we discuss only the Lagrangian as opposed to the Hamiltonian approach. On a euclidean cubic lattice one is dealing with quantities defined on lattice sites, bonds and plaquettes. A gauge field is associated with bonds; a mapping $b \to g_b \in G$, such that the inverse group element is assigned to the reversed bond defines a field configuration $\{g_b\}$ [4].

The most essential step of establishing a theory consists in fixing a lattice action. Given a character $\chi$ one gets for the Wilson loop variable and for a curve $C = \partial P$ being the boundary $\partial P \doteq \{b_1, b_2, b_3, b_4\}$ of a plaquette P:

$$W(\partial P) = \chi(g_{b_1} g_{b_2} g_{b_3} g_{b_4}) =: \chi(g_{\partial P}) \ . \tag{4}$$

Replacing $A_\mu$ in equ.(1) by $g_u A_\mu$, $g_u$ being the unrenormalized coupling constant and expanding formally (4) in terms of the lattice constant gives [1]:

$$\mathrm{Re}\ \chi(g_{\partial P}) \underset{a\to 0}{\simeq} d - \frac{a^4 g_u^2}{2} \mathrm{Tr}\ F^2 + O(a^6) \ , \qquad d = \chi(e) \tag{5}$$

where e denotes the unit element of the group. This justifies Wilson's definition for an action which leads to a gauge invariant generalization of Ising models first studied by Wegner [8]:

$$S_\Lambda(\{g\}) = \sum_{P\subset\Lambda} [d - \mathrm{Re}\ \chi(g_{\partial P})] \ , \tag{6}$$

$\Lambda$ denotes space time volume. Expectation values of local observables (depending on finitely many bond variables) are then defined by integrating over all possible field configurations:

$$\langle F\rangle_\Lambda = Z_\Lambda^{-1} \int \prod_{b\in\Lambda} dg_b\ F(\{g_b\})\ \exp\left[-\frac{1}{g_u^2} S_\Lambda(\{g\})\right] , \tag{7}$$

$dg_b$ denotes the Haar measure of the group and $Z_\Lambda$ the partition function. Identifying $1/g_u^2 = \beta$ with an inverse temperature makes contact with the classical statistical mechanics interpretation of euclidean lattice theories; the two languages can therefore be identified [9], so for instance a phase transition corresponds to a change of the vacuum, an exponential decrease of correlations corresponds to the existence of a mass gap; defects correspond to nontrivial topological configurations etc.

After taking the thermodynamic limit $\Lambda \to \mathbb{R}^4$ one tries to check the above mentioned requirements: a), b'), and e) are fulfilled, so the most essential question concerns the positivity property [4]. More explicitly one defines a $t = 0$ plane which is half way between lattice planes; this allows to define algebras $A_\pm$ of local observables defined on bond variables out of $\Lambda_\pm$, which are the subspaces of bonds with $t > 0$ or $t < 0$. One then asks for an antilinear mapping $\Theta$ of the field algebra $A$ to $A$ with

$$\Theta F(\{g_{xy}\}) = \overline{F(\{g_{rx,ry}\})} \tag{8}$$

where the bar means complex conjugation and r means the reflection on the plane $t = 0$; so $\Theta|A_\pm = A_\mp$. O.S. positivity is then the requirement that

$$\langle F\cdot\Theta F\rangle_\Lambda \geq 0 \qquad \forall\ F \in A_+ \ . \tag{9}$$

Since (9) implies on the one hand that the underlying space carries a positive definite scalar product and on the other hand one obtains a positive transfer matrix (besides chess-board estimates), requirement c) seems to be essential. It has been shown in [4] that Wilson's action fulfills (9).

Here we note that in a theory based on an alternative action proposed by Manton [3], which has been used recently in a number of calculations [10] and seems to allow for a smoother continuum limit, positivity is violated:

Theorem: Assume that the gauge group contains a U(1) subgroup and take as an action

$$S_\Lambda^M(\{g\}) = \sum_{P\subset\Lambda} D^2(e,g_{\partial P}) \tag{10}$$

where $D(e,g_{\partial P})$ denotes the length of the smallest geodesic connecting the unit element to $g_{\partial P}$; then O.S. positivity is violated.

Remarks: Clearly a violation for some subgroup implies the same for the larger group, so we may restrict ourselves to the U(1) case. The next step consists in choosing a gauge in which all bond variables for bonds crossing the t = 0 plane are set equal to unity, which implies a factorization of the integrations involved in (9). Positivity is implied by showing that exp $\{-\beta D^2(e,g_{\partial P})\}$ is a function of positive type on the group

$$\int dg \int dh\ F^*(g)\ e^{-\beta D^2(e,gh^{-1})}\ F(h) \geq 0 \tag{11}$$

which means that all fourier coefficients have to be positive. For the U(1) case it is simple to establish that the coefficients $I_n(\beta)$

$$I_n(\beta) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi}\, e^{in\alpha}\, e^{-\beta\alpha^2} \underset{\beta\to 0}{\simeq} (-)^{n+1}\, \frac{2\beta}{n^2} + O(\beta^2)\ , \quad n > 0\ , \tag{12}$$

change sign depending on whether n is even or odd, so that (11) is violated.

In the next chapter we will mainly concentrate on the observable energy per plaquette:

$$E(\beta) = \langle\chi(g_{\partial P})\rangle = -\frac{1}{6}\frac{\partial F}{\partial\beta}\ , \qquad F = \lim_{N\to\infty}\frac{\ln Z_\Lambda}{N}\ , \tag{13}$$

with N being the number of lattice points. For an attempt to match the strong coupling behaviour with asymptotic freedom results the string tension as defined in equ.(3) is of great importance too, although it is not a local observable.

May be it is interesting to mention two general results: It has been shown in [4] that the cluster expansion for local observables has a finite radius of convergence, implying analyticity of these quantities in the complex $\beta$ plane for $|\beta|$ small. In addition the cluster property d) is verified with a generated mass $m \geq c(\ln g_u)/a$ for large $g_u$. Furthermore Wilson's area law is realized in that phase.

There is actually a general proof saying that the potential $V(L)$ defined in (2) is bounded by the linear potential [11] $V(L) \leq cL$.

## III. Phase Transitions

A physical mass has to be a function of the form $m_{ph} = f(g_u)/a$ (in a pure gauge theory); so since $a \to 0$ corresponds to $g_u \to 0$ the continuum limit corresponds to an approach of a critical point of the theory. Clearly the phase, in which the theory will be for large $\beta = 1/g_u^2$, will determine the properties of the continuum theory. Since for large coupling constant all lattice theories show confinement, one likes to have one phase transition in the U(1) theory, while none should be present for SU(2) and SU(3). In this context it is interesting to note that a phase transition has been observed for the SU(5) theory [12].

### a) Abelian gauge theories:

Some time ago A. Guth [13] obtained after a work of Glimm and Jaffe a rigorous estimate on the Wilson loop variable for the U(1) theory implying that the potential is nonconfining for large $\beta$. Together with our previous remarks this implies the existence of at least one phase transition.

In addition to rigorous results one has been able to obtain further insight into the phase structure of lattice theories by using Monte Carlo simulations. In that way one starts from an initial configuration, goes through the lattice a number of times and tries to determine an equilibrium configuration which allows to determine expectation values approximately:

$$\langle F \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T d\tau \; F(\{g_\tau\}) \; . \qquad (14)$$

The first result for U(1) obtained by Creutz, Jacobs and Rebbi [5] indicated exactly one phase transition. Additional support was obtained by studying discrete subgroups of U(1).

For $Z_2$, $Z_3$ and $Z_4$ one observes one phase transition at the inverse temperatures determined by duality. $Z_2$ and $Z_4$ are actually very special theories. Their critical points fulfill $\beta_c^{Z_4} = 2\beta_c^{Z_2}$. In addition it was known [14] that the two spin systems in two dimensions are actually equivalent up to a scale transformation. So it was natural to look for a similar relation between the gauge theories [15].

With the help of the strong coupling expansion [16] we have been able to show equality between two noninteracting $Z_2$ theories putting them onto the same lattice and one $Z_4$ theory with a scaled temperature. For the partition functions and any volume $\Lambda$ we obtained equality

$$Z_\Lambda^{Z_4}(2\beta) = Z_\Lambda^{Z_2}(\beta)^2 \tag{15}$$

by identifying terms in the high temperature expansion. The actual proof is unfortunately technical and has up to now not been generalized to other theories.

Going up to $Z_n$ with $n \geq 5$ one observes two phase transitions. One moves rapidly out to low temperature, the other approaches the critical point of the U(1) theory.

b) Non-abelian theories:

One could ask why one believes to learn something about the phase structure of a theory through the study of theories where one takes a discrete subgroup as a gauge group. One result pointing in that direction was obtained first by Mack and Petkova [17] and generalized further in Ref. [18] and says that confinement within the $Z_n$ theories implies (up to a scale transformation) confinement in the SU(n) case.

For SU(2) one expects a roughening transition around $\beta \approx 2$, the region where one observes a turn-over from the strong coupling behaviour to the asymptotic freedom behaviour. A study of the non-abelian subgroups (there exist finitely many) shows one transition moving out up to $\beta \approx 6$ [19] for the icosahedral group which has 120 elements.

So for SU(2) one is in a satisfying position: Not only is it possible to parametrize the group manifold themselves easily; through the study of subgroups it is also possible to study the range of $\beta$ values up to $\beta \simeq 6$ which includes the roughening point.

For SU(3) things are not in such a good shape: Besides attempts to calculate directly the averaged action per plaquette [20] only a few points for the Wilson loop variable have been obtained. We asked ourselves the question how much one can learn by taking discrete nonabelian subgroups as a gauge group.

At first that program seems to be promising. There exist infinitely many nonabelian subgroups of SU(3) [21]. Besides finitely many crystal-like groups two infinite sequences of subgroups $\Delta(3n^2)$ and $\Delta(6n^2)$ exist, which are actually semidirect products of $Z_n \times Z_n$ with $Z_3$ and $S_3$:

$$\Delta(3n^2) = Z_n \times Z_n \underset{s}{\times} Z_3 , \qquad \Delta(6n^2) = Z_n \times Z_n \underset{s}{\times} S_3 . \tag{16}$$

The multiplication laws can be written down compactly and all irreducible representations and characters are known in closed form. Certain sequences of subgroups together with their representations have been studied recently by the Bonn group [22].

We have performed the familiar thermal cycles in order to get insight into the phase structure of these theories and have compared the results with high temperature and low temperature expansions [6]. For all investigated groups (we took first $\Delta(3n^2)$) the Monte Carlo results agree nicely with series expansions up to a certain value of $\beta$; then a phase change occurs. But by increasing n beyond 5 we observed two phase transitions, one moving out to large values of $\beta$, the other one stays around $\beta \simeq 2.2$ (see figures 1 and 2). Unfortunately it is clear from [20], that a turn-over from the strong to the weak coupling regime occurs around $\beta \simeq 6$, so that the nice feature of the SU(2) case is not reproduced here.

In the meantime we have investigated the $\Delta(6n^2)$ groups [23] and observed a similar situation (with a transition around 2.6). Since Bhanot and Rebbi [24] obtained a phase change for the largest crystal-like subgroup S(1080) around $\beta \simeq 3.0$ we have to conclude that through the study of these subgroups it is not possible to get information about the turn-over point.

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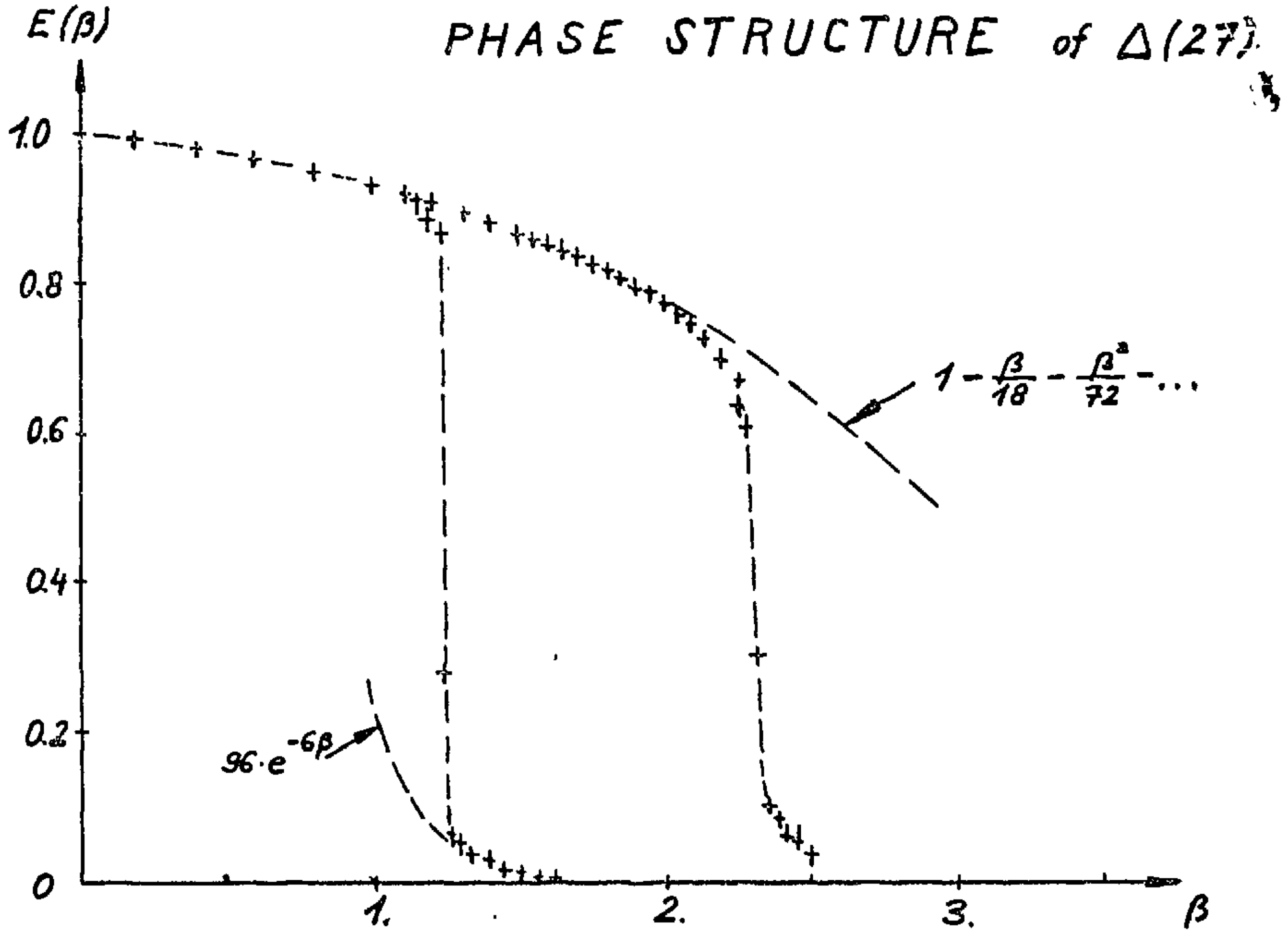


Figure 1

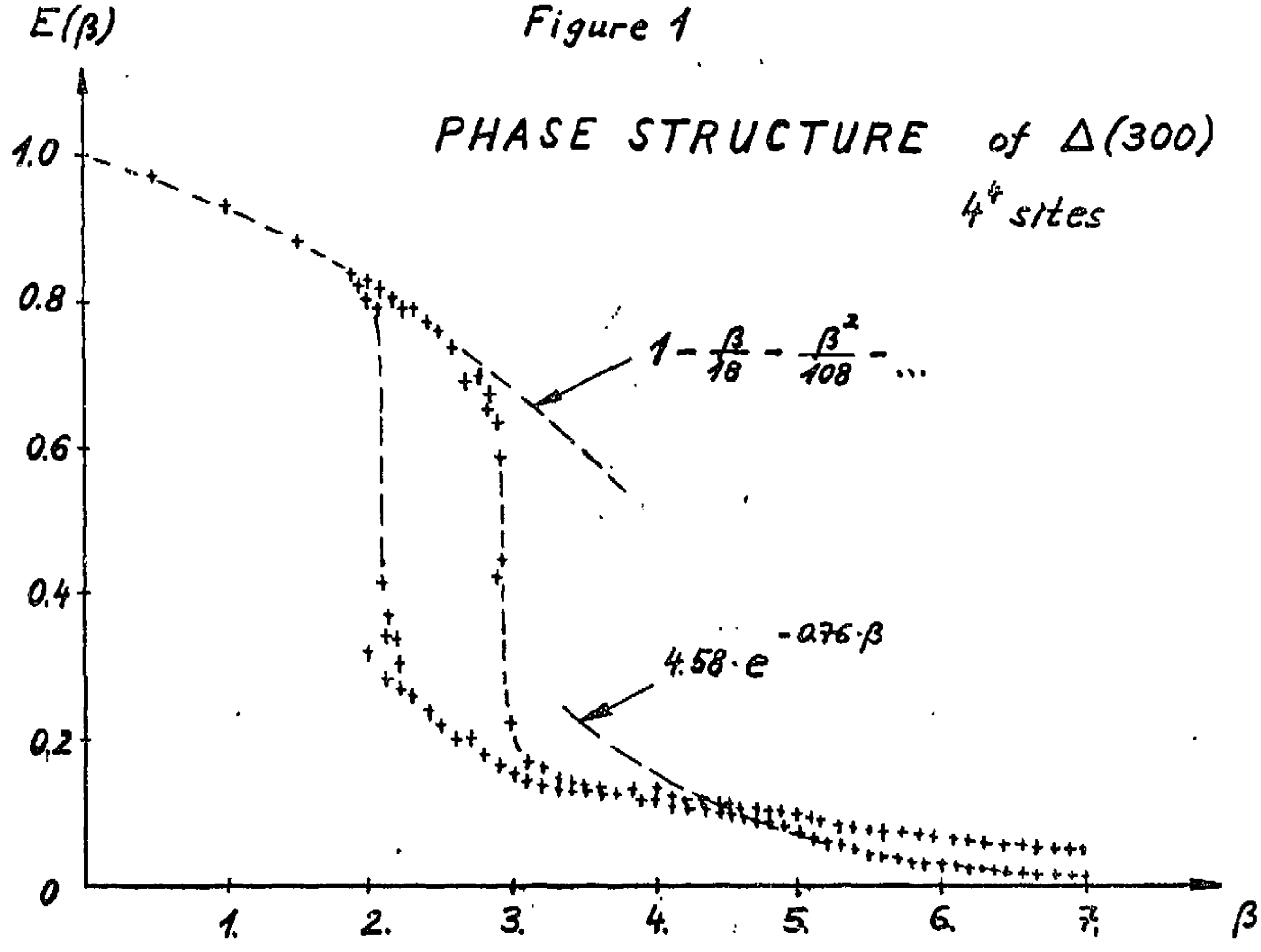


Figure 2