

**MASTER**

THE COMPUTATION OF INVERSE MAGNETIC CASCADES\*

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ABSTRACT

Inverse cascades of magnetic quantities for turbulent incompressible magnetohydrodynamics are reviewed, for two and three dimensions. The theory is extended to the Strauss equations, a description intermediate between two and three dimensions appropriate to tokamak magnetofluids. Consideration of the absolute equilibrium Gibbs ensemble for the system leads to a prediction of an inverse cascade of magnetic helicity, which may manifest itself as a major disruption. An agenda for computational investigation of this conjecture is proposed.

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## I. INTRODUCTION

### A. Background

Over the last fifteen years, the term "inverse cascade" has been applied to a class of turbulent processes in which the small spatial scales (high wavenumbers) feed some quantity, by nonlinear processes, to the large spatial scales (low wavenumbers). A variety of quantities can be inversely cascaded, depending upon the medium and upon the geometry: the process can look quite different in different cases. Inverse cascades have been most intensively studied in Navier-Stokes fluids and incompressible magnetofluids. Most progress has been numerical, and the accurate computation of an inverse cascade is a strenuous numerical challenge because of the high spatial resolution required. This needed high spatial resolution considerably restricts the range of allowable Reynolds numbers, and the implications of these restrictions have yet to be felt in the plasma simulation community, though the hydrodynamicists are painfully aware of them.

The idea of an inverse cascade first appeared in connection with flows in two-dimensional<sup>1,2,3,4,5</sup> Navier-Stokes fluids, then in magnetofluids, first in three dimensions<sup>6,7,8,9</sup> and then in two.<sup>10,11,12</sup> Most of the work has assumed the simplest and most highly idealized boundary conditions: rectangular periodic ones. But now, possible relevance to the "magnetic dynamo" and to magnetic fusion confinement is exerting pressure for computations with fewer idealizations and more realism. Computations are crucial for demonstrating most of the theoretical consequences. Analytical indicators can be used to anticipate the existence of inverse cascades, but accurate prediction of any of their details seems inherently to require sophisticated numerics. No one has had much success proceeding solely with pencil and paper. The intention here is to clarify the implications of certain classes of possible computations.

The mechanism underlying an inverse cascade is essentially simple.<sup>3,13</sup> Two things seem to be necessary: (1) a dissipation which only becomes effective at small spatial scales, such as viscous or Ohmic dissipation; and (2) two or more non-dissipative integral invariants of the motion which are representable as sums which emphasize differently the large and small scales.

Take an example: two-dimensional Navier-Stokes flow at high Reynolds numbers (the same mathematics does as well for electrostatic guiding center plasmas). The two non-dissipative invariants, expressed in terms of the Fourier coefficients of the velocity field  $\underline{v}(\underline{k}, t)$ , are the energy per unit mass  $E$  and the mean-square vorticity or "enstrophy"  $\Omega$ :

$$E = \sum_{\underline{k}} |\underline{v}(\underline{k}, t)|^2, \quad \Omega = \sum_{\underline{k}} k^2 |\underline{v}(\underline{k}, t)|^2 .$$

The non-dissipative terms in the equations of motion are the nonlinear ones, and if they transfer excitations to the high  $|\underline{k}|$  part of the spectrum (as they surely must), the only way the simultaneous constancy of both  $E$  and  $\Omega$  can be preserved is for there to be an accompanying transfer to low  $|\underline{k}|$ . The additional constant of the non-dissipative motion  $\Omega$  (which is not conserved in three dimensions) demands that the non-dissipative (i.e., non-linear) terms in the Navier-Stokes equation must transfer excitations in both directions in wavenumber space, if there is any transfer at all. The paths in  $\underline{k}$  space by which the transfer occurs are a secondary issue, but there is nothing simple about them.

Subtleties arise when one begins to ask sharper questions, such as which quantity gets transferred in which direction and how fast. These questions have been earnestly addressed in the literature and do not lead to short, graphic answers, only to long, contingent ones. Relatively simple and persuasive answers can be given for the quasi-steady state, when a source of excitations, band-limited in wavenumber space, is regarded as injecting the excitations at a statistically steady rate and attention is directed toward the spectra and the transfer rates of the cascaded quantities. The word "source" can mean lots of things, from an externally-applied forcing field (such as an electric field or current field, in magneto-hydrodynamics) to some microscopic instability which might be present.

Computationally, one often works initial value problems starting from smooth initial conditions, and in that case, it may be useful to identify a particular unstable normal mode as associated with the onset of the turbulent fluctuations. In the laboratory one usually does not work an initial value problem starting from smooth initial conditions; rather, unstable systems are often created only with some level of excitations already in place. It may then be a less interesting question as to which "mode" is the "most unstable" one,

because the fully developed and observed state may bear little relation to the kind of smooth profiles on which any kind of stability calculations can be done. A hydrodynamic analogy is pipe flow far above the critical Reynolds number: there is hardly even an academic relation between the details of the flow and any stability calculation that can be done. Particularly in plasma confinement experiments, there may be many simultaneous sources of excitation: a gradient in virtually any mean field variable (pressure, magnetic field, electric current, flow velocity, density, temperature) is potentially a source of excitations. Many sources may act simultaneously. They may interfere with or reinforce each other.

What happens to the excitations once they are launched on their journey through  $k$ -space may be the more interesting and the more physically significant question. There is a clear difference between turbulent processes that transfer their disturbances to small rather than large scales from the point of view of thermonuclear confinement. In the former case, the worst that can happen is an enhanced transport, a disadvantage which may well be offset by an enhanced heating. (A tokamak which had no "directly cascaded" turbulence of this kind might not heat at all!) Transfer toward large scales can clearly be more serious: macroscopic bulk deformations can redistribute and disrupt the plasma variables and perhaps terminate the confinement. Disruptions in tokamaks have some of the flavor of what one might expect from an inverse cascade, though experiments with highly resolved enough diagnostics to study the small scales of what happens in a major disruption appear not to have been carried out. (A respectable beginning on the problem was made some time ago by Hutchinson<sup>14</sup> and Morton.<sup>15</sup>) For the near-term future, connections between tokamak disruptions and possible inverse cascades are likely to continue to be primarily a numerical subject.

## B. Numerical Examples

We briefly remark upon three examples of attempted computations of inverse magnetic cascades. The quantities expected to be inversely cascaded are mean square vector potential in two dimensions and magnetic helicity in three.

Figure 1, due to Pouquet,<sup>16</sup> displays the results of a closure calculation of an inversely cascading magnetic vector potential spectrum in two dimensions. The mechanical and magnetic Reynolds numbers are of the order of 4500. The excitations are injected at a wave number indicated by an arrow, and everything at lower wave numbers is the result of back transfer in wave number space. The computation is not a direct solution of the dynamical equations themselves, but rather a statistical, eddy-damped "closure" approximation to them which permits higher values of the Reynolds numbers than a direct computation could possibly permit with present-day computers.

Figure 2, due to Fyfe et al.,<sup>12</sup> shows a direct numerical solution of the two-dimensional magnetohydrodynamic equations, driven by a random magnetic forcing confined to the indicated wavenumber band. All the points below  $k^2 = 55$  correspond to inverse magnetic transfer to long wavelengths. This, incidentally, is a calculation in which spatial resolution was inadequate to the Reynolds number chosen, and as a consequence there is no "dissipation range", or region of precipitous fall-off at increasing  $k$ , apparent at the upper end.

Figure 3 is taken from Meneguzzi et al.,<sup>17</sup> and illustrates the results of a direct computation from the three-dimensional incompressible magnetohydrodynamic equations, again with a random forcing, but this time with a helical, mechanical one. A weak "seed" magnetic field amplifies with time as the injected kinetic energy converts into magnetic energy through dynamo action. The back transfer of magnetic helicity is apparent.

Additional direct inverse cascade computations which may be mentioned are those of Lilly<sup>18</sup> and Fyfe et al.,<sup>12</sup> for the two-dimensional Navier Stokes case; see also Pouquet et al.<sup>19</sup> for the corresponding closure computation. Pouquet and Patterson<sup>7</sup> have displayed three-dimensional magnetohydrodynamic direct computations. Pouquet et al.<sup>8</sup> have given three-dimensional inverse cascade closure computations for incompressible magnetohydrodynamics.

All the computations cited assume rectangular periodic boundary conditions with no net flux of any quantity through any cross section of the system.

## II. THE STRAUSS EQUATIONS; ABSOLUTE EQUILIBRIUM ENSEMBLE THEORY

Rectangular periodic boundary conditions, which characterize most of the computational efforts to study inverse cascades, idealize away several key features of real systems. In particular, they preclude net fluxes of such quantities as electric current through a cross-section of the region of computation. They also rule out all true boundary effects, which undoubtedly play a role in real situations. It is desirable to state the ideas associated with inverse cascades in a mathematics that corresponds to more realistic representations of actual plasmas.

The Strauss equations<sup>20</sup> are a set of magnetohydrodynamic equations that are close to those of incompressible two-dimensional magnetohydrodynamics, but which include some important three-dimensional effects. They are far more tractable than the full set of three-dimensional magnetohydrodynamic equations. What appears to the writer to be a more transparent derivation than Strauss's can be given, ending with the same equations; the derivation is not simple, however. Their most important feature is that of their near two-dimensionality: the variable magnetic fields and velocity fields are perpendicular to the z-direction (say), but the field variables are functions of all three spatial coordinates. The variable part of the magnetic field is  $\underline{B}$  and is expressed in terms of a vector potential A as  $\underline{B} = \nabla_{\perp} \times \hat{e}_z A$ . (The subscript " $\perp$ " will always mean perpendicular to  $\hat{e}_z$ .) The velocity field  $\underline{v}$  is expressed in terms of a stream function U as  $\underline{v} = \nabla_{\perp} \times \hat{e}_z U$ . There is a constant, uniform dc magnetic field in the z direction, of magnitude  $B_0 \gg |\underline{B}|$ . The vorticity  $\omega$  and the current density  $\underline{j}$  are in the z direction and are given by  $\nabla_{\perp}^2 U = -\omega$  and  $\nabla_{\perp}^2 A = -j$ , in terms of A and U.

The Strauss equations<sup>20</sup> are, in a familiar set of dimensionless variables in which flow velocities are measured in units of the Alfvén speed,

$$\frac{\partial A}{\partial t} + \underline{v} \cdot \nabla_{\perp} A - \mu \nabla_{\perp}^2 A = B_0 \frac{\partial U}{\partial z} \quad (1)$$

$$\rho \left( \frac{\partial \omega}{\partial t} + \underline{v} \cdot \nabla_{\perp} \omega \right) - \underline{B} \cdot \nabla_{\perp} j - \rho \nu \nabla_{\perp}^2 \omega = B_0 \frac{\partial j}{\partial z} \quad (2)$$

The uniform dimensionless mass density  $\rho$  can be consistently set equal to unity. The quantities  $\mu^{-1}$  and  $\nu^{-1}$  are essentially the magnetic and mechanical Reynolds numbers, respectively.

Except for the terms on the right hand sides of Eqs. (1) and (2) and the z-dependences, Eqs. (1) and (2) are identical with the equations of two-dimensional incompressible magnetohydrodynamics,<sup>10-12</sup> a system which is now getting to be rather well understood. Many interesting questions can be asked of Eqs. (1) and (2). For example, the question of possible stability thresholds for quiescent, slowly-decaying equilibria as  $\mu$  and  $\nu$  decrease towards zero is an analogue of the question of the stability of hydrodynamic pipe flow or Couette flow as the Reynolds number increases. Computational studies of Eqs. (1) and (2) have been initiated by Carreras, Hicks and others<sup>21,22,23,24</sup> in connection with tokamak confinement.

The following considerations are intended to facilitate this enterprise by sharpening the mathematical framework in which the interpretation of a major disruption as an inverse magnetic cascade may be numerically tested. In a paper published in 1977, we put forward the suggestion<sup>12</sup> that the inverse cascade of magnetic helicity might be responsible for major tokamak disruptions, and a test of this hypothesis seems now to be within reach. Related considerations are discussed in a forthcoming paper by Tetreault.<sup>25</sup>

All inverse cascade computations to date have been motivated by considering first a model problem in which the non-dissipative dynamical equations have been expanded in a set of orthogonal functions.<sup>3,13</sup> The expansion is then truncated at a large but finite number of terms. Statistical mechanical procedures are then performed in the phase space defined by the expansion coefficients. Somewhat surprisingly, absolute equilibrium canonical ensembles (Gibbs distributions) have proved to be accurate predictors of time averages of phase functions of the expansion coefficients. Considering the limiting behavior, as the number of expansion coefficients becomes infinite, highlights any tendencies which may exist for some invariant to migrate to long wavelengths.

The Gibbs distributions are constructed from those non-dissipative invariants which remain invariant after the truncation: i.e., are "rugged". For the Strauss equations there are apparently three such rugged invariants. They are the energy  $E$ , the "cross helicity"  $H_c$ , and the magnetic helicity  $H_m$ :

$$E = \int r dr d\theta dz [B^2 + v^2] \quad (3)$$

$$H_c = \int r dr d\theta dz \omega A \quad (4)$$

$$H_m = B_0 \int r dr d\theta dz A. \quad (5)$$

The integrations are over the three dimensional region  $0 \leq \theta < 2\pi$ ,  $0 \leq r < a$ ,

$0 \leq z < L_z$ , where periodicity in  $z$  with period  $L_z$  is assumed. Free-slip boundary conditions  $\underline{v} \cdot \hat{e}_r = 0$ ,  $\underline{B} \cdot \hat{e}_r = 0$  are assumed at  $r = a$ , the wall of the rigid cylinder. For added realism (but added complexity), we might add the "no slip" boundary conditions on the tangential components:  $(\nabla_{\perp} \times \underline{B}) \times \hat{e}_r = 0$  and  $\underline{v} \times \hat{e}_r = 0$ . These are superfluous, however, for the model problem at hand, and greatly complicate the mathematics.

Usually, only quadratic invariants are easily proved to be "rugged", and this can be achieved by the somewhat formal device of treating  $B_0$  itself as a phase space coordinate whose equation of motion just happens to be  $dB_0/dt = 0$ .  $B_0$  itself is then a rugged invariant, and all the terms in Eqs. (1) and (2) as well as the expressions (3)-(5) can be considered as quadratic. The ensemble chosen can be chosen to be sharp in  $B_0$ : i.e., microcanonical in  $B_0$  but canonical in the other invariants.

$A$  and  $U$  are expanded in the complete orthonormal set of eigenfunctions of the Laplacian,<sup>26</sup>

$$A_{nmq} \equiv C_{nmq} J_m(\gamma_{nmq} r) \exp(im\theta + ik_n z), \quad (6)$$

where  $k_n = 2\pi n/L_z$ ,  $m$  and  $n$  are integers, and  $\gamma_{nmq} a$  is the  $q$ th zero of  $J_m(x)$ . The normalization constant  $C_{nmq}$  is

$$C_{nmq} = (\pi L_z a^2)^{-1/2} / J_{m+1}(\gamma_{nmq} a). \quad (7)$$

If we define  $\lambda_{nmq}^2 = \gamma_{nmq}^2 + k_n^2$ ,  $\nabla^2 A_{nmq} = -\lambda_{nmq}^2 A_{nmq}$ , and  $\nabla_{\perp}^2 A_{nmq} = -\gamma_{nmq}^2 A_{nmq}$ . The  $A_{nmq}$  are eigenfunctions of both  $\nabla_{\perp}^2$  and  $\nabla^2$ .

We may write the infinite sums

$$U = \sum_{nmq} \eta_{nmq} A_{nmq} \quad (8)$$

$$A = \sum_{nmq} \xi_{nmq} A_{nmq} \quad (9)$$

and

$$E = \sum_{nmq} \gamma_{nmq}^2 (|\xi_{nmq}|^2 + |\eta_{nmq}|^2) \quad (10)$$

$$2H_c = \sum_{nmq} \gamma_{nmq}^2 \xi_{nmq}^* \eta_{nmq} + (\text{complex conjugate}) \quad (11)$$

$$H_m = 2B_0 \left( \frac{\pi}{L_z} \right)^{1/2} \sum_q \frac{\xi_{00q}}{\gamma_{00q}} \equiv \Lambda \sum_q \frac{\xi_{00q}}{\gamma_{00q}}, \quad (12)$$

where for economy of notation,  $\Lambda \equiv 2B_0 (\pi/L_z)^{1/2}$ . The sums are over large but finite sets of terms once the truncation is performed. Also to be truncated is the set of ordinary first-order differential equations for  $d\xi_{nmq}/dt$  and  $d\eta_{nmq}/dt$  that result when Eqs. (8) and (9) are inserted in Eqs. (1) and (2).

The Gibbs ensemble which is appropriate to the case of no initial correlation between  $\underline{y}$  and  $\underline{B}$  is the multivariate probability distribution

$$D_{\text{eq.}} = \text{const.} \times \exp \{-\alpha E - \beta H_m\}, \quad (13)$$

with reciprocal temperatures  $\alpha^{-1}$ ,  $\beta^{-1}$  chosen to match desired ensemble expectations  $\langle E \rangle$ ,  $\langle H_m \rangle$ .

Inserting Eqs. (10) and (12) into (13), the modal expectations are readily calculated. First  $\langle \xi_{nmq} \rangle = \langle \eta_{nmq} \rangle = 0$ ,  $m^2 + n^2 \neq 0$ , and

$$\langle |\xi_{nmq}|^2 \rangle = \langle |\eta_{nmq}|^2 \rangle = \frac{2}{\alpha \gamma_{nmq}^2}, \quad m^2 + n^2 \neq 0. \quad (14)$$

Also

$$\langle \xi_{00q} \rangle = -\frac{\beta \Lambda}{2\alpha} \frac{1}{\gamma_{00q}^3} \quad (15)$$

$$\langle (\xi_{00q} - \langle \xi_{00q} \rangle)^2 \rangle = \frac{1}{\alpha \gamma_{00q}^2}. \quad (16)$$

$\alpha$  and  $\beta$  are determined as the roots of

$$\langle E \rangle = \sum_{\substack{nmq \\ n^2+m^2 \neq 0}} \frac{4}{\alpha} + \sum_q \left\{ \frac{1}{\alpha} + \frac{\beta^2 \Lambda^2}{4\alpha^2 \gamma_{00q}^4} \right\}, \quad (17)$$

and

$$\langle H_m \rangle = -\frac{\Lambda^2 \beta}{2\alpha} \sum_q \frac{1}{\gamma_{00q}^4}, \quad (18)$$

which keep all the  $\langle |\xi_{nmq}|^2 \rangle$  and  $\langle |\eta_{nmq}|^2 \rangle$  positive.

If  $\langle E \rangle$  and  $\langle H_m \rangle$  are held fixed and the number of  $nmq$  modes is allowed to increase without limit, it is easy to show that  $\alpha \rightarrow \infty$ ,  $|\beta| \rightarrow \infty$  with  $|\alpha/\beta|$  a finite ratio. The sum in Eq. (18) is convergent as the maximum  $q \rightarrow \infty$ . Except for the excess energy distributed in infinitesimal increments over the  $m^2 + n^2 \neq 0$  modes, the fluid excitations freeze into the  $00q$  magnetic modes. The vector potential  $\langle A \rangle$  approaches a function

$$\begin{aligned} \langle A \rangle &\rightarrow \sum_{q=1}^{\infty} \langle \xi_{00q} \rangle A_{00q} \\ &= -\frac{\beta \Lambda}{2\alpha} (\pi L_z a^2)^{-1/2} \sum_{q=1}^{\infty} \frac{J_0(\gamma_{00q} r)}{\gamma_{00q}^3 J_1(\gamma_{00q} a)}, \end{aligned} \quad (19)$$

a universal function whose defining expression is manifestly convergent. For all  $m^2 + n^2 \neq 0$ ,

$$\frac{\langle |\xi_{nmq}|^2 \rangle}{\langle \xi_{00q} \rangle^2} \rightarrow 0$$

as the number of terms approaches infinity.

This behavior is slightly different from the corresponding inverse cascade behaviors found in previously examined cases, insofar as the helicity does not condense into the 001 mode alone but into the rapidly converging series defined by Eq. (19). The state (19) is in fact a uniform current density state, with  $\langle A \rangle$  varying proportionately to  $r^2$ . Nevertheless, the earmarks are there, and to anyone familiar with the previous history of the theory of inverse cascades, provide a basis for conjecturing that in the presence of dissipation and forcing, there would be an inverse cascade of  $H_m$  to long wavelengths, headed for the state defined by Eq. (19).

The uniform-current state (19) has the additional significance of being either the state of minimum energy for given helicity, or the state of maximal helicity for given energy. This can readily be seen from the Euler equation for the variational problem of minimizing  $E$  subject to a fixed value of  $H_m$ :  $\nabla_{\perp}^2 A =$  a constant. For the dissipative initial value problem, it has been previously demonstrated that under many circumstances, the directly cascadable invariants will "selectively decay" relative to the inversely cascadable ones, and their ratios will approach their theoretical lower bounds for large times.<sup>9,27</sup> For the Strauss equations, it appears that this uniform current state is the "selectively decayed" state, analogous to the "Taylor state"<sup>28</sup> or force-free state of the full set of magnetohydrodynamic equations.

If the more realistic "no slip" boundary conditions are invoked, requiring  $j = 0$  at  $r = a$ , the uniform current state is not attainable. At the least, a sharp current gradient must develop somewhere across the cross-section of the cylinder as the selective decay progresses, perhaps in the form of a boundary layer near  $r = a$ . (Something similar was seen in a recent selective decay calculation inside a compact toroid.<sup>29</sup>) This current gradient, which necessarily exists at the edge of a current-carrying plasma bounded by a conducting wall, looms as a rather universal and difficult-to-avoid source of "tearing mode turbulence."

This gradient in the current near the walls is one among many potential sources for the helicity which might be inversely cascaded. There is probably no single mechanism for supplying small-scale helicity. A second likely possibility is current filamentation<sup>12</sup> that may develop along local hot spots in the magnetofluid; the resistivity falls off with increasing temperature,

and may thus channel the current along hot tubes of force, resulting in still higher local heating. There is probably no single mechanism for supplying small-scale helicity, which basically results any time current flows along a field line in the presence of resistivity. Arguing in favor of particular drivers or sources may prove as fruitless an activity as the generation of debates that has surrounded linear instability theory. It appears imperative to produce major disruptions in the presence of as many parameter variations as possible, to begin to acquire discrimination among the variety of sources that may be operative.

### III. A POSSIBLE COMPUTATIONAL AGENDA

Numerical demonstration of the possible inverse cascade properties of Strauss's equations will be facilitated by taking advantage of earlier experience and conceptualizations gained in studying inverse cascades, and by resistance to letting the conceptual framework be circumscribed by linear stability analysis. The first limitation that will have to be confronted will be the limitation on the Reynolds numbers  $\nu^{-1}$ ,  $\mu^{-1}$ . A currently popular rule of thumb for fluid computations is that for every unit of Reynolds number, one grid point (or finite element, or expansion coefficient) is required in each spatial dimension. Thus a three-dimensional simulation at a Reynolds number of 30 requires about  $(30)^3$  grid points to resolve the smallest spatial scales. This limitation might be violated by a factor of two, but probably not by an order of magnitude. At present the smallest  $\mu$  and  $\nu$  that are feasible to compute with are between about  $10^{-2}$  and  $10^{-3}$  for two dimensions and  $10^{-1}$  to  $10^{-2}$  for three. The desired physical values, for real experiments, are likely to be considerably smaller than that (particularly  $\mu$ ). There is no simple way around this difficulty.

Progress can be made in perhaps only one of two ways. (1) It may be attempted to escape the connection between the aforementioned necessary spatial resolution and wave number requirements by introducing an artificial enhanced dissipation which only becomes effective at high wave numbers.<sup>17</sup> The hope is that small scale dissipation only provides a sink anyway, and that the large scale dynamics will become independent of the details of the sink. (2) Alternatively, one may settle for the qualitative demonstration of the physics and extrapolate crudely to the behavior at the very high Reynolds numbers. In the former choice, the ultimate validation can only be a comparison with the results of very high-resolution codes in which no anomalous dissipation is introduced. In the second choice, the ultimate recourse is most probably to experiments.

The most single important possibility, as far as major disruptions are concerned, has to do with the possible existence of thresholds in Reynolds numbers. Such thresholds, at sometimes surprisingly low values, have characterized<sup>17</sup> the three-dimensional computations on the dynamo problem: above certain critical Reynolds numbers, a given stirring mechanism will initiate an inverse cascade,

otherwise it will not.<sup>17</sup> Of particular interest to tokamaks is the possibility that because of the drop of resistivity with increasing temperature, critical Lundquist numbers (Alfvén speed x length scale/magnetic diffusivity) or magnetic Reynolds numbers (flow speed x length scale/magnetic diffusivity) will be crossed as the magnetofluid heats up, and an inverse cascade will start dramatically.

A spectral-method numerical solution in search of inverse helicity cascades might proceed as follows, for Eqs. (1) and (2). Pick  $\mu$ ,  $\nu \sim 1/50$  or so, with a resolution of the order of, say, 100 in the radial and azimuthal coordinates and perhaps 15 to 20 in the z coordinate. (It is unknown as to how rapidly small scales in z will multiply themselves, but if they develop as readily as the small scales in  $\theta$  and r, the Strauss equations probably are not useful, anyway.) There is no reason to believe less spatial resolution is required in  $\theta$  than in r. Invariance of the results to increased resolution in  $\theta$ , r, and z is a necessary check on their accuracy. An initial current profile which is thought to be close to experimental reality should be chosen and an external forcing term (probably from a random number generator) should be permitted to drive the magnetofluid at the small scales, either locally or randomly in space as well. This can be most effectively accomplished by adding a small random term to the right hand sides of Eqs. (1) and (2), in the manner of Lilly,<sup>18</sup> or Fyfe et al.<sup>12</sup>

The purpose of this first exercise would be simply to see if an inverse cascade with the features of a major disruption can be induced, either by lowering the dissipation coefficients or raising the strength of the random forcing. Once such an event has been shown to exist, an infinite variety of refinements of the calculation, such as eliminating the random forcing in favor of a resistivity which depends upon the local temperature T, are imaginable. T, for example, might be convected and grow locally due to Ohmic dissipation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\perp}\right) T = K \nabla_{\perp}^2 T + \eta j^2,$$

where  $\eta$  is a heating rate, and K is a thermal conductivity.

Finally, the most important question connected with major disruptions is not so much can they occur as what can be done about them. If their interpretation as an inverse cascade of helicity is correct, then at a formal level,

the remedy is apparent: feed the magnetofluid some helicity of opposite sign. This means in effect inducing in the plasma some current flowing in the opposite direction from the main toroidal current. Where and how to do this seems like a delicate matter. Not much help is to be expected from attempts at suppression by static external helical windings, which supply a vacuum helicity at a particular  $n,m$  mode, but do little to the bulk distribution over modes that provides the basis for an inverse helicity cascade.

The clear-cut demonstration of an example of inverse cascade behavior for the Strauss equations would undoubtedly stimulate many additional refinements and insights that are now hard to foresee. The time is overdue, also, when the plasma simulation community should begin to seek a conceptual framework for its computations in their hydrodynamic antecedents, and cease to expect that linear stability analyses will provide adequate insight into processes in which non-linear transfer is the dominant effect.

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#### FIGURE CAPTIONS

Fig. 1. Magnetic potential spectrum as a function of wave number for successive times, in two dimensions. Magnetic fluctuations and kinetic energy are injected at the wave number indicated by the arrow, and their transfer is approximated by an eddy-damped "closure" calculation. (Taken from Pouquet.<sup>16</sup>)

Fig. 2. Magnetic modal energies, averaged over angle, as function of  $|\underline{k}|$ , for a direct solution of the two-dimensional magnetohydrodynamic equations in two dimensions (taken from Fyfe et al.<sup>12</sup>). Magnetic fluctuations are randomly injected in the band between the arrows, and the spectrum is allowed to fill up. Initially, it is empty.

Fig. 3. Omni-direction magnetic energy spectrum for the three-dimensional case (taken from Meneguzzi et al.<sup>17</sup>). A small seed magnetic field is amplified by a driven velocity field which contains mechanical helicity,  $\int \underline{y} \cdot \underline{\omega} \, d^2x$ . The spectra are labeled by the appropriate values of the time. This is the most clear-cut computation of "dynamo" action to date.

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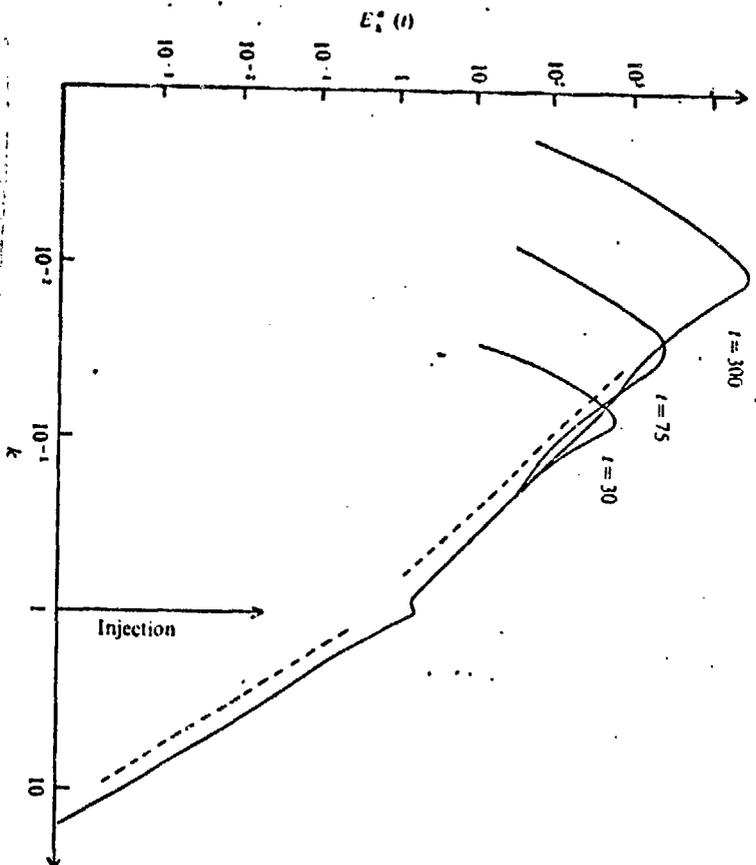


FIG. 1

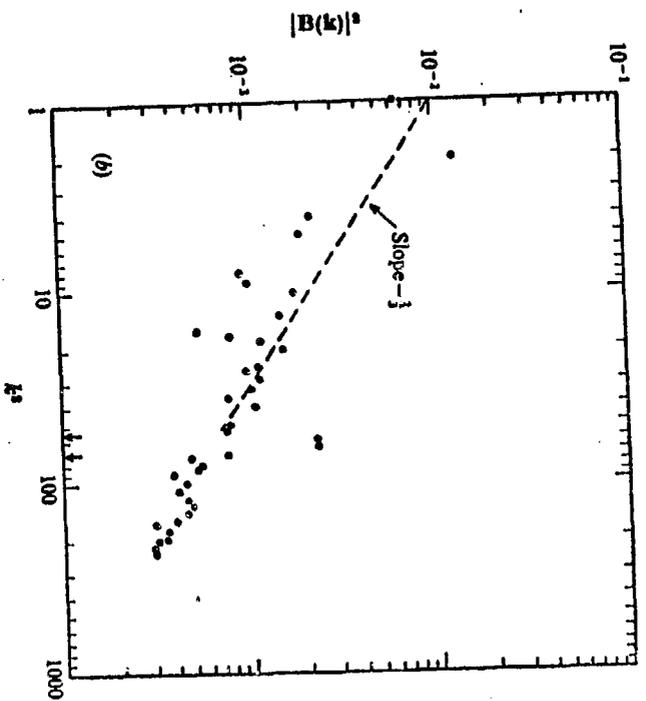


FIG. 2

# ENERGY SPECTRA

