MULTIPLET MASS SPLITTING IN A GRAVITATIONAL FIELD

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ABSTRACT

An expression for the mass splitting of particles belonging to the same spin multiplet defined in a space-time of general relativity is derived. The geometrical symmetry is a subgroup of $SO(r,s)$, $9 \geq r \geq 3$, $5 \geq s \geq 1$, the mass operator being proportional to the second order Casimir operator of that subgroup. A brief analysis of the calculated values as compared to the experimental data is included.

I. INTRODUCTION

Current models of quantum field theories on a curved background appear to lack appropriate symmetry principles similar to those existing in flat space-time theories. The consequence of transforming a Poincaré covariant quantum field equation into a general covariant equation is the substitution of the physically meaningful Poincaré symmetry for the difficult to handle manifold mapping group. Whereas the general covariance seems to be a valid principle from the classical point of view, quantum theories appear to be more symmetry demanding.

The purpose of this note is to investigate the extent in which the properties of the particle spectra can be used to indicate the type of space-time symmetry which is required to replace the Poincaré group, in a curved background situation, where that group is no longer a true symmetry. The resulting group should be compatible with the principle of general covariance, such as is the Lorentz group when regarded as the group of holonomy of the tetrads, or the de Sitter group when regarded as a cosmological symmetry. On the other hand the group should also be compatible with any internal symmetry which eventually appears in the theory. More specifically a combined symmetry arises and the results derived from this combination should agree with the observed particle spectra. As it is known the Poincaré group fails to be compatible with arbitrary combined symmetry schemes. This incompatibility results from the existence of several no go theorems which prove that the unification of the Poincaré group with internal symmetry groups cannot produce the observed mass diferences between particles belonging to the same spin multiplet [1], [2], [3]. There are three basic ways to avoid the mentioned theorems: (a) Assuming that the combined symmetry is an infinite Lie group; (b) Assuming that the combined symmetry is not a Lie group and (c) assuming that the geometric symmetry does not admit an Abelian normal subgroup from which the mass is derived. The alternative (a) may still be subjected to an extended theorem [4], while the alternative (b) has recently motivated substantial research [5]. The study of the potentialities of alternative (c) within the context of a curved space-time backgroung is the object of the present paper.

II. LOCAL SPACE-TIME SYMMETRY

Consider a space-time S locally and isometrically embedded in a flat space of $p \geq 4$ dimensions and with metric signature s(+) + s(-) denoted by M(r,s). Here the physical space is S and M(r,s) is regarded as a mere minimal container for S. However the proper homogeneous group of isometries of M(r,s), SO(r,s) has some physically interesting features. In the first place SO(r,s) contains a space-time subgroup $SO(r,s)|_S$ which generate all local homogeneous isometries of

S. Secondly SO(r,s) can be f it contracted into a group which contains the Poincaré group.

To see the above properties consider in M(r,s) a Gaussian coordinate system $\{x^\alpha\}$ based on S, constructed with the four arbitrary space-time coordinates x^i plus the p-4 coordinates x^A measured on the mutually orthogonal straight lines orthogonal to S at each point of S (Small case Latin indices run from 1 to 4 while capital Latin indices run from 5 to P. All Greek indices run from 1 to p). Thus $\{x^\alpha\} = \{x^i, x^A\}$.

An infinitesimal coordinate transformation of SO(r,s) in the Gaussian system is given by

$$x'^\alpha = x^\alpha + \xi^\alpha, \quad \xi^{(\alpha;\beta)} = 0, \quad \text{(fixed origin)} \tag{2.1}$$

where the covariant derivative is calculated with the metric affine connection of SO(r,s) (with Christoffel symbols written in Gaussian coordinates). The space-time subgroup $SO(r,s)\big|_S$ of SO(r,s) is defined by the equations

$$\begin{aligned} x'^\alpha &= x^\alpha + \xi^\alpha \\ \xi^{(\alpha;\beta)}\Big|_S &= 0 \\ \xi^A\Big|_S &= 0 \end{aligned} \tag{2.2}$$

where $\big|_S$ denotes the restriction to S (that is, calculated with $x^A = 0$). The last two equations imply that ξ^i is a Killing vector field in S whenever S admits isometries. Consequently $SO(r,s)\big|_S$ generates all possible homogeneous isometries of S [6].

The second mentioned property of SO(r,s) uses the definition of a local curvature radius for S. If $\eta^{\mu}_{(A)}$ denote the Cartezian components of the p-4 unit vectors orthogonal to S (and to themselves), then a point of the space M(r,s) not necessarily in the hypersurface S, has Cartezian coordinates

$$z^{\mu}(x^{i},\ x^{A}) = X^{\mu}(x^{i}) + x^{A}\ \eta^{\mu}_{(A)},$$

where $X^{\mu}(x^{i})$ are the Cartezian coordinates of a point in S and $\eta_{\mu\nu}$ denote the Cartezian components of the metric tensor of M(r,s). They are related to the Gaussian components by

$$g_{\alpha\beta} = z^{\mu}{}_{,\alpha}\ z^{\nu}{}_{,\beta}\eta_{\mu\nu} \quad .$$

Then it follows that

$$\begin{aligned} g_{ij}\Big|_{S} &= g_{ij}(S), \quad \text{(the metric tensor of S)}, \\ g_{iA} &= P_{ABi}x^{B}, \quad P_{ABi} + P_{BAi} = 0, \qquad (2.3) \\ g_{AB} &= \pm\ \delta_{AB}. \end{aligned}$$

The Christoffel symbols $\Gamma_{\alpha\beta\gamma}$ and $\Gamma^{\gamma}_{\alpha\beta}$ of the metric affine connection of M(r,s), written in the Gaussian system, are calculated as usual with the derivatives of $g_{\alpha\beta}$. In particular denote

$$b_{ijA} = \Gamma_{ijA}\Big|_{S} \ .$$

Corresponding to each principal direction given by a dispacement dx^k in S and each normal direction $\eta_{(A)}$, there is a local curvature radius given by the solutions $x^A = \rho^A_{(k)}$, of ([7]):

$$\det(g_{ij} + x^A b_{ijA}) = 0 .$$

Then, in the subspace orthogonal to S, there are four curvature radii defined by

$$\rho_{(k)} = g_{AB}\, \rho^A_{(k)} \rho^B_{(k)} .$$

It may happen that two or three of these radii have the same absolute value. If all four radii are equal, then S has constant curvature in the neighborhood of the point.

Since all components $\rho^A_{(k)}$ tend to infinity in the flat limit of S, they can be used as a contracting factors for SO(r,s). The Lie algebra of SO(r,s), written in the Gaussian frame reads

$$[L_{\alpha\beta}, L_{\gamma\delta}] = g_{\alpha\gamma} L_{\beta\delta} + g_{\beta\delta} L_{\alpha\gamma} - g_{\alpha\delta} L_{\beta\gamma} - g_{\beta\gamma} L_{\alpha\delta} .$$

The smallest of the curvature radii denoted by ρ_m with component ρ^A_m, corresponds to the principal direction where the curvature is predominant and therefore to the strongest local effect of the gravitational field. For the direction of predominant curvature denote

$$\pi^{(m)}_{(i)} = \pi_i = \frac{1}{\rho^A_m} L_{iA},$$

The Lie algebra of SO(r,s), in terms of π_i becomes

$$
\begin{aligned}
[L_{ij}, L_{k\ell}] &= g_{ik}L_{j\ell} + g_{j\ell}L_{iA} - g_{i\ell}L_{jk} - g_{jk}L_{i\ell},\\
[L_{ij}, \pi_k] &= g_{ik}\pi_j + \frac{1}{\rho_m^A} g_{jA}L_{ik} - \frac{1}{\rho_m^A} g_{iA}L_{jk} - g_{jk}\pi_i,\\
\frac{1}{\rho_m^A\rho_m^B}[L_{ij}, L_{AB}] &= \frac{1}{\rho_m^A} g_{iA}\pi_j + \frac{1}{\rho_m^B} g_{jB}\pi_i - \frac{1}{\rho_m^B} g_{iB}\pi_j - \frac{1}{\rho_m^A} g_{jA}\pi_i,\\
[\pi_i, \pi_j] &= \frac{1}{\rho_m^A\rho_m^B} g_{ij}L_{AB} + \frac{1}{\rho_m^A\rho_m^B} g_{AB}L_{ij} - \frac{1}{\rho_m^B} g_{iB}\pi_j - \frac{1}{\rho_m^A} g_{Aj}\pi_i,\\
[\pi_i, L_{BC}] &= \frac{1}{\rho_m^A} g_{iB}L_{AC} + \frac{1}{\rho_m^A} g_{AC}L_{iB} - \frac{1}{\rho_m^A} g_{iC}L_{AB} - g_{AB}\frac{1}{\rho_m^A} L_{iC},\\
[L_{AB}, L_{CD}] &= g_{AC}L_{BD} + g_{BD}L_{AC} - g_{AD}L_{BC} - g_{BC}L_{AD}.
\end{aligned}
$$

Taking the limit $\rho_m^A \to \infty$ and assuming that the embedding remains minimal along the limiting process, the following Lie algebra results

$$
\begin{aligned}
[\bar{L}_{ij}, \bar{L}_{k\ell}] &= \eta_{ik}\bar{L}_{j\ell} + \eta_{j\ell}\bar{L}_{ik} - \eta_{i\ell}\bar{L}_{jk} - \eta_{jk}\bar{L}_{i\ell}\\
[\bar{L}_{ij}, \bar{\pi}_k] &= \eta_{ik}\bar{\pi}_j - \eta_{jk}\bar{\pi}_i,\\
[\bar{\pi}_i, \bar{\pi}_j] &= 0.
\end{aligned}
$$

The other commutators do not appear due to the requirement that embedding remains always minimal. The bar over the operators indicates the flat limit situation. As it can be seen the resulting Lie algebra is that of the Poincaré group.

III. MASS AND MASS SPLITTING.

The most difficult consequence of the replacement of P by an alternative symmetry is to addapt the physical definitions which have been taylored to P. It has been observed that the mass operator of P is also the second order Casimir operator of that group. Hence the conjecture that in a de Sitter symmetric theory the mass operator could be replaced by the second order Casimir operator of the de Sitter group [8]. However, unless a scaling factor of cosmological origin is introduced, the masses and mass differences result to be too small to be compared to the experimental values. If a cosmological gravitational field is sufficient to bring the mass splitting from zero (in the absence of gravitation) to a small value proportional to the inverse of the radius of curvature of the universe, then it seems natural to suppose that a local gravitational field is capable of inducing a greater mass difference [9]. In the following, the above suggestion is considered, where the space-time symmetry is regarded as the space-time restriction of SO(r,s). It is interesting to notice that in the particular case of the de Sitter space-time, this restriction coincides with the group SO(4, 1) itself.

From the properties of SO(r,s) it follows that the mass operator can be tentatively taken for the principal direction dx^m in S as

$$M^2(\rho_m) = (K \frac{1}{\rho_m})^2 \, L^{\alpha\beta} L_{\alpha\beta} \Big|_S , \qquad (3.1)$$

where K is a mass dimension factor and

$$\frac{1}{\rho_m^2} = g^{AB} \frac{1}{\rho_m^A} \frac{1}{\rho_m^B} .$$

The choice of (3.1) for the mass operator is justified by the fact that it reduces to the Poincaré mass operator in the flat limit. In fact, separating $L_{\alpha\beta}$ in the components L_{ij}, L_{iA}, L_{AB}, it becomes after a simple algebraic transformation.

$$M^2(\rho_m) = (K \frac{1}{\rho_m})^2 \, (L_{ij} L^{ij} + L_{AB} L^{AB}) + 2\Pi_i \Pi^i$$

$$+ \sum_{A<B< \quad <D=5} \varepsilon^{AB \, \dots \, DEF} \frac{1}{\rho_m^E} L_{iF} , \qquad (3.2)$$

where $\varepsilon^{AB \, \dots \, DEF}$ is the complete antisymmetric p-4 symbol. Because of the antisymmetry of that symbol, the $\frac{1}{\rho_m^E}$ are not absorbed into L_{iF} and in the flat limit contraction the expression 3.2 reduces to

$$M^2(\rho_m)\Big|_{\text{flat contration}} = 2\bar{\pi}_i\bar{\pi}^i$$

In order to define a particle structure (or elementary system in the sense of Wigner,), unitary representations of $SO(r,s)|_S$ are required. Let $\{\lambda\} = \{\lambda_1\ \lambda_2\ \dots\ \lambda_n\}$ denote a set of eigenvalues of the n Casimir operators, $C_1, C_2 \dots C_n$ of $SO(r,s)|_S$. To each set $\{\lambda\}$ there is one associated unitary representation of $SO(r,s)|_S$ with a Hilbert representation space H_λ. The direct sum of all such spaces gives a larger Hilbert space H. Now a combined symmetry Lie group G of finite order can be defined, such that it contains $SO(r,s)|_S$ plus an internal symmetry I as subgroups and such that H is a representation space for G, completely reducible respect to $SO(r,s)|_S$. If each Casimir operator of $SO(r,s)|_S$ has in its domain a complete set of eigenstates and in particular $M\ (\rho_m)$ is Hermitian, then a multiplet of I in H is given by a base in each subspace H_λ of H. Under such conditions, the spectrum of eigenvalues of $M\ (\rho_m)$ may contain isolated points, so that the emergence of a mass-like spectrum is possible.

Now let E be an operator of G such that it contains at least one eigenstate $|b\rangle$ of $M^2(\rho_m)$ in H. If $|a\rangle$ is another eigenstate of $M^2(\rho_m)$, then a non zero transition probability $\langle a|E|b\rangle$ may be obtained. When $|a\rangle$ and $|b\rangle$ belong to distinct subspaces H_λ, then E cannot be taken as an operator of $SO(r,s)|_S$ because in that case the transition probability would be zero. Therefore, assume that E is not an operator of $SO(r,s)|_S$. In this case it follows from the spectral theory in Hilbert spaces that the difference betwen the squared eigenvalue of M^2 for two representations a, b is given by [9].

$$(m_a^2 - m_b^2)\Big|_S = \frac{\langle a|\left[M^2(\rho_m),\ F\right]|b\rangle}{\langle a|E|b\rangle}\Big|_S . \qquad (3.3)$$

Denoting an arbitrary Lie algebra generator of G not in $SO(r,s)|_S$ by M_{rs}, then $E = a^{rs} M_{rs}$, where the indices r,s in the coefficients a^{rs} run through the dimension of the complementary subgroup of $SO(r,s)|_S$ in G. On the other hand let $L_{\mu\nu}$ be the Lie algebra generators of SO(r,s) written in the Cartezian frame of $M'(r,s)$. Then the Lie algebra generators of $SO(r,s)|_S$ are given by

$$L_{ij} = Z^\mu_i Z^\nu_j L_{\mu\nu}\Big|_S, \quad L_{iA} = Z^\mu_i Z^\nu_A L_{\mu\nu}\Big|_S, \quad L_{AB} = Z^\mu_A Z^\nu_B L_{\mu\nu}\Big|_S .$$

Let $E_{\underset{\sim}{a}\underset{\sim}{b}}$ be an arbitrary generator of G which in particular may be $L_{\alpha\beta}$ or M_{rs} (the indices $\underset{\sim}{a}$, $\underset{\sim}{b}$, and r,s running through the respective subalgebras). From (3.1) it follows that

$$M(\rho_m) = (K\frac{1}{\rho_m})^2 \Big[g^{ik}(S)g^{j\ell}(S)L_{ij}L_{k\ell}+2g^{ij}(S)g^{AB}(S)L_{iA}L_{jB}$$

$$+ g^{AB}g^{CD}L_{AC}L_{BD}\Big].$$

Therefore,

$$[M^2(\rho_m),E] = (K\frac{1}{\rho_m})^2 \Big[g^{ik}(S)g^{j\ell}(S)(L_{ij}[L_{k\ell},E]+[L_{ij},E]L_{k\ell})+$$

$$+ 2g^{ij}(S)g^{AB}(L_{iA}[L_{jB},E] + [L_{iA},E]L_{jB}) +$$

$$+ g^{AB}g^{CD}(L_{AC}[L_{BD},E] + [L_{AC},E]L_{BD})\Big]. \qquad (3.4)$$

In terms of structure constants the commutators can be writen as (denoting $Z^{\mu}_{\alpha} = Z^{\mu}_{,\alpha}$)

$$[L_{\alpha\beta}M_{rs}] = Z^{\mu}_{\alpha}Z^{\nu}_{\beta}[L_{\mu\nu}M_{rs}] = C^{\underset{\sim}{a}\underset{\sim}{b}}_{\alpha\beta\, rs}\, E_{\underset{\sim}{a}\underset{\sim}{b}},$$

where

$$C^{\underset{\sim}{a}\underset{\sim}{b}}_{\alpha\beta rs} = Z^{\mu}_{\alpha}Z^{\nu}_{\beta}C^{\underset{\sim}{a}\underset{\sim}{b}}_{\mu\nu rs}$$

are the structure constants of G with the indices corresponding to $SO(r,s)\big|_S$ written in the Gaussian system.

Tnen the expression (3.4) becomes

$$[M^2(\rho_m),E] = (K\frac{1}{\rho_m})^2\ a^{rs}\Big[g^{ik}(S)g^{j\ell}(S)C^{\underset{\sim\sim}{ab}}_{ijrs}\{E_{\underset{\sim\sim}{ab}},L_{k\ell}\} +$$

$$2g^{ij}(S)g^{AB}\ C^{\underset{\sim\sim}{ab}}_{iA\ rs}\ \{E_{\underset{\sim\sim}{ab}},L_{jB}\}+g^{AB}g^{CD}\ C^{\underset{\sim\sim}{ab}}_{BD\ rs}\ \{E_{\underset{\sim\sim}{ab}},L_{AC}\}\Big], \tag{3.5}$$

where { , } denotes the anticommutator.

For a given state vector $|a\rangle$ belonging to a representation of G, define the "spin" coefficients by

$$S^a_{ij} = \langle a|L_{ij}|a\rangle = Z^{\mu}_{i}Z^{\nu}_{j}\ \ \langle a|L_{\mu\nu}|a\rangle,$$

while the "momentum" coefficients are defined by

$$P^a_{iA} = \langle a|L_{iA}|a\rangle = Z^{\mu}_{i}Z^{\nu}_{A}\langle a|L_{\mu\nu}|a\rangle.$$

Finally "normal spin" coefficients may also be introduced:

$$J^a_{AB} = \langle a|L_{AB}|a\rangle = Z^{\mu}_{A}Z^{\nu}_{B}\langle a|L_{\mu\nu}|a\rangle.$$

In terms of these coefficients and using (3.5), expression (3.3) reads:

$$(m^2_a - m^2_b)\Big|_S = (K\frac{1}{\rho_m})^2\ a^{rs}\Big[g^{ik}(S)g^{j\ell}(S)\ C^{\underset{\sim\sim}{ab}}_{ijrs}\ (S^a_{k\ell} + S^b_{k\ell})$$

$$+ 2g^{ij}(S)g^{AB}\, C^{\tilde{a}\tilde{b}}_{iA\ rs}\,(P^{a}_{jB} + P^{b}_{jB}) + g^{AB}g^{CD}\, C^{\tilde{a}\tilde{b}}_{BD\ rs}(J^{a}_{AC}, J^{b}_{AC})\Big]\Big|_{S}\cdot$$

$$\cdot\ \frac{\langle a|E_{\underset{\sim}{ab}}|b\rangle}{\langle a|E|b\rangle}. \qquad (3.6)$$

Alternatively, a simpler expression may be derived. From (3.6), replacing the coefficientes by their corresponding expressions:

$$(m^2_a - m^2_b)\Big|_S = (K\,\frac{1}{\rho_m})^2\, a^{rs}\, C^{\tilde{a}\tilde{b}}_{\mu\nu\ rs}\Big[g^{ik}g^{j\ell}Z^{\mu}_{i}Z^{\nu}_{j}Z^{\rho}_{k}Z^{\sigma}_{\ell}$$

$$+ 2g^{ij}g^{AB}Z^{\mu}_{i}Z^{\nu}_{A}Z^{\rho}_{j}Z^{\sigma}_{B} + g^{AB}g^{CD}Z^{\mu}_{B}Z^{\nu}_{D}Z^{\rho}_{A}Z^{\sigma}_{C}\Big]\Big|_S\ (\langle a|L_{\rho\sigma}|a\rangle +$$

$$+ \langle b|L_{\rho\sigma}|b\rangle)\frac{\langle a|E_{\underset{\sim}{ab}}|b\rangle}{\langle a|E|b\rangle}. \qquad (3.7)$$

Introducing the space-time functions

$$f^{\mu\nu} = g^{ik}Z^{\mu}_{i}Z^{\nu}_{k},\quad h^{\mu\nu} = g^{AB}Z^{\mu}_{A}Z^{\nu}_{B} \qquad (3.8)$$

and the group representation functions

$$U_{\rho\sigma}(ab) = \frac{1}{2}(\langle a|L_{\rho\sigma}|a\rangle + \langle b|L_{\rho\sigma}|b\rangle) \qquad (3.9)$$

$$V_{\rho\sigma}(ab) = \frac{1}{2}(\langle a|L_{\rho\sigma}|a\rangle - \langle b|L_{\rho\sigma}|b\rangle), \qquad (3.10)$$

then (3.7) may be simplified to

$$(m_a^2 - m_b^2)\Big|_S = 2(K\frac{1}{\rho_m})^2 \left[f^{\mu\nu}f^{\rho\sigma} + 2f^{\mu\rho}h^{\nu\sigma}+h^{\mu\rho}h^{\nu\sigma}\right]\Big|_S U_{\rho\sigma}\cdot$$

$$\frac{a^{rs}\ C^{\underset{\sim}{a}\underset{\sim}{b}}_{\mu\nu rs}\langle a|E_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|E|b\rangle}$$

$$= 2(K\frac{1}{\rho_m})\ \left[(f^{\mu\rho}+h^{\mu\rho})(f^{\nu\sigma}+h^{\nu\sigma})\right]\Big|_S U_{\rho\sigma}\frac{a^{rs}\ C^{\underset{\sim}{a}\underset{\sim}{b}}_{\mu\nu rs}\langle a|E_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|E|b\rangle}. \tag{3.11}$$

Since $|a\rangle$, $|b\rangle$ belong to distinct representations of SO(r,s),

$$a^{rs}\ C^{\underset{\sim}{a}\underset{\sim}{b}}_{\mu\nu rs}\langle a|E_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle = a^{rs}\ C^{pq}_{\mu\nu rs}\langle a|M_{pq}|b\rangle +$$

$$+\ a^{rs}\ C^{\tau\varepsilon}_{\mu\nu rs}\langle a|L_{\tau\varepsilon}|b\rangle$$

$$= a^{rs}\langle a|C^{pq}_{\mu\nu rs}\ M_{pq}|b\rangle = a^{rs}\langle a|\left[L_{\mu\nu},M_{rs}\right]|b\rangle.$$

Therefore

$$a^{rs}\ C^{\underset{\sim}{a}\underset{\sim}{b}}_{\mu\nu rs}\ \frac{\langle a|E_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|E|b\rangle} = \langle a|L_{\mu\nu}|a\rangle - \langle b|L_{\mu\nu}|b\rangle = 2V_{\mu\nu}(ab).$$

On the other hand, from the equation (2.3) and from the fact that $g^{iA}\big|_S = 0$, it follows that

$$(f^{\mu\rho} + h^{\mu\rho})\big|_S = \eta^{\mu\rho}.$$

Therefore (3.4) becomes

$$(m_a^2 - m_b^2)\Big|_S = (K \frac{1}{\rho_m})^2 \eta^{\mu\rho}\eta^{\nu\sigma} U_{\rho\sigma}(ab)V_{\mu\nu}(ab),$$

or better

$$(m_a^2 - m_b^2)\Big|_S = (K \frac{1}{\rho_m})^2 U^{\mu\nu}(ab)V_{\mu\nu}(ab), \qquad (3.12)$$

where $U^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} U_{\rho\sigma}$.

In the flat limit the group functions remain finite while the coefficient $\frac{K}{\rho_m}$ tend to zero so that the mass splitting vanishes as expected. On the other hand, when SO(r,s) is a normal subgroup of G, then $\langle a|[L_{\mu\nu}, E]|b\rangle = 0$ so that the mass splitting also vanishes.

IV - CONCLUSIONS

The most interesting aspect of the mass splitting formula (3.12) is that the space-time dependence rests only on the curvature radius ρ_m. That expression generalizes an expression previously derived a de Sitter space-time with the difference that now it is only locally defined and depends of the local gravitational field [10]. The space-time subgroup $SO(r,s)\big|_S$ of SO(r,s) has disapeared alltogether. The only remaining group dependence of the mass splitting rests in the functions $V_{\rho\sigma}(ab)$, $V_{\mu\nu}(ab)$ which are determined by the representations of SO(r,s). Since this group is a property of the embedding class, rather than of S, it holds true for all space-time minimally and isometrically embedded in M(r,s). However, ρ_m depends of each particular space-time and of the specific curvature direction dx^m. There are four possible such directions and the one corresponding to the smallest ρ_m will provide the most significant mass splitting. In some particular regions of space-time where the curvature is uniform in all directions a single curvature radius appear. This is the case of space-times with constant curvature with curvature radius R. In this case the expression (3.12) reads

$$(m_a^2 - m_b^2) = \frac{K^2}{R^2} U^{\mu\nu}(ab)V_{\mu\nu}(ab),$$

where the functions $U^{\mu\nu}$, $V_{\mu\nu}$ are derived from the representations of SO(4,1) or SO(3,2). For example taking the case of the de Sitter space-time with $R = 10^{28}$ cm and assuming that $U^{\mu\nu}V_{\mu\nu} \sim 10$ then

$$(m_a^2 - m_b^2)\Big|_{\text{de Sitter}} \sim K^2\, 10^{-55} . \qquad (4.1)$$

This value is far too small as compared with the observed values. It can been conjectured that the factor K contains a minimum mass value which is somehow associated with the de Sitter cosmological model [10]. It is possible that such assumption is a valid one, but it should be derived from additional dynamical considerations. It might also be conjectured that the small value obtained with (4.1) is a consequence of the weakness of the gravitational field of a cosmological model and that a local gravitational field would produce more significant results. This is indeed the case, because the factor ρ_m^{-2} is much greater than R^{-2}. For example, in a regular point of Schwarzschild space-time the significant curvature radius is approximately [11]

$$\rho_m \sim \frac{r^{3/2}}{(2m)^{1/2}} .$$

Then, again admitting that $U^{\mu\nu}V_{\mu\nu}$, obtained from the representations of SO(4,2), is of the order of ten, it follows that

$$(m_a^2 - m_b^2)\Big|_{\text{Schwarzschild}} \sim K^2 \cdot 10 \frac{1}{\rho_m^2}$$

At the surface of the Earth $\rho_m \sim 2\ 10^{13}$ cm. so that the above mass splitting will be of the order of $K^2 \cdot 2{,}5 \cdot 10^{-26}$. In order to achieve the observed values at the surface of the Earth, K should be of the order of 10^{15} Mev.

Whatever be the value of K, expression (3.12) shows that the gravitational field plays a significant role in the mass splitting. In the absence of gravitation, the Poincaré symmetry and the mentioned theorems imply in zero mass splitting. In the case of a cosmological model, the mass splitting is non zero, but it is too small and it increases as local gravitational fields are considered. Therefore, it appears to make sense to conjecture that the mass splitting is induced by the gravitational field.

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