SOME ASPECTS OF TRANSFORMATION OF THE NONLINEAR PLASMA EQUATIONS TO THE SPACE- INDEPENDENT FRAME

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ABSTRACT : Relativistically correct transformation of nonlinear plasma equations are derived in a space--independent frame. This transformation is useful in many ways because in place of partial differential equations one obtains a set of ordinary differential equations in a single independent variable. Equations of Akhiezer and Polovin (1956) for nonlinear plasma oscillations have been generalised and the results of Arons and Max (1974), and others for wave number shift and precessional rotation of electromagnetic wave are recovered in a space-independent frame.

I. Introduction : Winkles and Eldridge [1], Clemmow [2,3,4], Lerche [5], Chian and Clemmow [6], Kennal and Pellat [7], Decoster [8] and others have shown that Lorentz transformations are very useful to transform the field equations from space-time dependent frame S to a space independent frame S', where solutions of nonlinear higher order plasma equations can be found out easily, because in a space--independent frame nonlinear partial differential equations become ordinary differential equation. Sometimes, it is found very difficult to obtain solutions of some problems of nonlinear plasmas in the closed form because refractive index in the right circular polarisation becomes inhomogeneous. For this reason, Chakraborty et al [9] used WKB approximation in deriving the expressions for nonlinear Faraday Effect. But if from the beginning, equations are transformed relativistically into a space independent frame, it is expected that difficulties arising from the nonlinearly developed refractive index can be bypassed mathematically because in place of partial differential equations only ordinary differential equations have to be solved.

In the present paper, we use transformation of the cold, relativistic plasma equations from a laboratory inertial frame to a space independent frame. Using these transformed equations we make generalisation of the nonlinear equations to derive the expressions of wave number shift and precessional rotation of an elliptically polarised electromagnetic wave in a cold, relativistic, collisionless, unmagnetised plasma. Wave number shift and precessional rotation of a strong laser beam has been numerically calculated. Another consequence is a generalisation of some important results obtained first by Akhiezer and Polovin [10].

II. Assumptions and basic equations : We assume (i) Plasma is stationary, cold and homogeneous; (ii) The incident wave strong enough for the occurence of relativistic electron motion; (iii) For radiation intensities less than approximately 3×10^{22} W/cm^2, the ion velocity remains below the electron velocity [11]. The ion motion has little effect on the propagation of electromagnetic waves and we neglect this motion; (iv) Incident electromagnetic wave is transverse, elliptically polarised and sinosoidal,

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(v) Plasma is below a certain threshold power limit so that self-focussing and self-trapping mechanisms are insignificant. Under the threshold power Stimulated Raman Scattering is minimum; (vi) Self-action effects arising out from the pondermotive force and thermal instabilities are negligible; (vii) The forces arising due to collision and gravitation are negligibly small in comparison with other forces present in the medium.

Under the above assumptions the field equations in an inertial frame 'S' can be written as

$$\left(\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}\right)\vec{p} = -e\{\vec{E}+(\vec{v}\times\vec{B})/c\} \quad \dots (1a)$$

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial z}(N v_z) = 0. \quad \dots (1b)$$

$$\frac{\partial E_y}{\partial z} = \frac{1}{c}\frac{\partial B_y}{\partial t}, \quad \frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial B_y}{\partial t}. \quad \dots (2)$$

$$\left.\begin{aligned}&\frac{\partial B_y}{\partial z} = -\frac{1}{c}\frac{\partial E_x}{\partial t}+\frac{4\pi e N v_x}{c}, \quad \frac{\partial B_x}{\partial z} = \frac{1}{c}\frac{\partial E_y}{\partial t}-\frac{4\pi e N v_y}{c}\\ &\frac{\partial E_z}{\partial t} = 4\pi e N v_x.\end{aligned}\right\} \quad \dots (3)$$

$$\frac{\partial E_z}{\partial z} = 4\pi e(N_i-N), \quad \frac{\partial B_z}{\partial z} = 0. \quad \dots (4)$$

with $p = m_0\vec{v}/\sqrt{1-v^2/c^2}$. $\quad \dots (5)$

where, the wave is propagating along z-direction, Ni and N are the number density of ions and electrons, m_0 and e are the rest mass and charge of an electron, other parameters have their usual meanings.

III. Space independent frame and transformation of some quantities to it :

Lorentz transformation from the S-frame to the S'-frame moving relatively with velocity V_0 parallel to the z-axis is given by

$$x = x', \; y = y', \; z = (z'+V_0 t')\lambda_0, \; t = (t'+V_0 z'/c)\lambda_0 \quad \dots (6)$$

where, $\lambda_0 = (1-\beta_0^2)^{-1/2}$, $\beta_0 = V_0/c$.

The phase transformation is

$$(\omega t-kz) = (\omega\cosh\psi_0 - kc\sinh\psi_0)t' - (k\cosh\psi_0 - c^{-1}\sinh\psi_0)z' \quad \dots (7)$$

where k and ω are the wave number and frequency, $\beta_0 = \tanh\psi_0$.

Following Winkles and Eldridge [1] we assume $V_0 = c^2/V = kc^2/\omega$, $V (=\omega/k)$ being the phase velocity of the wave. So, (7) reduces to

$$(\omega t-kz) = \omega\sqrt{1-\beta_0^2}\,t' = \omega'\tau. \quad \dots (8)$$

where, $\omega' = \omega\sqrt{1-\beta_0^2} = \omega/\lambda_0$, $t' = \tau$. This transformation enables us to change the variables from the space-time dependent frame S to the space independent frame S'. We can now write,

$$v_z = \dot{z} = V-(V/\lambda_0)\frac{\partial\tau}{\partial t}, \quad \frac{\partial}{\partial t} = \lambda_0\frac{\partial}{\partial\tau}, \quad \frac{\partial}{\partial z} = -\frac{\lambda_0}{V}\frac{\partial}{\partial\tau} \quad \dots (9)$$

The transformation of important physical variables from the space-time dependent S-frame to our space-independent S'-frame are

$$\begin{aligned}&v_x = v_x'/\lambda_0(1+\beta_0 v_z'/c), \; v_y = v_y'/\lambda_0(1+\beta_0 v_z'/c), \; v_z = (v_z'+\beta_0 c)/(1+\beta_0 v_z'/c)\lambda_0,\\ &E_x = \lambda_0(E_x'+\beta_0 H_y'), \; E_y = \lambda_0(E_y'-\beta_0 H_x'), \; E_z = E_z',\\ &H_x = \lambda_0(H_x'-\beta_0 E_y'), \; H_y = \lambda_0(H_y'+\beta_0 E_x'), \; H_z = H_z', \quad \dots (10)\\ &m' = m_0\lambda_0(1+\beta_0 v_z'/c), \; p_x = p_x', \; p_y = p_y', \; p_z = \lambda_0(p_z'+m_0 V_0\lambda'),\\ &N = N'\beta_0(1+\beta_0\beta_z').\end{aligned}$$

IV. Transformation of the field equations to the space-independent frame :

Considering the transformations of Section-II, we obtain from equation(1a)

$$\frac{dp_x'}{d\tau} = -eE_x' - \frac{eH_y'V}{c(\lambda_0 V-V_0)}\left[V_0(\lambda_0-1) - \frac{V v_z'}{\lambda_0^3\{(\lambda_0-1)v_z'+\lambda_0 V-V_0\}}\right]$$

$$\frac{dp_y'}{d\tau} = -eE_y' + \frac{eH_x'V}{c(\lambda_0 V-V_0)}\left[V_0(\lambda_0-1) - \frac{V v_z'}{\lambda_0^3\{(\lambda_0-1)v_z'+\lambda_0 V-V_0\}}\right]$$

and

$$\begin{aligned}\frac{dp_z'}{d\tau} = &-\frac{eVE_z'}{\lambda_0\{\lambda_0 V-V_0+v_z'(\lambda_0-1)\}} + \frac{eVp_z'E_z'}{m_0\lambda_0\lambda' V_0\{\lambda_0 V-V_0+v_z'(\lambda_0-1)\}}\\ &+\frac{e(p_x'E_x'+p_y'E_y')[V(\lambda_0^2 V-\lambda_0 V_0-V_0)+v_z'\{V(\lambda_0^2-\lambda_0+1)-V_0\}]}{m_0\lambda_0\lambda' V_0(V+v_z')\{\lambda_0 V-V_0+v_z'(\lambda_0-1)\}}\dots\\ &+\frac{e(p'\times B')_z V\{v_z'V(V+V\lambda_0^4-V_0\lambda_0^5)+\lambda_0^4(\lambda_0 V-V_0)^2\}}{m_0\lambda_0^3\lambda' V_0(V+v_z')\{\lambda_0 V-V_0+v_z'(\lambda_0-1)\}}.\end{aligned} \quad \dots (11)$$

The equation of continuity 1(b) becomes

$$d_0 \frac{d}{dT}\left[\frac{N'(1-\beta_0^2)}{\beta_0}\right] = 0 \quad \text{and so, } N' = N_0 \qquad \ldots (12)$$

where N_0 is a constant.

From the first equation of (2), the equation for B'_x is

$$\left(\frac{1}{\beta_0} - \beta_0\right) \frac{dB'_x}{dT} = 0 \qquad \text{and so, } B'_x = \text{Constant} \qquad \ldots (13a)$$

Similarly, from second equation of (2), we obtain

$$B'_y = \text{Constant} \qquad \ldots (13b)$$

Introducing transformation relations for the space independent frame, equation (3), yield,

$$\frac{dE'_x}{dT} = 4\pi e N_0 v'_x / d_0(1+\beta_0\beta'_z),$$

$$\frac{dE'_y}{dT} = 4\pi e N_0 v'_y / d_0(1+\beta_0\beta'_z), \qquad \ldots (14)$$

and

$$\frac{dE'_z}{dT} = 4\pi e N_0 (v'_z + V_0)/d_0(1+\beta_0\beta'_z).$$

V. Derivations of nonlinear equations : Differentiating equations of (11) with respect to the time-variable T of the space-independent frame-S' and then using the relations of (14), we obtain

$$\frac{d^2 q'_x}{dT^2} + \frac{V\Omega'_y}{d_0^3(d_0V-V_0)} \frac{d}{dT}\left[\frac{\beta'_z}{(d_0-1)c\beta'_z + d_0V - V_0}\right] + \frac{\omega_p^2 \beta'_x}{d_0(1+\beta_0\beta'_z)} = 0$$

$$\frac{d^2 q'_y}{dT^2} - \frac{V\Omega'_x}{d_0^3(d_0V-V_0)} \frac{d}{dT}\left[\frac{\beta'_z}{(d_0-1)c\beta'_z + d_0V - V_0}\right] + \frac{\omega_p^2 \beta'_y}{d_0(1+\beta_0\beta'_z)} = 0$$

and

$$\frac{d}{dT}\left[\left(\frac{dq'_z}{dT}\right) \cdot \frac{\{(d_0V-V_0)+c(d_0-1)\beta'_z\}}{\beta_0 d_0^2 - q'_z(1-\beta'^2)^{-1/2}}\right] - \frac{c}{d_0V_0}\frac{d}{dT}\left[\frac{V(d_0^2V-d_0V_0-V_0)+c\beta'_z\{V(d_0^2-d_0+1)-V_0\}}{\beta_0 d_0^2(1-\beta'^2)^{-1/2} - q'_z}\right.$$

$$\left.\cdot\left\{\frac{d}{dT}(q'^2_x + q'^2_y) + (V/c)(q'_x\Omega'_y - q'_y\Omega'_x)\left(V_0(d_0-1) - \frac{cV\beta'_z}{d_0^3\left((d_0-1)c\beta'_z + d_0V - V_0\right)}\right)\right\}\right]$$

$$+ \frac{cV}{d_0^3V_0}\frac{d}{dT}\left[\frac{(q'\times\Omega')_z\{cV(V+Vd_0^6-V_0d_0^5)\beta'_z + d_0^4(d_0V-V_0)^2\}}{(V+c\beta'_z)\{\beta_0 d_0^2(1-\beta'^2)^{-1/2} - q'_z\}}\right]$$

$$+ \frac{\omega_p^2 V}{V_0 d_0^4}\frac{(c\beta'_z + V_0)}{(1+\beta_0\beta'_z)} = 0 \qquad \ldots (15)$$

where, $\omega_p^2 = 4\pi N_0 e^2/m_0$, $\vec{q}' = \vec{P}'/m_0c$, $\vec{\Omega} = e\vec{H}'/m_0c$.

Akhiezer and Polovin [10] derived some nonlinear equations including Lagrangian and Momentum from which nonlinear oscillations of longitudinal and transverse waves are well understood. Nonlinear equations derived here are more generalised than that of Akhiezer and Polovin. It is to be remembered that oscillations of longitudinal wave can be studied, assuming $q'_x = q'_y = 0$ and for transverse wave, q'_z is equal to zero.

VI. Wave number shift, and precessional rotation of electromagnetic wave : Frequency and wave number of waves are changed due to nonlinear effects in plasmas. Montogomery and Tidman [12] obtained an expression for the frequency shift (as nonlinear correction) for both travelling and standing waves in plasmas. Using a plane polarised transverse wave, Sluijter and Montogomery [13] derived the expression for the shift of frequency and wave number considering nonlinear interaction of a wave with a plasma which has static ions, moving electrons and no magnetic field. Tidman and Stainer [14], Das [15, 16], Boyd [17], Chandra [18, 19], Winkles and Eldridge [1], Goldstein and Salu [20], Schindler and Janick [21] and others also obtained the results of frequency and wave number shift under different situations in plasma. Some non-relativistic results for the frequency shift were earlier obtained by Sturrock [22], Dawson [23] and Jackson [24].

Recently Arons and Max [25] reconsidered the problem of Sluitjer and Montogomery [13] using elliptically polarised wave and showed that the nonlinear effects give rise to precession of polarisation ellipse without affecting the ellipticity. Later, Katz et al. [26] and Lie and Wonnacott [27] described the precession of polarisation axis with the aid of usual technique of classical mechanics. Chakraborty [28] developed the theory of precessional effects due to interaction of four waves in a cold unmagnetized plasma. He found that when the plasma is subjected to two strong fields sharp band of monochromatic noise transform into growing continuous spectra. Chakraborty and Chandra [29], Khan and Chakraborty [30], Bhattacharyya and Chakraborty [31], Chandra [32], Decoster [8] also derived the expressions of the frequency and wave number shift including precessional rotation of electromagnetic waves due to nonlinear effects in plasmas.

Rotations of polarisation ellipse of laser beam have been experimentaly detected in the medium like liquid-filled absorption cell [33], fused quartz, borosilicate crown glass and crystals [34,/35, 36]. Wong and Shen [37] studied experimentally the pre-transitional behaviour of laser-filled induced molecular alignment of two isotropic neumatic compounds (viz. MBBA and EBBA). For electron plasma, waves propagating in the presence of axial magnetic field frequency shift and wave number shift have been experimentally observed [38, 39, 40]. But for precessional frequency frequency of electromagnetic wave-interaction in plasmas there is no report of experimental investigation.

In the following section, we have derived the expressions of wave number shift and precessional frequency of an elliptically polarised electro-magnetic wave due to nonlinear effects in a cold, collisionless, unmagnetised relativistic plasma in a space-independent frame.

In a space-independent frame, nonlinear equations correct upto third order of small quantities are derived in the following manner

$$\dot{v}_\pm = P_\pm - eE_\pm/m_0\lambda_0 . \qquad \ldots (16)$$

where, $v_\pm = v_x \pm i v_y$, $E_\pm = E_x \pm iE_y$,

$$P_\pm = \frac{\lambda_0^2}{V}\dot{v}_\pm \delta v_z^1 + \frac{\lambda_0^2}{V}\frac{d}{dt}(v_\pm \delta v_z^1) - \lambda_0^2 \frac{d}{dt}(v_\pm^2 v_\mp/2c^2) + \frac{e\beta_0\lambda_0}{m_0 c}E_\pm \delta v_z^1 .$$

v_x and v_y are the first order velocity components in the direction parallel to OX and OY, δv_z^1 is the 2nd order velocity along OZ, subscripts with plus (+) and minus (-) signs indicate right and left circular polarisation. Now assuming

$$E_\pm = [(a \pm b)e^{i\theta_0} + (a \mp b)e^{-i\theta_0}]/2 = a\cos\theta_0 \pm ib\sin\theta_0 .$$

We obtain, $v_\pm = e(-a\sin\theta_0 \pm ib\cos\theta_0)/m_0\omega_p\lambda_0$.

and $\delta v_z^1 = -e^2(a^2-b^2)\omega^2\cos 2\theta_0/m_0^2 V\lambda_0^2\omega_p^2(3\omega^2+k^2c^2)$.

Now, the nonlinear equation forelectric field correct upto third order is

$$e\dot{E}_\pm + \omega_p^2 e E_\pm = \lambda_0 m_0 \omega_p^2 P_\pm . \qquad \ldots (17)$$

From these relations nonlinear dispersion relation for left and right polarised wave can be derived as

$$\omega_\pm'^2 - \omega_p^2 = \frac{k^2c^2}{2(4-X)}(\alpha \mp \beta)^2 - \frac{\omega_p^2}{8}\{3(\alpha^2+\beta^2) \mp 2\alpha\beta\} \qquad \ldots (18)$$

where, $\alpha = ea/m_0\omega_p c$, $\beta = eb/m_0\omega_p c$, $X = \omega_p^2/\omega^2$.

To get the results in the laboratory inertial frame we put $\omega_\pm' = \omega_\pm/\lambda_0$ from our previous transformation relation, where $\lambda_0^2 = \omega_\pm^2/(\omega_\pm^2 - K_\pm^2 c^2)$.

If we now consider the spatial evolution problem we can write, $K_\pm = K + \delta k_\pm$, $\omega_\pm = \omega$. Therefore, $\omega_\pm'^2 - \omega_p^2 = -2k\,\delta k_\pm c^2$,

So, $$\frac{\delta k_\pm}{k} = -\frac{X}{4}\left[\frac{(\alpha\mp\beta)^2}{4-X} - \frac{3}{n^2}(\alpha^2+\beta^2) \pm \frac{1}{2n^2}(\alpha\beta)\right] \qquad \ldots (19)$$

Where, $n^2 = 1 - X$.

Therefore,

$$\frac{\delta k_+ + \delta k_-}{K} = \frac{X}{4(1-X)}\left[\frac{3}{4} - \frac{1-X}{4-X}\right](\alpha^2+\beta^2) \qquad \ldots (20)$$

and $$\frac{\delta k_+ - \delta k_-}{K} = \frac{X}{2(1-X)}\left[\frac{1-X}{4-X} - \frac{1}{4}\right]\alpha\beta . \qquad \ldots (21)$$

Expressions (20) and (21) give the wave number shift and precessional frequency for the elliptically polarised electromagnetic wave. These are identical to the results of Arons and Max [23]. Equation(21) also shows the birefrigence effect in a plasma medium for an electro-magnetic travelling wave. Moreover, if $\beta = 0$, equation (20) leads to the formula for wave number corrections of Sluijter and Montogomery [13].

Numerical Estimation : We assume, laser beams of neodium-glass having wave length = 1.06 μm, w = 1.78×10^{15}/sec. propagate through a dense plasma $N_0 = 5 \times 10^{20}/cm^3$. In this case, laser flux (P) is 10^{16} W/cm^2 which is less than the threshold power ($W_{Th} = 10^{20}$ W/cm^2) for generating self-focussing and other nonlinear effects. Under this situation $\alpha^2 = \beta^2 = .005$. So, the electromagnetic wave will have wave number shift 6.64×10^2 per cm.(i.e. wave length shift = 9.46×10^{-3} cm) and 1° precessional rotation will be created through a distance 5.504×10^{-3} cm.

REMARKS : In the cases, where effects of collision, gravitation, Kinetic temperature, static magnetic field etc.are considered, these nonlinear effects e.g. predessional rotatiom will be more interesting and it would give some important results in the study of nonlinear Faraday Rotation, inverse Faraday Effect etc. We desire to investigate these problems in future.

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