OMNIGENOUS MAGNETIC FIELDS

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ABSTRACT: In omnigenous magnetic fields particles' drift surfaces coincide with plasma magnetic surfaces. In this paper we formulate equations of omnigenous magnetic fields in natural curvilinear coordinates. An analysis of fields which are omnigenous only in the paraxial approximation is presented.

1. INTRODUCTION

Proposed a few years ago, tandem mirrors[1,2] retain the advantages of conventional mirrors, at the same time allowing considerable increase in longitudinal ion confinement in comparison with simple mirrors. Yet as was shown in Ref.[3-5], nonaxisymmetry of these traps may result in substantial increase of transverse transport as compared to the classical case. The reason is that drifts in the nonaxisymmetric field shift a particle's guiding center from a magnetic surface.*) The magnitude and direction of the drifts depend on the particle's energy and magnetic moment, and Coulomb collisions, changing these integrals of motion, result in particle having random jumps from one magnetic surface to another. In the most unfavorable regimes this mechanism of transport (which is usually refered to as neoclassical) may lead to reduction of the plasma confinement time to an ion-ion collision time.

The most radical way to suppress neoclassical transport would be to create a magnetic field in which all the plasma particles do not leave their magnetic surfaces in the course of drifting. This field is often called "omnigenous"[6] and an idea for using such a field for transport suppression was first proposed in Ref.[7].

An additional advantage of omnigenous fields is that in a scalar-pressure plasma embedded in this field longitudinal currents are not generated[8,9]. This can be important for long tandem mirrors where even small asimuthal and radial fields created by the parallel current strongly distort magnetic surfaces[10,11].

In this paper we formulate equations of omnigenous magnetic fields[12] in curvilinear coordinates related to the magnetic field. An analysis of fields which are omnigenous only in the paraxial approximation is presented.

*) By magnetic surfaces we mean herein the surfaces of constant pressure.

2. OMNIGENITY CONDITION

We consider a potential magnetic field $\vec{B}$ satisfying the equations

$$\mathrm{rot}\,\vec{B} = 0, \quad \mathrm{div}\,\vec{B} = 0. \tag{1}$$

Particles in potential fields, as is well known, drift in the direction of the binormal to the field line (see, for example[13]):

$$\vec{v}_{dr} \propto [\vec{B} \times \nabla B]. \tag{2}$$

For a magnetic field to be omnigenous, the drift velocity $\vec{v}_{dr}$ and parallel velocity $\vec{v}_{\parallel}$ $(\vec{v}_{\parallel} \propto \vec{B})$ at a given point should not displace the particle out of the magnetic surface. This means that these vectors are to be tangential to the magnetic surface passing through the point considered, their cross product $[\vec{v}_{dr} \times \vec{v}_{\parallel}]$ being a normal to the surface. Defining a vector $\vec{m}$ directed along $[\vec{v}_{dr} \times \vec{v}_{\parallel}]$,

$$\vec{m} = [\vec{B} \times [\vec{B} \times \nabla B]] \tag{3}$$

and introducing a function $F(\vec{r})$ so that $F(\vec{r})$ = const is an equation for an omnigenous magnetic surface, we see that the vectors $\vec{m}$ and ∇F must be co-linear:

$$\vec{m}\,\frac{|\nabla F|}{|\vec{m}|} = \pm \nabla F. \tag{4}$$

Taking the curl of both sides of the last equation and multiplying the result by $\vec{m}$ yield a condition which an omnigenous magnetic field should satisfy:

$$\vec{m} \cdot \mathrm{rot}\,\vec{m} = 0. \tag{5}$$

Although we have derived this equation as a necessary condition one can show that it is also a sufficient one. Inserting eq. (3) into eq. (5) gives the omnigenity condition in terms of $\vec{B}$:

$$[\vec{B} \times \nabla B] \cdot \nabla (\vec{B} \cdot \nabla B) = 0, \tag{6}$$

which together with (1) completely determines potential omnigenous fields. It was first derived by Panov[12] (in another way) who gave also its geometrical interpretation. The high nonlinearity of eq. (6) makes its analysis very difficult. We shall see in the following section that in a special curvilinear coordinate system it takes a form which in some respect is simpler.

Though our consideration is restricted to potential fields we note here that a more general class of omnigenous fields is determined by eq. (6) together with the following equations

$$\vec{B} \cdot \mathrm{rot}\,\vec{B} = 0, \quad \mathrm{div}\,\vec{B} = 0 \tag{7}$$

(instead of (1)). The first, from eq. (7), means the absence of parallel currents in the plasma. In this case the drifts are also directed in the binormal direction[13] and the derivation of (6) given above remains valid. On the other hand, if $\vec{B} \cdot \mathrm{rot}\,\vec{B} \neq 0$, the drift velocity has both normal and binormal components and the field is necessarily nonomnigenous. Thus, eqs. (6) and (7) determine also the most general case.

3. CURVILINEAR COORDINATES

Henceforth we again restrict our consideration to potential fields only.

Let α, β, φ be curvilinear coordinates defined by a transformation $\alpha = \alpha(x,y,z)$, $\beta = \beta(x,y,z)$, $\varphi = \varphi(x,y,z)$. The inverse transformation from α, β, φ to x, y, z we denote by the vector function $\vec{F} = \vec{F}(\alpha, \beta, \varphi)$.

The function $\varphi(\vec{F})$ is chosen to be a potential of the magnetic field,

$$\nabla\varphi = \vec{B} \tag{8}$$

and α and β are flux coordinates[14],

$$[\nabla\alpha \times \nabla\beta] = \vec{B}, \tag{9}$$

constant along field lines. Of course, α and β are not uniquely determined by eq. (9); any transformation $\tilde{\alpha} = \tilde{\alpha}(\alpha, \beta)$, $\tilde{\beta} = \tilde{\beta}(\alpha, \beta)$ with a unit Jacobian,

$$\frac{\partial(\tilde{\alpha}, \tilde{\beta})}{\partial(\alpha, \beta)} = 1 ,$$

is tolerable because it does not change the cross product $[\nabla\alpha \times \nabla\beta]$. For example, in an omnigenous

field one can always choose the surfaces of constant α to be magnetic surfaces.

In the thus defined curvilinear coordinate system, covariant components of $\vec{B}$ are determined by eq. (8):

$$B_i = (0, 0, 1) \tag{10}$$

and its contravariant components by eq. (9):

$$B^i = (0, 0, g^{-1/2}), \tag{11}$$

where $g = \det\|g_{ik}\|$ and g_{ik} is the metric tensor. Since these components are related by $B_i = g_{ik}B^k$, from the expressions (10) and (11) it immediately follows that

$$g_{13} = g_{23} = 0 \;, \quad g_{33} = \sqrt{g} . \tag{12}$$

Conversely, if a curvilinear coordinate system has the property (12), a vector field $\vec{B}$ having covariant components (10) (and contravariant components (11)) is a solution of eqs. (1).

Let us now consider how the omnigenity condition looks in the coordinate system defined above. Instead of using eq. (6) we employ another equivalent approach requiring that a function $f(\alpha, \beta)$ should exist, so that f = const is an equation for an omnigenous magnetic surface. Since f does not depend on φ, field lines lie on the surfaces of constant f and the motion along $\vec{B}$ does not displace particles from magnetic surfaces. In order that the drift leaves particles on magnetic surfaces it must be tangential to them:

$$\nabla f \cdot [\vec{B} \times \nabla B] = 0. \qquad (13)$$

It easy to check that eq. (13) is equivalent to the requirement that the component g_{33} depends on α and β through the function f :

$$g_{33} = g_{33}(f(\alpha,\beta), \varphi). \qquad (14)$$

Thus we have formulated eqs.(1) and (6) as constraints on the metric tensor of the curvilinear system. Noting that

$$g_{13} = \vec{r}_\alpha \cdot \vec{r}_\varphi\,, \; g_{23} = \vec{r}_\beta \cdot \vec{r}_\varphi\,, \; g_{33} = \vec{r}_\varphi^{\,2}\,, \sqrt{g} = \vec{r}_\varphi \cdot [\vec{r}_\alpha \times \vec{r}_\beta]$$

one can rewrite them as equations for the function $\vec{r}(\alpha,\beta,\varphi)$. From (12) it follows that

$$\vec{r}_\varphi = [\vec{r}_\alpha \times \vec{r}_\beta] \qquad (15)$$

where the subscriptes denote differentiation with respect to the corresponding variable. The omnigenity condition (14) means that only those solutions of (15) should be selected for which depends on in the following manner:

$$\vec{r}_\varphi^{\,2}(f(\alpha,\beta), \varphi). \qquad (16)$$

Another useful form of (16) is obtained by calculating the derivative:

$$\frac{\partial}{\partial\varphi}\left(\frac{\partial \vec{r}_\varphi^{\,2}/\partial\alpha}{\partial \vec{r}_\varphi^{\,2}/\partial\beta}\right) = 0\,, \qquad (17)$$

and is fully equivalent to (16).

The general solution of the system (15), (16) has not thus far been obtained. In the next section we present its approximate solution in the paraxial approximation for tandem mirrors.

3. PARAXIAL APPROXIMATION

Let choose a Cartesian coordinate system so that z - axis is directed along the magnetic axis and the xz and yz planes are the symmetry planes of the tandem mirror. Eq. (15) then has the following form:

$$\begin{aligned} x_\varphi &= y_\alpha z_\beta - y_\beta z_\alpha\,, \\ y_\varphi &= z_\alpha x_\beta - z_\beta x_\alpha\,, \\ z_\varphi &= x_\alpha y_\beta - x_\beta y_\alpha\,. \end{aligned} \qquad (18)$$

Taking into account the symmetry of the tandem field we choose α, β coordinates so that reflection in the xz plane changes the sign of α and reflection in the yz plane changes the sign of β :

$$\begin{aligned} x(\alpha,\beta,\varphi) &= x(\alpha,-\beta,\varphi) = -x(-\alpha,\beta,\varphi)\,, \\ y(\alpha,\beta,\varphi) &= -y(\alpha,-\beta,\varphi) = y(-\alpha,\beta,\varphi)\,, \\ z(\alpha,\beta,\varphi) &= z(\alpha,-\beta,\varphi) = z(-\alpha,\beta,\varphi). \end{aligned} \qquad (19)$$

The line $\alpha = \beta = 0$ is the magnetic axis.

Near the axis α and β are small and the functions x, y, z can be expanded on powers of α and β

$$x = \alpha X_1(\varphi) + \ldots$$

$$y = \beta Y_1(\varphi) + \ldots \qquad (20)$$

$$z = Z_0(\varphi) + \tfrac{1}{2} Z_2^{\alpha}(\varphi)\alpha^2 + \tfrac{1}{2} Z_2^{\beta}(\varphi)\beta^2 + \ldots$$

In these expansions eqs. (19) are taken into account. Inserting (20) into (18) it is easy to check that the functions X_1 and Y_1 can be chosen arbitrary and $Z_0, Z_2^{\alpha}, Z_2^{\beta}$ are given by the following formulae

$$\frac{dZ_0}{d\varphi} = X_1 Y_1 \,, \quad Z_2^{\alpha} = -\frac{X_{1\varphi}}{Y_1} \,, \quad Z_2^{\beta} = -\frac{Y_{1\varphi}}{X_1} \qquad (21)$$

Arbitrariness of X_1 and Y_1 simply corresponds to the fact that in the first paraxial approximation the tandem field is determined by two components: the strengths of the axially-symmetric field on the axis and the quadrupole component.

In the approximation (19)

$$F_\varphi^2 = Z_{0\varphi}^2 + \alpha^2\left(X_{1\varphi}^2 + Z_{0\varphi} Z_2^{\alpha}\right) + \beta^2\left(Y_{1\varphi}^2 + Z_{0\varphi} Z_2^{\beta}\right)$$

and the omnigenity condition (17) gives

$$\frac{\partial}{\partial\varphi}\,\frac{X_{1\varphi}^2 + Z_{0\varphi} Z_2^{\alpha}}{Y_{1\varphi}^2 + Z_{0\varphi} Z_2^{\beta}} = 0 .$$

To simplify it we introduce instead of functions $X_1(\varphi)$ and $Y_1(\varphi)$ the functions $X(z)$ and $Y(z)$ by the definition

$$X\left(Z_0(\varphi)\right) \equiv X_1(\varphi) \,, \quad Y\left(Z_0(\varphi)\right) \equiv Y_1(\varphi)$$

They have a simple meaning: $x = \alpha X(z)$ is the equation of field lines in the xz plane; $y = \beta Y(z)$ gives field lines in the yz plane. Simple algebra yields

$$X\frac{d^2X}{dz^2} = c\, Y\frac{d^2Y}{dz^2} \,, \qquad (22)$$

where c is a constant. We see that, for the field to be omnigenous, one of these functions can be arbitrary; the other is to be determined from eq. (22).

The function $f(\alpha, \beta)$ is easily found in the approximation considered:

$$f(\alpha,\beta) = c\alpha^2 + \beta^2 = c\frac{x^2}{X^2(z)} + \frac{y^2}{Y^2(z)} .$$

This expression shows that sign of the constant c is very important. Positive constants correspond to nested magnetic surfaces with elliptical cross-sections; negative ones correspond to hyperbolic cross-sections. For particles to be confined near the axis of the device only the first case is suitable, of course.

4. CONCLUSION

We have shown that in the first approximation of the paraxial expansion there exists a wide class of omnigenous fields. Their determination consists of finding two functions, X and Y, related by one equation (22). A question which arises is whether is it possible to improve omnigenity by considering the next order of paraxiality (retaining terms up to order α^4 and β^4 in expansion (19)). It turns out that in this approximation two additional functions appear (so that there are four functions together with X in Y) and three new equations (in addition to eq. (22)) for these functions must be satisfied. In yet higher orders the number of equations becomes larger than the number of functions. This means that in higher orders only some degenerate solutions are possible.

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