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ATOMIC ENERGY COMMISSION

COUPLED CONVECTIVE AND CONDUCTIVE HEAT TRANSFER BY UP-WIND FINITE ELEMENT METHOD

by

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Descriptors

CONVECTION

THERMAL CONDUCTION

FINITE ELEMENT METHOD

HEAT EXCHANGERS

PLATES

PIPES

LAMINAR FLOW

WEIGHTING FUNCTION

BOUNDARY CONDITIONS

ONE DIMENSIONAL CALCULATIONS

NAVIER-STOKES EQUATION

DIFFUSION

STEADY STATE CONDITIONS

PRIMARY COOLANT CIRCUITS

SODIUM

COUPLED CONVECTIVE AND CONDUCTIVE HEAT TRANSFER BY UP-WIND FINITE ELEMENT METHOD

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1.0 INTRODUCTION :

The finite element method has been applied to a class of initial boundary value problems that are commonly referred to as diffusion-convection problems. Convective and conductive heat transfer both coupled /1/, /2/ as well as decoupled /3/ have been solved by Gartling, Taylor and Ijam and Hsu and Nickel. Convective problems are divided into two categories depending upon the force responsible for fluid motion. The momentum equation and energy equation are one way coupled in forced convection problems and two way coupled in natural convection problems. The fluid velocities may be determined by solving Navier-Stoke's equations in case of forced convection problems.

The combined heat transfer due to convection and conduction in a region consisting of both solid body and moving fluid is of the great interest in engineering practice, a typical example in nuclear reactor are heat transfer in component of heat exchanger, primary heat transport pipe in fast breeder reactor. The current practice in transient temperature distribution is obtained on the basis of an estimated heat flux at the solid/fluid interface. An assumed fluid temperature distribution and an average convective heat transfer coefficient are normally used for heat flux calculations.

The analyst is advised to use "accepted heat transfer practice", to find the temperature distribution by the ASME Boiler and Pressure Vessel code and is instructed to use the average wall temperature in "primary stress" calculations, the average gradient through the wall for, "equivalent linear stress" calculations and to lump the rest into "local thermal stress" calculations. Whether or not current analytical practice is conservative with regard to these quantities is uncertain.

In order to eliminate the uncertainty and possible errors introduced by the interface heat flux estimate, the coupled convective and conductive heat transfer has been analysed.

The transportive velocity field of the moving fluid, in decoupled problem may be assumed to be known prior to analysis. The scheme for determining the fluid velocity is described in Appendix-1.

The numerical solution of convective-diffusion equation present serious difficulties when the convective term is dominent. This difficulty stem from the combination of essentially elliptic and parabolic nature of two terms present in energy equation and yield a oscillatory solution, whenever mesh size exceed a certain critical value. Roache /7/ has described the spatial oscillations as "Wiggles" and has given the condition on the Cell Reynold Number in case of one-dimensional vorticity equation. In finite difference method this difficulty has been removed by using an upwind differencing formulation for advection terms /4/, /5/. A possible way to overcome these difficulties in the context of finite element was suggested by Zienkiewicz et al /6/ on the same ground as given by Spadling and Runchal.

In essence the procedure applies the finite element method using weighting functions of non-symmetric forms different from those originally used in shape functions. By suitable choice of such shape functions stability can be established with accuracy loss that is inherent in finite difference upwinding /7/.

2.0 FINITE ELEMENT FORMULATION :

Neglecting dissipation of energy in the system, the governing partial differential equation of the coupled convective and conductive heat transfer problem in 2-dimension is

$$\rho C \frac{\partial T}{\partial t} + \rho C u \frac{\partial T}{\partial x} + \rho C v \frac{\partial T}{\partial y} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + Q \quad (1)$$

subjected to the boundary conditions:

$$\left.\begin{aligned} T &= T_s \quad \text{on } \Gamma_1 \\ -K_x \frac{\partial T}{\partial x} l_x - K_y \frac{\partial T}{\partial y} l_y &= q_s \quad \text{on } \Gamma_2 \end{aligned}\right\} \quad 2(a,b)$$

Where T is unknown space and time dependent temperature in the region T_s and q_s are specified temperature and heat flux on the boundary surface Γ_1 and Γ_2 respectively; ρ is mass density of fluid and C is specific heat of fluid, K_x and K_y is the thermal conductivities in x and y directions respectively, Q represent the volumetric heat generation. The material properties may be inhomogeneous and transportive velocity field of the moving fluid u and v in x and y direction are known. l_x and l_y are component of outward normal vector on the boundary surface at a given point. In solid region the advection term is absent.

The domain Ω is mathematically subdivided into an arbitrary number of geometrically simple regions, the modified weight residual procedure transform the governing partial differential equation into a set of ordinary differential equations and leads to a finite element formulation of the coupled heat transfer problem.

The equation (1) is written in operator form as

$$\mathcal{L}(T) - Q = 0 \quad (3)$$

Where

$$\mathcal{L} = \rho c \frac{\partial}{\partial t} + \rho c u \frac{\partial}{\partial x} + \rho c v \frac{\partial}{\partial y} - \frac{\partial}{\partial x}\left(k_x \frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial}{\partial y}\right)$$

and let us assume a temperature distribution within element

$$\hat{T} = \sum_{i=1}^{node} N_i T_i \qquad (4)$$

If assumed temperature is substituted in (3), the equation will not be satisfied and an error vector $\{\epsilon\}$ will be generated.

$$\mathcal{L}(\hat{T}) - Q = \epsilon \qquad (5)$$

Now construct an orthogonal projection of each of this residual spaces onto subspace spanned by appropriate weighting functions. Constructing an inner product we get

$$\langle \epsilon, W_i \rangle = \int_\Omega \epsilon \, W_i \, d\Omega = 0 \qquad (6)$$

or

$$\int_\Omega \left[\rho c \frac{\partial \hat{T}}{\partial t} + \rho c \tilde{u} \frac{\partial \hat{T}}{\partial x} + \rho c \tilde{v} \frac{\partial \hat{T}}{\partial y} - \frac{\partial}{\partial x}\left(k_x \frac{\partial \hat{T}}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial \hat{T}}{\partial y}\right) - Q\right] W_j \, d\Omega \qquad (7)$$

Where w_i is suitable weighting function ,

Through the application of divergence theoram, this can be written as

$$\int_\Omega \left[\rho c \frac{\partial \hat{T}}{\partial t} W_j + \rho c \tilde{u} \frac{\partial \hat{T}}{\partial x} W_j + \rho c \tilde{v} \frac{\partial \hat{T}}{\partial y} + k_x \frac{\partial \hat{T}}{\partial x} \frac{\partial W_j}{\partial x} + k_y \frac{\partial \hat{T}}{\partial y} \frac{\partial W_j}{\partial y} - Q W_j\right] dx\,dy - \int_{\Gamma_2} \left[k \frac{\partial \hat{T}}{\partial n} n + q_s\right] W_j \, ds = 0 \qquad (8)$$

Substituting (4) in (8) and carrying out integration, the governing matrix equation of the coupled heat transfer problem can be written as

$$[C]\{\dot{T}\} + [K]\{T\} + [N(V)]\{T\} = \{R\} \qquad (9)$$

Where

$$[C] = \sum_{e=1}^{n} C^e \;;\; [K] = \sum_{e=1}^{n} K^e \;;\; [N(V)] = \sum_{e=1}^{n} N^e(v) \text{ and } \{R\} = \sum_{e=1}^{n} R^e \qquad (9a\text{-}d)$$

n is total number of elements.

The matrix $[C]$ and $[K]$ are heat capacity and conductivity matrix reppectively, $[N(V)]$ is advection matrix which depends upon the fluid velocity is due to mass transfer. $\{R\}$ is system of flux vector. The element matrices are given by

$$C_{ij}^{e} = \int_{\Omega} N_i \, \rho c \, W_j \, dx\, dy$$

$$K_{ij}^{e} = \int_{\Omega} \left(\frac{\partial N_i}{\partial x} k_x \frac{\partial W_j}{\partial x} + \frac{\partial N_i}{\partial y} k_y \frac{\partial W_j}{\partial y} \right) dx\, dy$$

$$N_{ij}^{e}(v) = \int_{\Omega} \left(\tilde{u} \frac{\partial N_i}{\partial x} W_j + \tilde{v} \frac{\partial N_i}{\partial y} W_j \right] dx\, dy$$

$$\{R_j^{e}\} = \int_{\Omega} Q \, W_j \, dx\, dy$$

3.0 **WEIGHTING FUNCTION FOR ISOPARAMETRIE LINEAR EIEMENTS :**

The weighting function for two-dimensional, isoparametrix element can be constructed in terms of product of one dimensional function /8/. The shape function for node Ni for node $\xi = \eta = -1$ is

$$N_i = \left(\frac{1-\xi}{2}\right)\left(\frac{1-\eta}{2}\right) = n_i(\xi) \cdot n_i(\eta)$$

Where

$$n_i(\xi) = \frac{1-\xi}{2} \quad \text{and} \quad n_i(\eta) = \frac{1-\eta}{2}$$

Similarly the weighting function can be written as

$$W_i = W_i(\xi) \; W_i(\eta)$$

Where

$$W_i(\xi) = n_i(\xi) + \alpha_{il} F(\xi)$$

$$W_i(\eta) = n_i(\eta) + \alpha_{im} F(\eta)$$

$$F(\xi) = 3(1-\xi^2)/4$$

The suffix 1 (or m) is that of adjacent node laying along the same element side and

$$|\alpha_{il}| = |\alpha_{li}|$$

$$|\alpha_{im}| = |\alpha_{mi}|$$

The sign of α coefficient dependent on the direction of the velocity vector, The velocity vector will be variable in general, hence the average velocity vector along the element side ij i.e., V_{ij} and given to α a sign identical to that of vector (see Fig 1-a).

The nodal velocity vector V_i will be available and we evaluate V_{ij} as

$$V_{ij} = (V_i + V_j) \, I_{ij}/2$$

In case of nine noded element, the mean velocity is calculated from

$$\bar{V}_{ij} = (V_i + 4V_m + V_j)/6$$

Where V_i and V_j are corner node velocities and V_m is velocity of mid node.

I_{ij} is unit vector specifying the direction of the side which is calculated from the nodal coordinates. The four independent values of α can be given an absolute value ranging from 0 (no upwinding) to 1 (full upwinding). However, an optimal values of α for non-oscillator solution can be obtained based on mesh size and local velocity /8/ and expressed as

$$\alpha = \coth(P/2) - 2/P$$

Where

$$P_x = \frac{\rho c \tilde{u} h_x}{k_x} \quad \text{and} \quad P_y = \frac{\rho c \tilde{v} h_y}{k_y}$$

$\tilde{u}$ and $\tilde{v}$ are average velocity along an element side length hx and hy in x and y direction respectively. It has been observed that optimal parameter gives better results than full upwinding used in finite difference scheme.

4.0 WEIGHTING FUNCTION FOR PARABOLIC ISOPARAMETRIC ELEMENT :

The upwind nine-noded element has been proved to achieve stability in all the situation than eight noded element. Hence, in present analysis, only 9 noded element have been considered to model fluid region and eight noded elements to model solid region. The idea of previous section has been extended for quadratic elements.

The shape function for the Ni node $\xi = \eta = -1$ is

$$N_i = n_i(\xi)\, n_i(\eta)$$

$$n_i(\xi) = \tfrac{1}{2}\xi(\xi-1)\ ; \quad n_i(\eta) = \eta(\eta-1)/2$$

and weighting function can be written as

$$W_i = W_i(\xi)\, W_i(\eta)$$

$$W_i(\xi) = \xi(\xi-1)/2 - \alpha_{ij} F(\xi)$$

$$W_i(\eta) = \eta(\eta-1)/2 - \alpha_{ik} F(\eta)\ ; \quad F(\xi) = 5\xi(\xi^2-1)/8$$

For midside node, at $\xi = 0\ ; \eta = -1$ the shape function

$$N_K = n_K(\xi)\, n_K(\eta)$$

$$n_K(\xi) = 1-\xi^2\ , \quad n_K(\eta) = (1-\eta)/2$$

and weighting function can be written as

$$W_k = W_k(\xi)\, W_k(\eta)$$

$$W_k(\xi) = 1-\xi^2 + \beta_{km}\, G(\xi)$$

$$W_k(\eta) = (1-\eta)/2 + \beta_{kl}\, G(\eta) \quad ,$$

$$G(\xi) = 5\xi(\xi^2-1)/2$$

and optimal values of parameters are

$$\beta = \coth(P/4) - 4/P$$

and $\alpha = 2\tanh(P/2)\cdot\left(1 + 3\beta/P + 12/P^2\right) - 12/P - \beta$

which is derived from the exact solution of one dimensional diffusion-convection equation. The sign convection for the above parameters is shown in Fig 1-b.

5.0 SOLUTION PROCEDURE :

It can be seen by inspection that the advection matrix $[N(V)]$ due to moving fluid is in general, asymmetric. Tay and Davis /9/ solved the assymmetric system directly, using a general banded Gaussian elimination procedure.

Three different alternatives can be used to solve (9) :

1. A general, banded equation solver may be used to factor the asymmetric system.

   The matrix $[N(V)]$ can be placed on the right hand side of (9), multiplied by some appropriate temperature field and treated as an 'Initial flux' vector.

3. The symmetric part of $[N(V)]$ may be added to $[K]$ and skew-symmetric part is treated as in 2.

   Gartling has suggested to use the first method on the basis of lesser oscillations at higher Peclet number. Therefore, the present development has concentrated on solving as an unsymmetric system. To find transient solution to (9), modified crank-Nicolsm method was used. The modified crank-Nicolson algorithm is given in ref. /10/ as

$$[K_{eff}]\{T\}_{t+\Delta t/2} = \{F_{eff}\}_{t+\Delta t/2} \qquad (15)$$

where

$$[K_{eff}] = [K] + [N(v)] + 2/\Delta t\,[C]$$

$$\{F_{eff}\}_{t+\Delta t/2} = \frac{2}{\Delta t}[C]\{T\}_t + \{R\}_{t+\Delta t/2}$$

...8/-

6.0 SELECTION OF MESH SIZE AND TIME STEP :

In the solution of steady state vorticity transport equation with a mesh size of h , the solution will be nonoscillatory if local or element pecklet number is less than 1. i.e., $P = \frac{\rho c \bar{u} h}{K} < 1$

The above restriction can be violated in case of upwind scheme. Similarly for transient analysis the time step Δt should be less than $\frac{\rho c h^2}{2K}$. These are valid for one dimensional problem only. The above criterion has been nicely described in reference /7/ for two dimensional problems.

7.0 EXAMPLES :

7.1 SOLUTION OF ONE DIMENSIONAL TRANSPORT EQUATION

The one dimensional steady state vorticity equation (diffusion-convection equation) is

$$u \frac{\partial \xi}{\partial x} = \alpha \frac{\partial^2 \xi}{\partial x^2}$$

Where u is advection speed and α is generalised diffusion coefficient. Although vorticity does not exist in one dimensional situation but this equation model some aspects of multidimensional equations. The advection speed is constant or may be function of x. The study of one dimensional vorticity equation is convenient to study its limitations.

Let us consider in the first problem the solution of steady state equation described above. For u=0, the finite element solution gives exact continuum solution, as shown in Fig.2a. As wind blows harder i.e., $u > 0$, the ξ profile is blown downstream, as in the continuum solution. The solutions were obtained at various values of u=0 and α/u = 0.1,0.2...0.01 The solution matches very well for u=0 and α/u = 0.1, ...0.05 /7/. At α/u = 0.01 severe oscillations have been observed as shown in Fig.2b and the wiggles are totally absent in case of upwind finite element solution. Of course, at higher advection velocity, the wiggles will be more pronounced and upwinding is the only remedy. The results shown in Fig.2b for $\alpha/u = 0.001$ clearly indicates that no oscillation are present.

The second problem is the solution of one dimensional concentration problem reported in reference /11/. The governing differential equation is

$$\frac{\partial c}{\partial t} = D' \frac{\partial^2 c}{\partial x^2} - u \frac{\partial c}{\partial x}$$

Subjected to conditions

$$C(0,t) = C_1 \quad , t > 0$$
$$C(\infty,t) = 0 \quad , t > 0$$
$$C(x,0) = C_0 \quad , x \geq 0$$

Where $C(x, t)$ is concentration of solute in the fluid, D' is a dispersion coefficient, u is a constant average bulk fluid velocity, C_1 is the input concentration and C_0 is the initial concentration of solute in the fluid. Fig.3 shows the finite element solution. The finite element solution and exact solution matches very well.

7.2 COMBINED CONDUCTION/CONVECTION BETWEEN PARALLEL PLATE AND CIRCULAR PIPE :

In this example a fully developed parabolic laminar flow of sodium between two thick parallel steel plates/pipes have been assumed (see Fig.4). A finite element mesh has been shown in Fig.5 and linear isoparametric elements in the fluid region and in solid region have been used. Both inlet and outlet temperatures were specified and outer surfaces of plate is assumed to be perfectly insulated. The inlet velocities were assumed to be known. Fig.5 also shows the isothermal temperature lines in fluid region as well as in solid region. At higher Peclet Number, the oscillations have been observed and then upwinding with optimal parameter were used to remove the oscillations. Fig.6 shows the temperature plot along the flow center line. The effect of upwinding at higher Peclet Number is worth noticing. The Table 1 compare the temperature values at maximum velocity of 80 ft/sec and 1000 ft/sec respectively. The same problem has been solved using mixed elements (Lagrangian and Serendipity). The fluid region is discritized using 8 nine noded Lagrangian elements and solid region is discritized by 12 eight noded serendipity elements as shown in Fig.7. The temperature distribution along the flow center line is shown in Fig.8. Oscillations can be seen at maximum velocity of

100 ft/sec and oscillations are absent when upwind formulation has been used. Fig.9 shows the temperature distribution along the interior surface of plate without and with upwinding. It is evident that upwinding is necessary to remove the oscillation an advection dominated flow.

8.0 CONCLUSIONS :

Application of standard Garlerkin procedures of weighting function posses certain difficulties while solving non-self adjoint energy equation. The difficulty will not be present if Nueman type boundary conditions at exit are specified, or upwinding finite element scheme is used at high Peclet Number. In this report oscillations have been overcome by using optimal upwind scheme both in linear and quadratic elements.

9.0 ACKNOWLEDGEMENTS :

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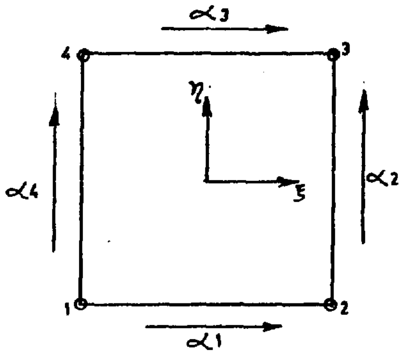
TABLE NO.1 : TEMPERATURE AT X = 0·0 WITH INCREASING FLUID VELOCITY

Y	TEMPERATURE °F, U_{MAX} = 50 ft/Sec	TEMPERATURE °F, U_{MAX} = 1000·0 ft/Sec
0·0	100·0*	100·0*
0·2	99·955	100·0
0·4	99·797	100·0
0·6	99·408	100·0
0·8	98·547	99·996
1·0	0·0*	0·0*

* INDICATES THE SPECIFIED TEMPERATURE



IF $P < 20$

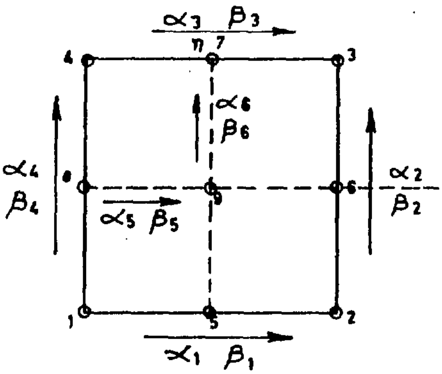
$$\alpha = \coth\left(\frac{P}{2}\right) - 2/P$$

IF $P > 20$

$$\alpha = 1 - 2/P$$

α_1 IS POSITIVE IF $u_1 > u_2$

FIG 1a VELOCITY SIGNS CONVENTION FOR LINEAR ELEMENT



IF $P < 10$

$$\beta = \coth\,(P/4) - \frac{4}{P}$$

AND

$$\alpha = 2\,\tanh\,(P/2)\,\left(\frac{1+3\beta}{P} + \frac{12}{P^2}\right) - \frac{12}{P} - \beta$$

IF $P > 10$

$$\beta = 1 - 4/P$$

$$\alpha = 1 - 2/P$$

FIG. 1b. VELOCITY SIGNS CONVENTION FOR NINE NODED ELEMENTS

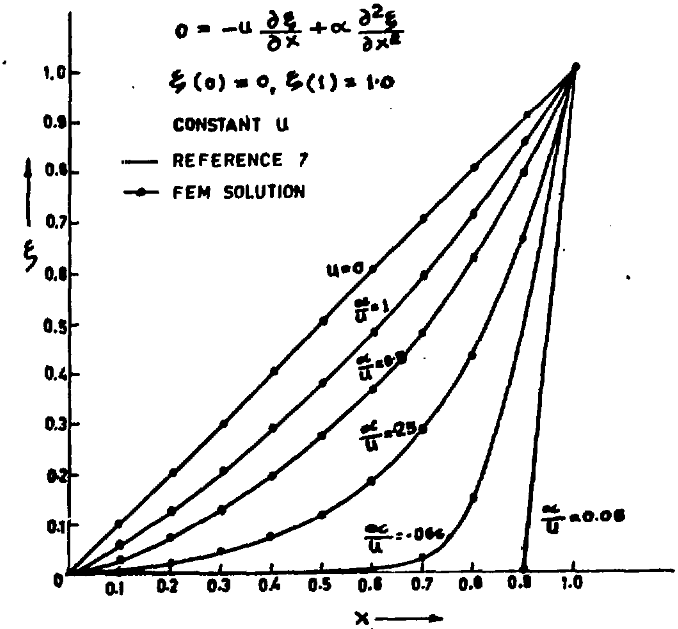


FIG.2 SOLUTION TO THE STEADY STATE LINEAR MODEL ADVECTION DIFFUSION EQUATION.

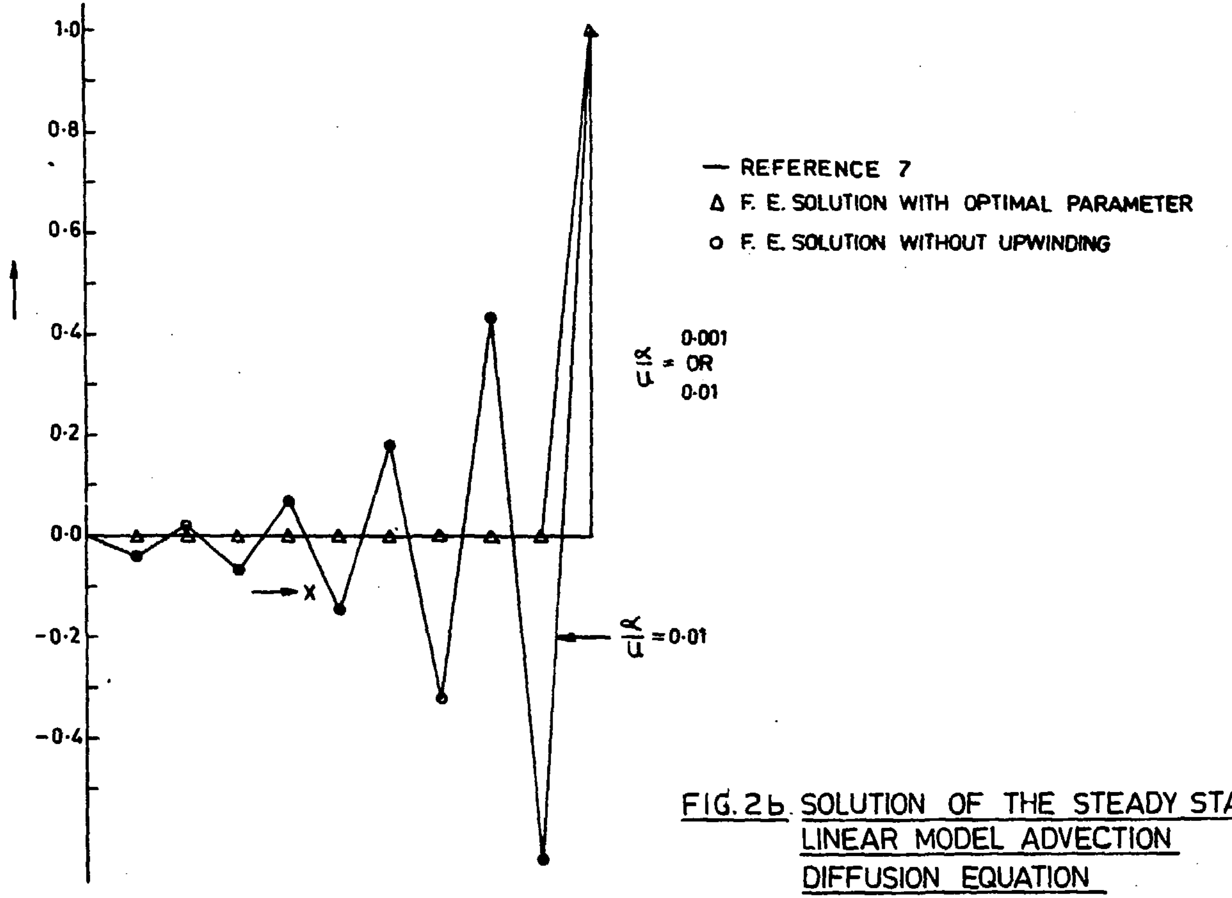


FIG. 2b. SOLUTION OF THE STEADY STATE LINEAR MODEL ADVECTION DIFFUSION EQUATION

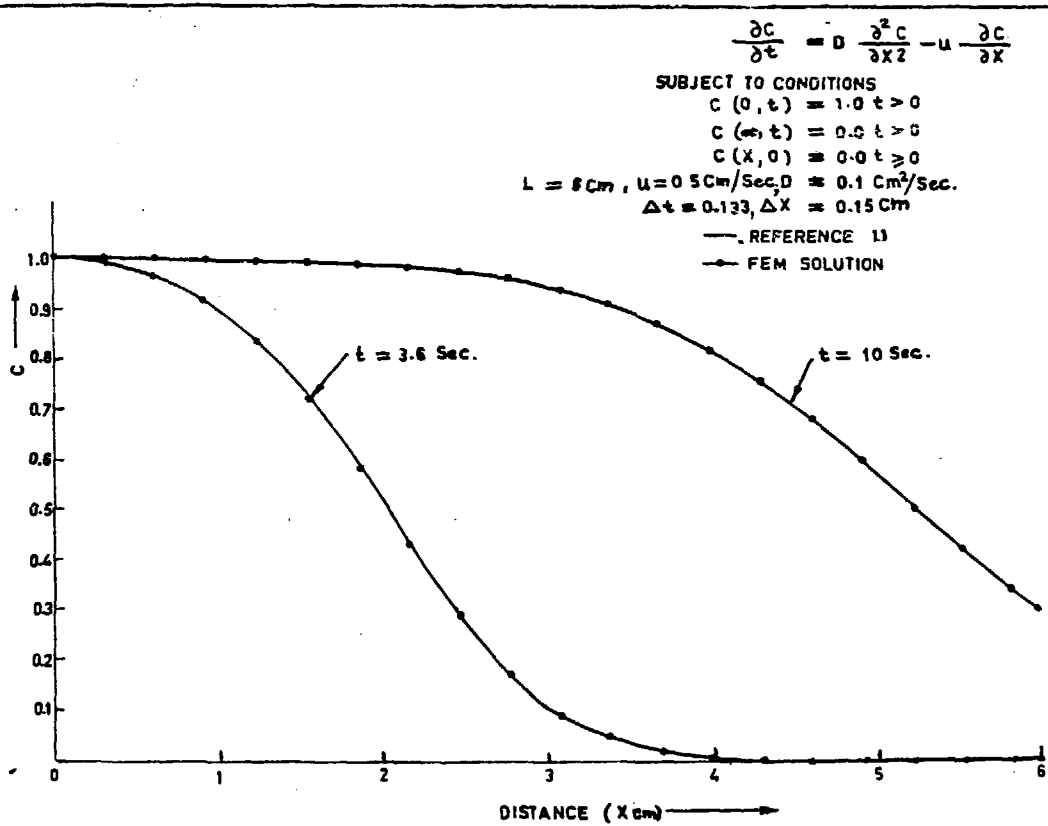


FIG.3. ONE DIMENSIONAL DIFFUSION CONVECTION PROBLEM

ρ_{SODIUM} = 58 lbm/ft^3
$C_{P SODIUM}$ = 33 Btu/lbm-°F
μ_{SODIUM} = 1.69 lbm/ft-hr
K_{SODIUM} = 49.8 Btu/ft-hr °F
ρ_{STEEL} = 49.0 lbm/ft^3
K_{STEEL} = 24.8 Btu/ft-hr-°F
$C_{P STEEL}$ = 0.14 Btu/lbm-°F

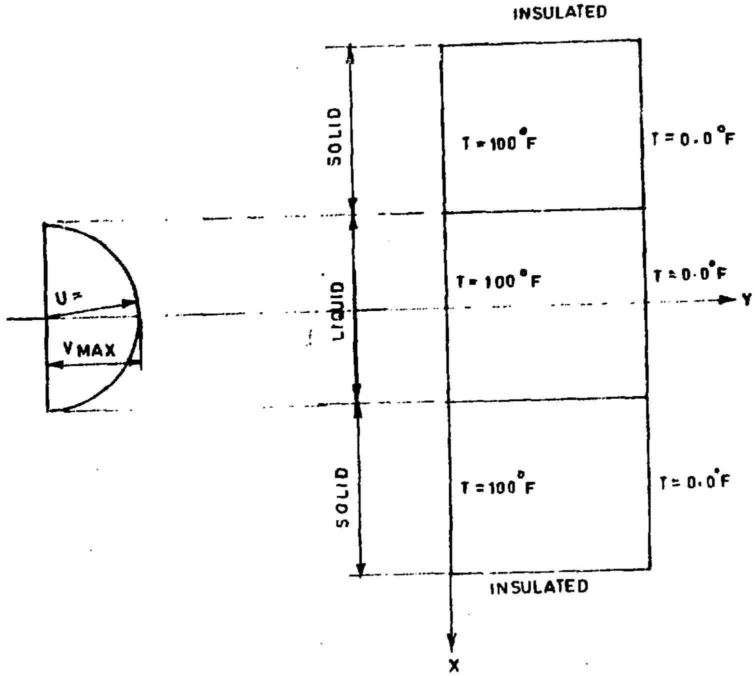


FIG. 4. THE GEOMETRY AND BOUNDARY CONDITIONS

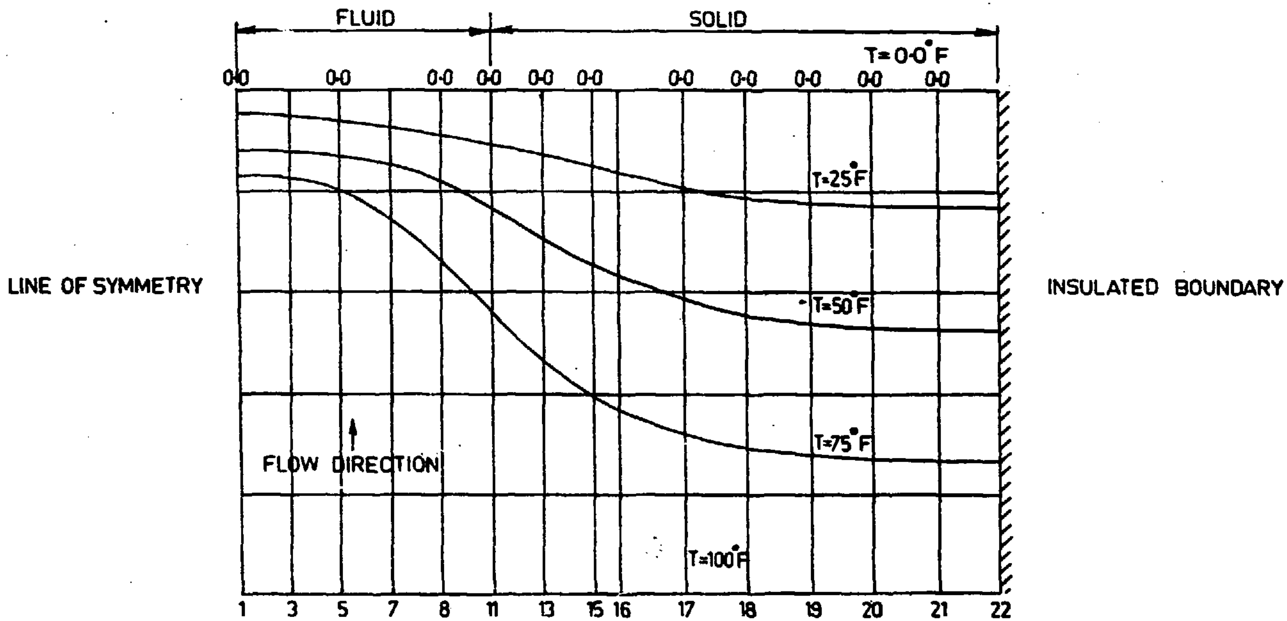


FIG.5 ISOTHERM LINES IN THE SOLID AND FLUID REGION AT PECLET NO. 15

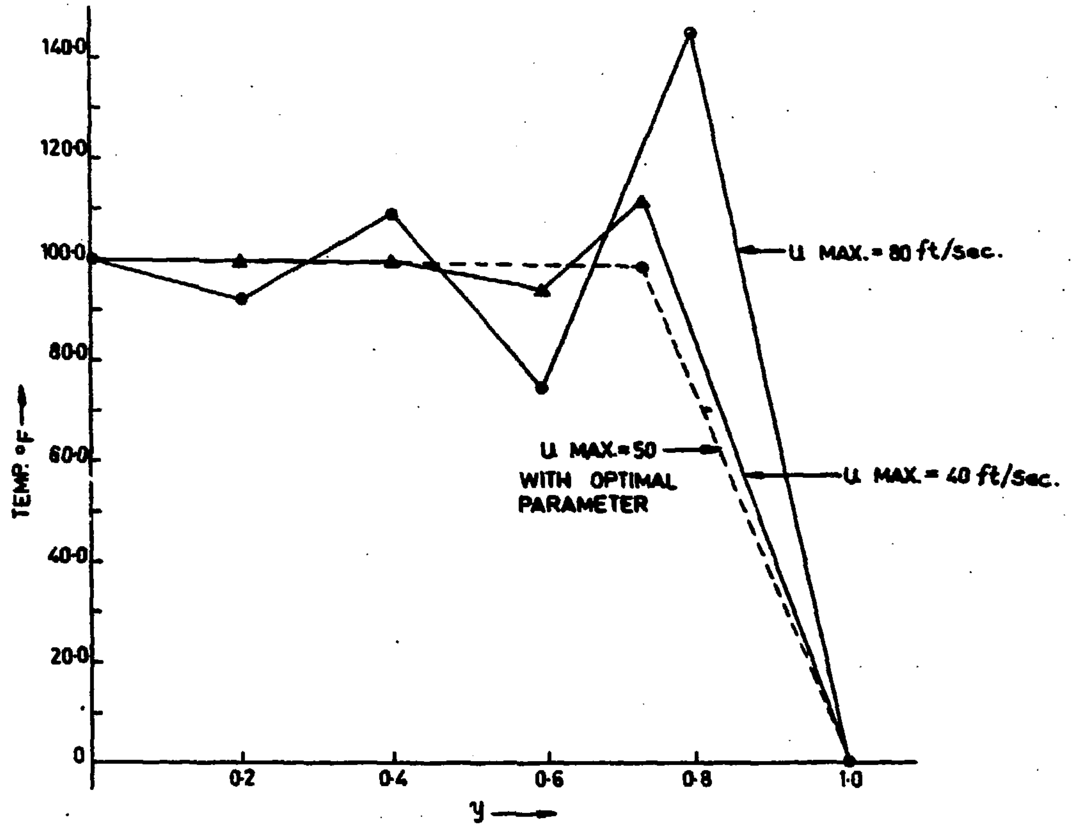


FIG.6. TEMPERATURE DISTRIBUTION IN FLUID AT X = 0·0 INCH

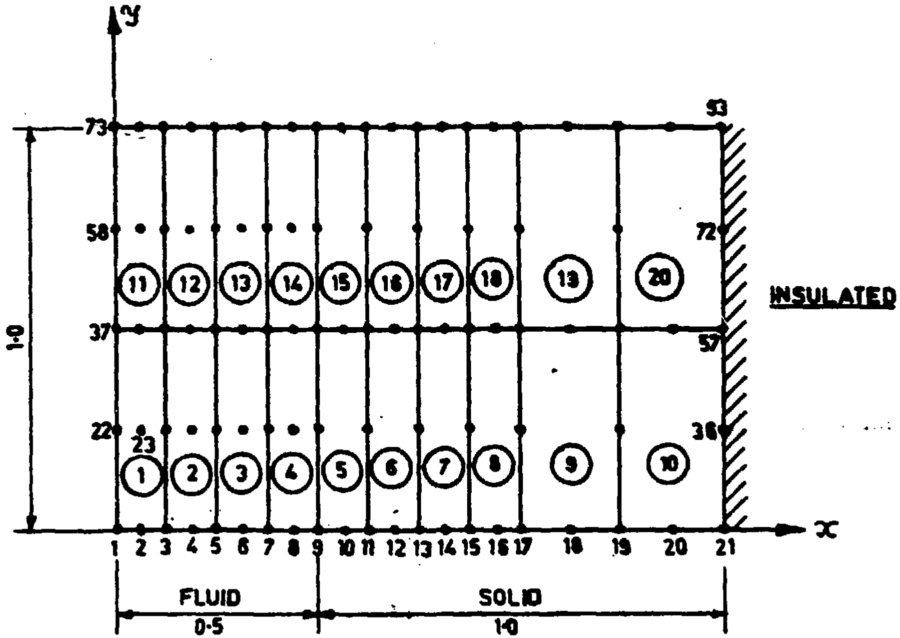
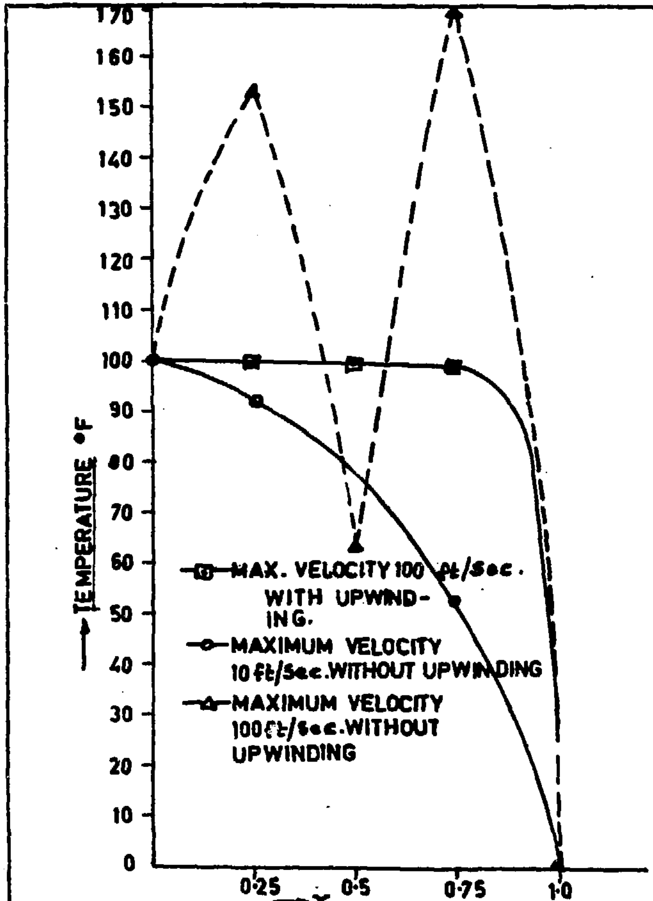


FIG. 7 DISERITIZATION OF FLUID AND SOLID REGION USING MIXED ELEMENT [LAGRANGIAN AND SERENDIPITY]



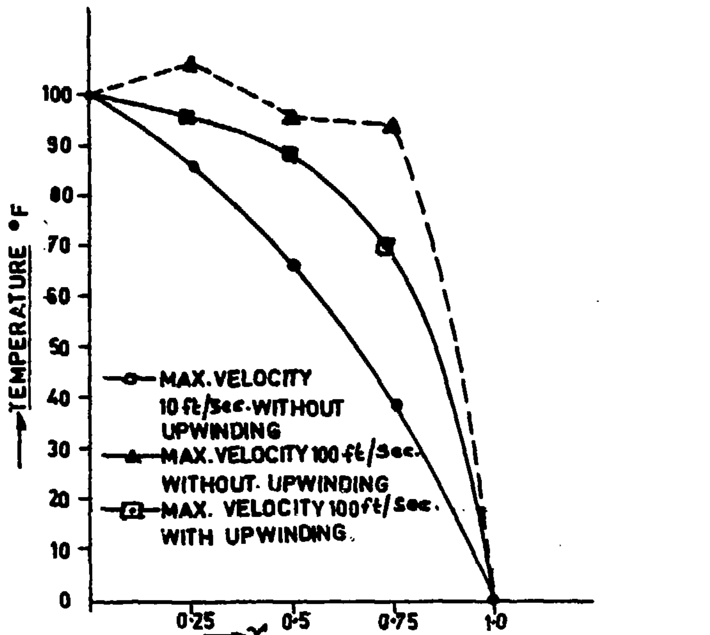


FIG 8. TEMPERATURE DISTRIBUTION ALONG FLOW CENTER LINE.

FIG 9. TEMPERATURE DISTRIBUTION ALONG INTERIOR SURFACE

APPENDIX-1

SOLUTION OF NAVIER-STOKE'S EQUATION :

Fluid mechanics is rich in nonlinearity. It is rich in mixed hyperbolic and elliptic partial differential equations /1/.

The mathematical description of motion of body of Viscous fluid was developed in the early 19th Century by Navier; Poisson and Stoke's and the first numerical solution was obtained by Thom in 1933. With the development of digital computer in late 1940's more emphasis was given to numerical solution.

Throughout the development of numerical solution in fluid mechanics, the finite difference method was more popular. The finite difference method allowed a great deal of sophistication in modeling fluid flow problems. However, there are several areas in which finite difference method prove to be inconvenient. The some of difficulties are to account the complex boundary shape, inaccurate specification of boundary conditions along nonstraight boundaries and an ability to employ nonuniform, nonrectangular meshes.

The finite element method has been successfully used for solid mechanics problems and letter was extended to other areas of mathematical physics.

The solution of Navier-Stoke's equation was presented by Oden in 1970 for general Stokesian fluid using energy ballance approach. In 1972, Cheng and Olson independently proposed restricted variational principles for Viscous flow problems using vorticity and steam function approach but the solutions were obtained at low Reynold Number. Bratanow et al employed a linearized restricted Variational principle in finite element analysis at high Reynold Number. The Galerkin's method was used by Baker on the vorticity transport equation. Taylor and Hood, Kawahara et al, Gartling and Gresho et al solved Navier Stoke's equations using primitive variable approach (mixed formulation).

The most popular and attractive method in which pressure is eliminated as an explicit variable is penality - function approach. It leads to the simple and effective finite element implementation of incompressibility. There are many references in solid mechanics and in fluid mechanics where penality function formulation has been used successfully. Penality function approach for incompressible fluid flow problems have been nicely described in references /7/, /8/. The penality function method is also described in this report.

BASIC EQUATIONS :

The two dimensional Navier-Stoke's equation assuming the fluid motion is laminar, steady, isothermal and incompressible can be written as

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j} \qquad (1)$$

Where

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (2)$$

is a contitutive relationship and constrained condition from the conservation of mass is

$$\frac{\partial v_i}{\partial x_i} = 0 \qquad (3)$$

The solution should satisfy the following boundary conditions

$$u_i = \bar{u}_i \quad \text{on } \Gamma_1 \qquad (4a)$$

$$\tau_{ij} n_j = h_i \quad \text{on } \Gamma_2 \qquad (4b)$$

Where V_i represent the velocity component in X_i direction, ρ is density, μ

is Viscosity f_i is body force vector, p is pressure, τ_{ij} is stress tensor and δ_{ij} is the Kronecker delta. The specification of boundary conditions for the region of interest in term of V_i or τ_{ij} as given by equation (4) completes the boundary value problem.

The choice of dependent variable depnds upon the problem but in the present report for mixed formulation, the velocity and pressure will be used. A comparison between U-V-P and stream-vorticity as dependent variable has been given in ref /2/.

FINITE ELEMENT METHOD :

It has been mentioned by several investigator, first by Millikan /3/ and recently by Deshpande /4/ that no classical variational principle exists for the Navier-Stoke's equations and hence several alternative method have been employed. However, as pointed out by Finlayson and Scriven, these nonclassical variational statement are essentially the Galerkin method or another form of weighted residuals.

Note that in an incompressible fluid the pressure is an intrinsic and independent variable of the motion and is not related to any thermodynamic equation of state, it is an implicit variable which instantaneous "adjust itself" in such a way that the incompressibility constant (continuity equation) remains satisfied. This is one characteristic of an incompressible flow that invariably makes the problem difficult to solve, another is the nonlinear advection term $\underset{\sim}{V} \cdot \nabla \underset{\sim}{V}$.

The Navier-Stoke's equations contains highest differential of pressure which appear in the first. Therefore, polynomial function for the velocities must be parabolic while that for the pressure is linear.

The element velocity and pressure are approximated as

$$\hat{V} = \sum_{j=1}^{N} N_j(x,y)\, V_j(t) \qquad (5)$$

$$P = \sum_{j=1}^{M} H_j(x,y)\, P_j(t) \qquad (6)$$

Where there are N velocity nodes and M pressure nodes in the discritized domain and N_j and H_j are parabolic and linear shape functions describe elsewhere /5/. Consider the residual space of the momentum and continuity equations as

$$\rho \dot{\hat{V}} + \rho\, \hat{V}_{i,j}\, V_j - \rho f_i - \tau_{ij,j} = \epsilon_i^{(1)} \qquad (7)$$

$$\hat{V}_{i,i} = \epsilon^{(2)} \qquad (8)$$

Now construct the orthogonal projection of these residual spaces onto subspace spanned by appropriate weighting functions. The inner product must yield the spatial invarient for momentum equation as the obvious variable is the velocity.

$$I_1 = \langle \epsilon_i^{(1)}, V_i \rangle = \int_\Omega \epsilon_i^{(1)}\, V_i\, d\Omega \qquad (9)$$

and for the continuity equation, the pressure is the subspace.

$$I_2 = \langle \epsilon^{(2)}, P \rangle = \int_\Omega \epsilon^{(2)}\, P\, d\Omega \qquad (10)$$

The invarient I_1 is energy due to velocity field whereas invarient I_2 represents the energy required to maintain the incompressibility of the fluid. For arbitrary nodal values of velocity and pressure the above equations reduces to

$$\int_\Omega \epsilon_i^{(1)}\, N_j\, d\Omega = 0 \qquad (11)$$

$$\int_\Omega \epsilon^{(2)}\, H_j\, d\Omega = 0 \qquad (12)$$

After algebraic simplification, we get

$$[M]\{\dot{V}\} + [K]\{V\} + [N(V)]\{V\} + [C]\{P\} = \{F\} \qquad (13)$$

and

$$[C]^T\{P\} = 0 \qquad (14)$$

Where

$$[K] = \mu \begin{bmatrix} 2k_{11}+k_{22}; & k_{21}; & -\frac{Q_1}{\mu} \\ k_{12}; & 2k_{22}+k_{11}; & -\frac{Q_2}{\mu} \\ -\frac{1}{\mu}Q_1^T; & -\frac{1}{\mu}Q_2^T, & 0 \end{bmatrix}; [N(v)] = \rho \begin{bmatrix} N_1(V_1)+N_2(V_2); & 0, & 0 \\ 0; & N_1(V_1)+N_2(V_2), & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{V\} = \begin{Bmatrix} V_1 \\ V_2 \\ P \end{Bmatrix}; \quad N_1(V_1) = \int_\Omega N N^T \tilde{V}_1 \frac{\partial N^T}{\partial x_1} d\Omega$$

$$N_2(V_2) = \int_\Omega N N^T \tilde{V}_2 \frac{\partial N^T}{\partial x_2} d\Omega; \quad Q_1 = \int_\Omega \frac{\partial N}{\partial x_1} H^T d\Omega$$

$$Q_2 = \int_\Omega \frac{\partial N}{\partial x_2} H^T d\Omega, \quad k_{11} = \int_\Omega \frac{\partial N}{\partial x_1} \cdot \frac{\partial N^T}{\partial x_1} d\Omega$$

$$k_{22} = \int_\Omega \frac{\partial N}{\partial x_2} \frac{\partial N^T}{\partial x_2} d\Omega, \quad k_{12} = \int_\Omega \frac{\partial N}{\partial x_1} \frac{\partial N^T}{\partial x_2} d\Omega$$

$$k_{21} = \int_\Omega \frac{\partial N}{\partial x_2} \frac{\partial N^T}{\partial x_1} d\Omega, \quad F = \rho \int_\Omega [N]^T f \, d\Omega + \int_{\Gamma_2} N^T \tau_{ij} n_j \, ds$$

and $M = \rho \int_\Omega N N^T d\Omega$

The equations (12) and (13) written together as

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{V} \\ \dot{P} \end{Bmatrix} + \begin{bmatrix} K+N(V) & C \\ C^T & 0 \end{bmatrix} \begin{Bmatrix} V \\ P \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad (15)$$

For steady state problem, Equation (14) reduces to

$$\begin{bmatrix} K+N(V) & C \\ C^T & 0 \end{bmatrix} \begin{Bmatrix} V \\ P \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad (16)$$

or

$$[H]\{\phi\} = \{F\}$$

where

$$\{\phi\} = \begin{Bmatrix} V \\ P \end{Bmatrix} \; ; \; \{F\} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad \text{and} \quad [H] = \begin{bmatrix} K + N(v) & C \\ C^T & 0 \end{bmatrix}$$

PENALITY - FUNCTION - FORMUIATION :

In this method, the constitutive equation (2) is replaced by

$$\tau_{ij}^{(\lambda)} = -p^{(\lambda)} \delta_{ij} + 2\mu \left(\frac{\partial V_i^{(\lambda)}}{\partial x_j} + \frac{\partial V_j^{(\lambda)}}{\partial x_i} \right) \qquad (17)$$

in which

$$p^{(\lambda)} = -\lambda \frac{\partial V_i^{(\lambda)}}{\partial x_i}$$

Where $\lambda > 0$ is known as penality parameter. Now equation (3) will be dropped in the finite element formulation of Navier-Stoke's equation. The problem may be restated as follows.

$$\rho V_j^{(\lambda)} \frac{\partial V_i^{(\lambda)}}{\partial x_j} = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

or

$$\rho V_j^{(\lambda)} \frac{\partial V_i^{(\lambda)}}{\partial x_j} = \rho f_i - \frac{\partial p^{(\lambda)}}{\partial x_j} + 2\mu \frac{\partial}{\partial x_j} \left(\frac{\partial V_i^{(\lambda)}}{\partial x_j} + \frac{\partial V_j^{(\lambda)}}{\partial x_i} \right) \qquad (18)$$

Let W_j denote the test functions for the velocity V_j, the weak form of (18) may be written as

$$\int_\Omega W_i V_j \frac{\partial V_i}{\partial x_j} d\Omega + 2\mu \int \frac{\partial W_i}{\partial x_j} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) d\Omega +$$

$$\lambda \int_\Omega \frac{\partial W_i}{\partial x_i} \frac{\partial V_j}{\partial x_j} d\Omega = \rho \int_\Omega W_i f_i \, d\Omega - \int_{\Gamma_2} W_i \tau_{ij} n_j \, d\Gamma \qquad (19)$$

Now equation (4) will be used to expand V_i and Galerkin formulation where test function are chosen the same as weighting function results in the following matrix equation. In equation (19) the superskript (λ) has been dropped.

$$[A]\{v\} + \{Q(v)\} = \{R\} \tag{20}$$

where

$$[A] = \sum [A]^e$$

$$\{Q(v)\} = \sum \{Q(v)\}^e \quad \text{and} \quad \{R\} = \sum \{R\}^e$$

$$[A]^e = \int_\Omega [B]^T [D\lambda][B]\, d\Omega + \int_\Omega [B]^T [D\mu][B]\, d\Omega$$

$$\{Q(v)\}^e = \int_\Omega \rho\, [N]^T\, v_j \frac{\partial v_i}{\partial x_j}\, d\Omega$$

$$[B_i] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} ; \; [D\lambda] = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \; [D\mu] = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

λ must be large enough so that compressibility and pressure errors are negligible. For all computational purpose we can choose $\lambda = 10^7$.

$$\{R\}^e = \rho \int_\Omega [N]^T \{f_i\}\, d\Omega + \int_{\Gamma_2} [N]^T \{h_i\}\, d\Gamma$$

The most effective elements used so far in penality-function-formulation are Lagrangian isoparametric elements. The bilinear (4-node) and biquadratic (9-node) elements have been incorporated in the computer program. The integrating order for 4-node element for λ-term is 1-point and for μ-term is 2 x 2. The integrating order for 9-node element for λ-term is 2 x 2 and for μ-term is 3 x 3. These are selective Gauss - Legendre integration rules for 2-D isoparametric elements. The element pressure may be determined by computing the values of

$$p = -\lambda \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \tag{21}$$

by using 2 x 2 integrating rule. Pressure field obtained using above equation will be discontinuous and smoothing procedure is desirable for plotting purpose.

SOLUTION SCHEME :

Two solution scheme have been used in the present analysis are namely.

(i) Picard iteration for mixed formulation.

(ii) Incremental Newton-Raphson method for penality-function-formulation.

(i) PICARD ITERATION METHOD :

The solution of Equation (17) is

$$\{\phi\} = [H]^{-1}\{F\} \qquad (22)$$

The matrix $[H]$ is function of $\{\phi\}$ and first solution is obtained by neglecting the nonlinear term $[N(V)]$. The successive solution are obtained by

$$\{\phi^{r+1}\} = [H(\phi^{r})]^{-1}\{F\} \qquad (23)$$

(ii) INCREMENTAL NEWTON RAPHSON METHOD :

Equation (20) can be written as the sum of linear and nonlinear terms

$$[A]\{V\} + \{Q(V)\} = \{R\} \qquad (24)$$

In the above scheme, the density is used as "loading parameter" within each load level we iterate until convergence is obtained. The converge solution is used as the initial guess for the next load level. In the beginning, the Stokes solution is obtained by neglecting $\{Q(V)\}$.

In typical load level, the iterates are computed from,

$$[A^{*}(\tilde{V})]\{\Delta V\} = \{\Delta R\} \qquad (25)$$

$$\tilde{V} \leftarrow \tilde{V} + \Delta V$$

Where $\tilde{V}$ is the latest approximation to the solution and unballanced forced is calculated as

$$\{\Delta R\} = \{R\} - [A]\{\tilde{V}\} - \{Q(\tilde{V})\} \qquad (26)$$

$$[A^{*}] = [A] + [DQ(\tilde{V})] \qquad (27)$$

Where $[DQ(V)]$ is tangent convection matrix. The convergence is achieved for any load level when ratio of $\{\Delta R\}$ to its original value is less than some tolerance say 10^{-2}. The tangent convection matrix for each individual element may be written as

$$[DQ(\bar{V})]^e = \rho \int Ne \left[\frac{\partial \bar{v}_i}{\partial x_j} . Nm + \delta_{ij} \bar{V}_k \frac{\partial Nm}{\partial x_k} \right] d\Omega \quad (28)$$

Where l and m are element node number (local).

EXAMPLES :

1.0 SLIDER BEARING :

The geometry of slider bearing is shown in Fig.1. The lower plane AB moves with U = 30 ft/sec velocity relative to the upper plane CD. The boundary conditions are shown in Fig.1. The viscosity and density were 0.0008 lb sec/ft^2 and 1.0 respectively. The velocity at two boundaries are shown in Fig.2. The pressure profile is shown in Fig.3. The result matches very well with classical solution given in ref /6/.

2.0 COUETTE FLOW :

In this example, a fully developed plane parallel flow, between two parallel plates have been considered. The upper plaete moves with a relative velocity u=3 units. The u velocity at y=0 is 0 and at y=h is 3 units. The velocity at any point is given by

$$u = \frac{y}{h} U - \frac{h^2}{2\mu} \frac{dp}{dx} \frac{y}{h} (1 - y/h)$$

The density and viscosity of fluid was taken as unity. Fig.4 shows the velocity profile for different values of pressure gradient at inlet and exit.

CONCLUSION :

In this report some of concept relating to finite element formulation of the Navier-Stoke's equations using mixed formulation and Penality formulation have been discussed. The two different approaches for solution of nonlinear differential equations for two different types of formulation have been described. Incremental Newton Raphson method can also be applied to mixed formulation.

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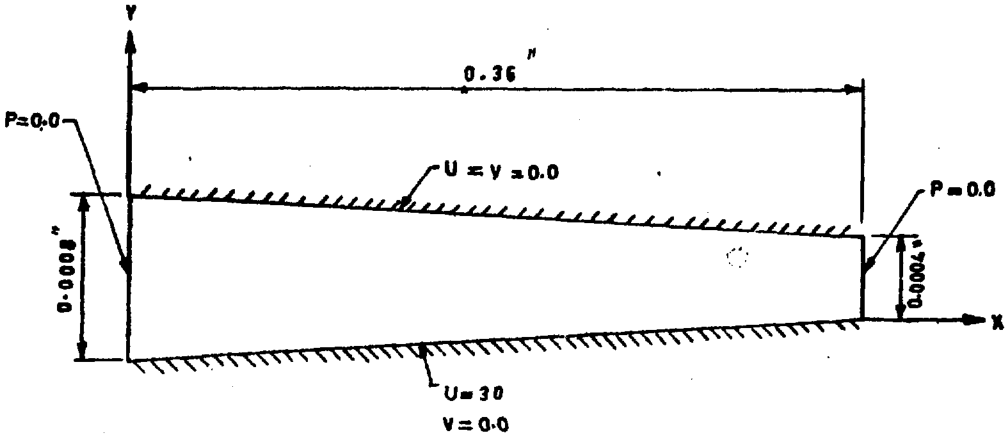

FIG. 1 LUBRICATION IN A SLIDER BEARING — BOUNDARY DATA

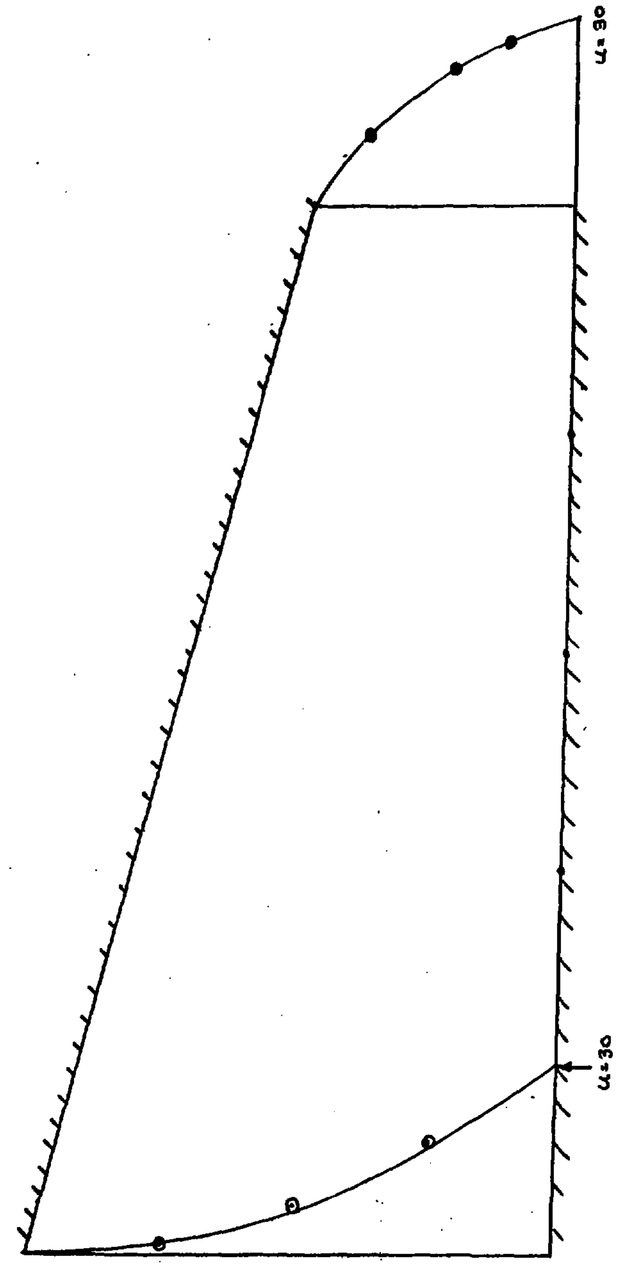

FIG.2 VELOCITY VECTOR & PRESSURE PLOT FOR SLIDER BEARING

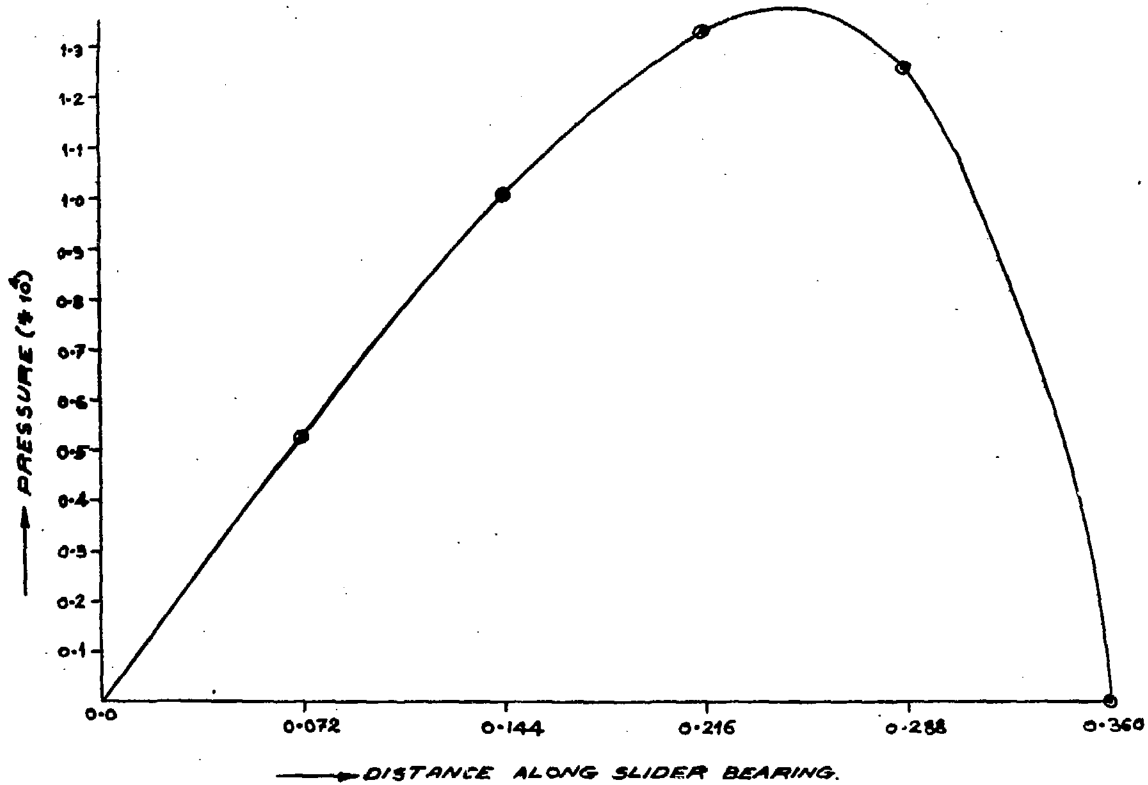

FIG.3 PRESSURE ON A SLIDER BEARING FACE

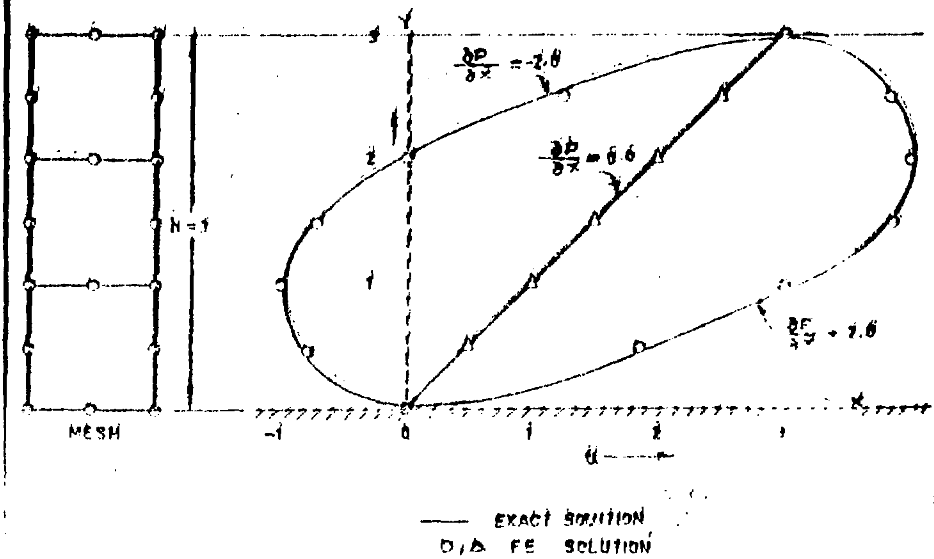

FIG.4 COUITE FLOW BETWEEN FLAT PLATE