AECL-7428

ATOMIC ENERGY OF CANADA LIMITED

L'ÉNERGIE ATOMIQUE DU CANADA LIMITÉE

SAMPLING FROM THE NORMAL AND EXPONENTIAL DISTRIBUTIONS

Échantillonnage à partir de la distribution normale ou exponentielle

K.R. CHAPLIN and C.A. WILLS

Chalk River Nuclear Laboratories   Laboratoires nucléaires de Chalk River

Chalk River, Ontario

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Résumé

On décrit des méthodes permettant d'engendrer des nombres aléatoires à partir de la distribution normale ou exponentielle. Il s'agit de diviser chaque fonction en sous-régions pour chacune desquelles on développera une méthode d'échantillonnage faisant appel à la technique de l'acceptation-rejet. En faisant l'échantillonnage à partir de la distribution normale ou exponentielle, chaque sous-région fournit la valeur aléatoire requise avec une probabilité égale au rapport existant entre sa zone et la zone totale. Des procédures écrites en FORTRAN et destinées au système CYBER 175/CDC 6600 sont fournies pour que l'on puisse mettre en oeuvre les deux algorithmes.

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ABSTRACT

Methods for generating random numbers from the normal and exponential distributions are described. These involve dividing each function into subregions, and for each of these developing a method of sampling usually based on an acceptance rejection technique. When sampling from the normal or exponential distribution, each subregion provides the required random value with probability equal to the ratio of its area to the total area. Procedures written in FORTRAN for the CYBER 175/CDC 6600 system are provided to implement the two algorithms.

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TABLE OF CONTENTS

LIST OF FIGURES

SAMPLING FROM THE NORMAL AND EXPONENTIAL DISTRIBUTIONS

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1. INTRODUCTION

This paper presents a modified version of an existing algorithm for sampling from a normal distribution and a new algorithm for sampling from an exponential distribution. The next section describes how to sample from several specific probability density functions and the powerful acceptance rejection technique for sampling from an arbitrary distribution. The third section describes the algorithm for sampling from the normal distribution. The fourth section presents the algorithm for sampling the exponential distribution. The fifth section gives timing comparisons for subroutines implementing these algorithms.

2. GENERATING RANDOM QUANTITIES

Sampling from a distribution function F(x) is defined as generating a random value X where the probability that $X \leq x$ is F(x). The function F(x) must, therefore, be monotonically increasing from 0 to 1:

If $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$
$F(-\infty) = 0$
$F(\infty) = 1$

The probability density function (p.d.f.) of X is defined by

$$f(x) = \frac{d}{dx} F(x) \tag{2.1}$$

so

$$F(x) = \int_{-\infty}^{x} f(x)dx \tag{2.2}$$

The rest of this section describes methods of sampling which will be used in the algorithms for the standard normal and the exponential distributions. Section 2.1 shows that the sum of two uniformly distributed random variables has a p.d.f. with the shape of a symmetric trapezoid. Section 2.2 defines a variable with a p.d.f. having the shape of an isosceles triangle. Section 2.3 defines a technique which will generate a random variable X when given the p.d.f. f(x). Section 2.4 describes modifications which make this method more efficient.

2.1 Sampling from a Symmetrical Trapezoid

Take two uniformly distributed variables $u_1 \in (-k,k)$ and $u_2 \in (-h,h)$ where $k \geq h > 0$. Then the probability density functions for u_1 and u_2 are

$$f(u_1) = \begin{cases} \frac{1}{2k} & -k \leq u_1 \leq k \\ 0 & |u_1| > k \end{cases} \tag{2.3}$$

$$g(u_2) = \begin{cases} \frac{1}{2h} & -h \leq u_2 \leq h \\ 0 & |u_2| > h \end{cases} \tag{2.4}$$

Then the joint p.d.f. of u_1 and u_2 is defined by

$$h(u_1,u_2) = f(u_1)g(u_2)$$

$$= \begin{cases} \frac{1}{4hk} & |u_1| \leq k \text{ and } |u_2| \leq h \\ 0 & |u_1| > k \text{ or } |u_2| > h \end{cases} \tag{2.5}$$

Define a new variable $u = u_1 + u_2$. Now, u has p.d.f.

$$\phi(u) = \int_{-\infty}^{\infty} h(u-u_2,u_2)du_2$$

$$= \int_{-\infty}^{\infty} f(u-u_2)g(u_2)du_2 \tag{2.6}$$

When $|u_2| > h$ or $|u-u_2| > k$, the value of (2.6) is 0 so it can be rewritten as

$$\phi(u) = \begin{cases} \int_{-h}^{h} f(u-u_2)g(u_2)du_2 & |u-u_2| \leq k \\ 0 & \text{otherwise} \end{cases} \tag{2.6a}$$

The condition involving u can be rewritten as $-k+u \leq u_2 \leq k+u$, and so there are three separate cases for evaluating the integral. Note also that the two conditions require that $|u| \leq k+h$ for (2.6) to be non-zero.

(1) When $k+u \leq h$ and $u-k \leq -h$

$$\phi(u) = \int_{-h}^{u+k} f(u-u_2)g(u_2)du_2$$

$$= \int_{-h}^{u+k} \frac{1}{4kh}\, du_2$$

$$= \frac{u+k+h}{4kh} \qquad -h-k \leq u \leq k-h \qquad (2.6b)$$

(2) When $k+u \geq h$ and $-k+u \leq -h$ or $-k+h \leq u \leq k-h$

$$\phi(u) = \int_{-h}^{h} f(u-u_2)g(u_2)du_2$$

$$= \int_{-h}^{h} \frac{1}{4kh}\, du_2$$

$$= \frac{1}{2k} \qquad -k+h \leq u \leq k-h \qquad (2.6c)$$

(3) When $k+u \geq h$ and $u-k \geq -h$

$$\phi(u) = \int_{u-k}^{h} f(u-u_2)g(u_2)du_2$$

$$= \int_{u-k}^{h} \frac{1}{4kh}\, du_2$$

$$= \frac{h+k-u}{4kh} \qquad k-h \leq u \leq k+h \qquad (2.6d)$$

A plot of $\phi(u)$ is shown in Figure 1.

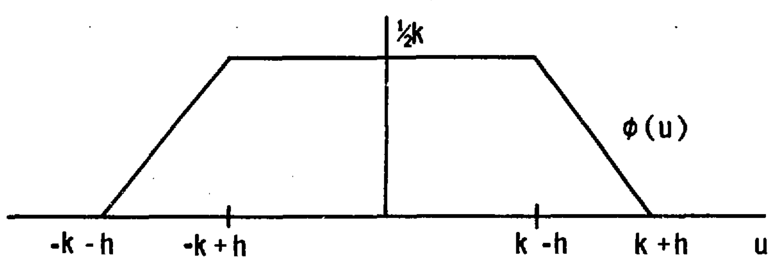

Figure 1

Probability Density Function of u

Thus the p.d.f. of the variable $u=u_1+u_2$ where u_1 is uniformly distributed on (-k,k) and u_2 is uniformly distributed on (-h,h) is a trapezoid.

2.2 Sampling from an Isosceles Triangle

In Figure 1, if the points $-k+h$ and $k-h$ are set to zero, the variable $u = u_1+u_2$ for $u_1 \in (-k,k)$, $u_2 \in (-h,h)$ will be a sample from an isosceles triangle. This gives

$$-k+h = k-h = 0$$
$$h = k \tag{2.7}$$

The triangle then looks like:

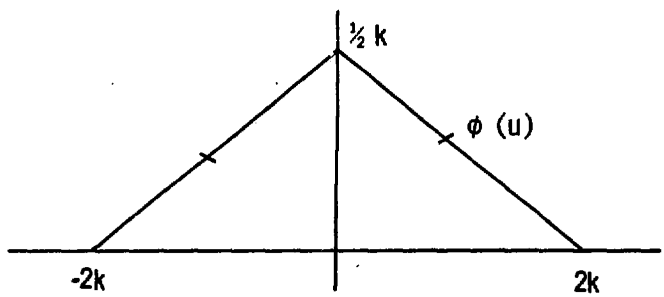

Figure 2

Sampling From an Isosceles Triangle

So the variable $u = u_1+u_2$ $u_1 \in (-k,k)$ and $u_2 \in (-k,k)$ has the p.d.f. $\phi(u)$ shown in Figure 2.

2.3 The Acceptance Rejection Technique

A random variable X is to be generated for a p.d.f. $f(x)$ for $-\infty \le x \le \infty$. A function $g(x)$ is selected so that $f(x) \le g(x)$ for all values of x and $\int_{-\infty}^{\infty} g(x)dx = 1/K$, $0 < K \le 1$. The probability density function $h(x)$ is defined by $h(x) = Kg(x)$. It can be shown [5] that if X is sampled from $h(x)$ and a random number r uniformly distributed on the interval (0,1) is selected, then if $r \le (f(X)/g(X))$, X is from the density function $f(x)$. The probability that $r \le (f(x)/g(x))$ is K, also called the sampling efficiency. Therefore, the probability function of X will have a negative binomial distribution, since a number of trials may have to be performed before a value of X is accepted. The expected number of pairs of random numbers which must be generated before an X value is accepted is 1/K.

The most basic acceptance rejection (AR) algorithm involves the function $f(x)$ defined on the interval $[0,a]$. The function is enclosed in the smallest possible rectangle as shown in Figure 3.

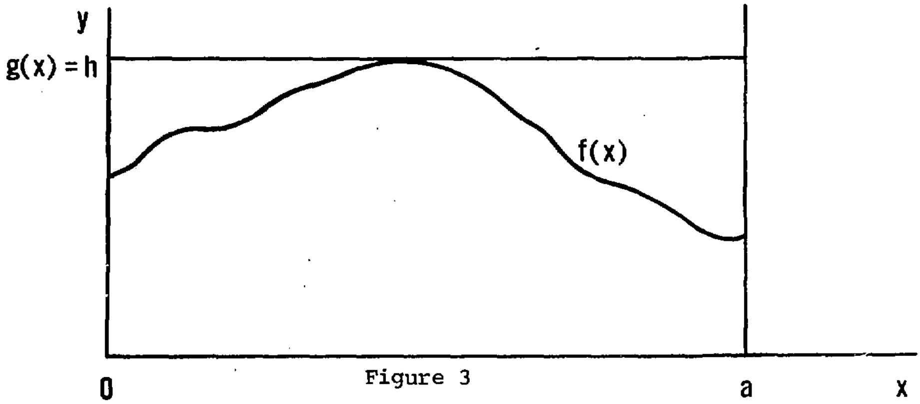

Figure 3

Basic Acceptance Rejection Technique

The height of this box is h and so $g(x) = h$. Two uniform random numbers are chosen, r_1 on the interval $(0,a)$ and r_2 on the interval $(0,h)$. If r_2 lies below $f(r_1)$ in Figure 3, then r_1 is accepted as a sample from $f(x)$. Otherwise the process is repeated. Note that, on the average, 2ah uniform random numbers must be generated, ah function evaluations made, and ah comparisons done to accept one random value.

2.4 Improvements to the Acceptance Rejection Technique

The reflected acceptance rejection technique extends the ordinary AR technique by making two tests on each pair of random numbers before rejecting it. Again $f(x)$, the probability density function to be sampled from, is defined on $[0,a]$. This time the function $f(x) + f(a-x)$ is enclosed in the smallest possible rectangle. Again the height of the rectangle is $g(x) = h$ (see Figure 4). The method of the reflected AR technique is as follows. Two random numbers are chosen $r_1 \in (0,a)$ and $r_2 \in (0,h)$. The value r_1 is accepted as a sample from $f(x)$ if $r_2 \leq f(r_1)$. If $r_2 > f(r_1)$ then it can be shown [5] that if $f(r_1) \leq r_2 \leq f(r_1) + f(a-r_1)$, then $a-r_1$ can be accepted as a sample from the distribution $f(x)$. If $r_2 > f(r_1) + f(a-r_1)$, then the process is repeated.

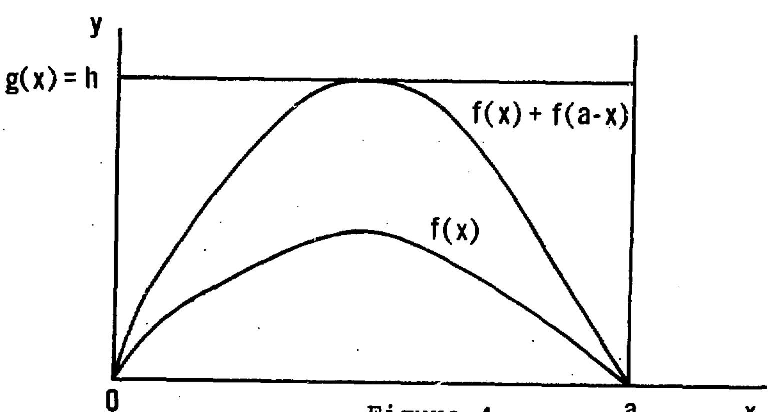

Figure 4

Reflected Acceptance Rejection Technique

The probability that $r_2 \leq f(r_1) + f(a-r_1)$ is 2/ah. Again, the probability function of $\bar{x}$ has a negative binominal distribution. The expected number of pairs of random numbers which must be generated before one is accepted is ah/2. The expected number of random numbers to be generated is, then, ah. On the average, ah-1/2 function evaluations and ah-1/2 comparisons will have to be made.

Depending on the probability density function $f(x)$, the reflected AR technique can be very efficient. If $f(x)$ is approximately antisymmetric about a/2, then a very large portion of the rectangle will be included in the region $r_2 \leq f(r_1) + f(a-r_1)$.

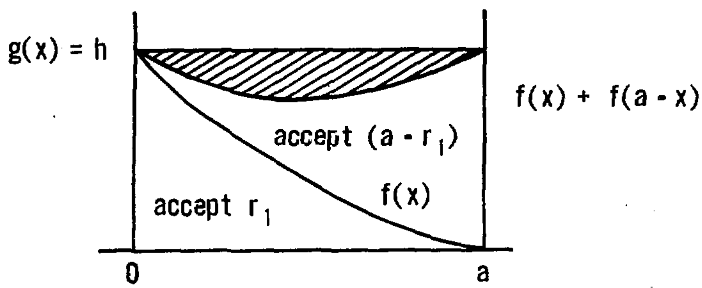

Figure 5

Region of Rejection for Reflected Acceptance Rejection Technique

Only pairs of random numbers in the shaded region of Figure 5 would be rejected, whereas in the basic AR method pairs of random numbers in over 1/2 of the area of the rectangle would be rejected. The efficiency of the reflected AR method is

$$\frac{\int_0^a (f(x) + f(a-x))dx}{\int_0^a g(x)dx} = \frac{2}{ha}$$

In the case shown in Figure 5, this is twice the efficiency of the basic AR method.

The function shown in Figure 4 is not a good candidate for the reflected AR method. However, it is a good candidate for the translated AR technique. When a function $f(x)$ is (nearly) symmetric about a/2, then the function $f(x) + f(a-x)$ in the second step of the reflected AR method is replaced by $f(x) + f(a/2+x)$ on $0 \leq x \leq a/2$ and $f(x) + f(a/2-x)$ on $a/2 \leq x \leq a$. The rest of the algorithm remains exactly the same. The shaded area in Figure 6 contains the pairs of random numbers which would be rejected by this method.

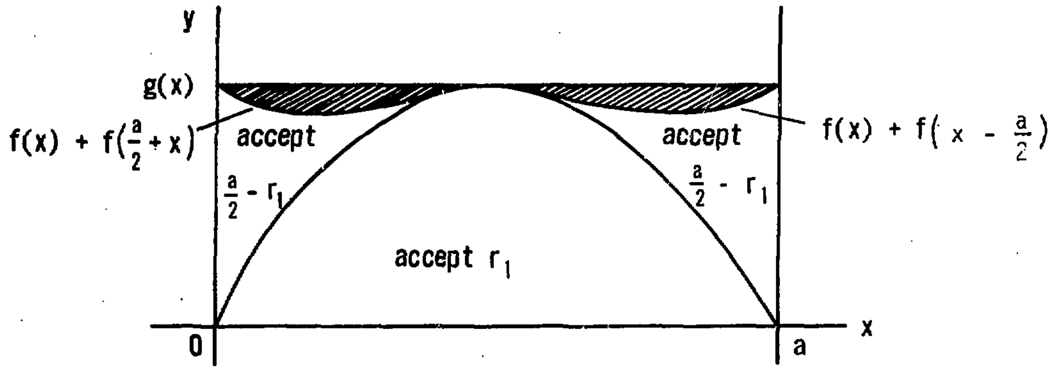

Figure 6

Translated Acceptance Rejection Technique

An improvement to the reflected AR method may be made by observing that if the minimum value of $f(x) + f(a-x) = \beta$ and if $r_2 \le \beta$, then either r_1 or $a-r_1$ will always be accepted. If $r_2 \le f(x)$ then r_1 is accepted and if $f(x) < r_2 \le \beta$ then $(a-r_1)$ is accepted so only one function evaluation is required.

A final improvement may be made in the reflected AR method because $f(x) + f(a-x)$ is symmetric about $x = a/2$ and so only the region $[0,a/2]$ or $[a/2,a]$ for r_1 needs to be considered. By choosing the region in which the area under $f(x)$ is the largest, the number of function values and decisions is decreased. In this case the region of rejection for the function shown in Figure 5 is shown in Figure 7.

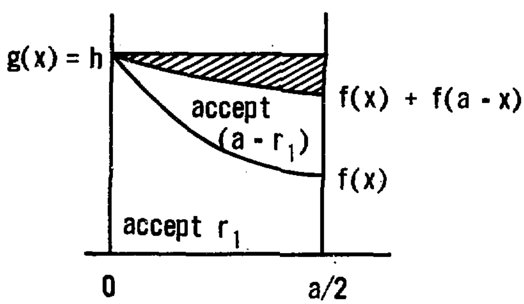

Figure 7

Improved Reflected Acceptance Rejection Technique

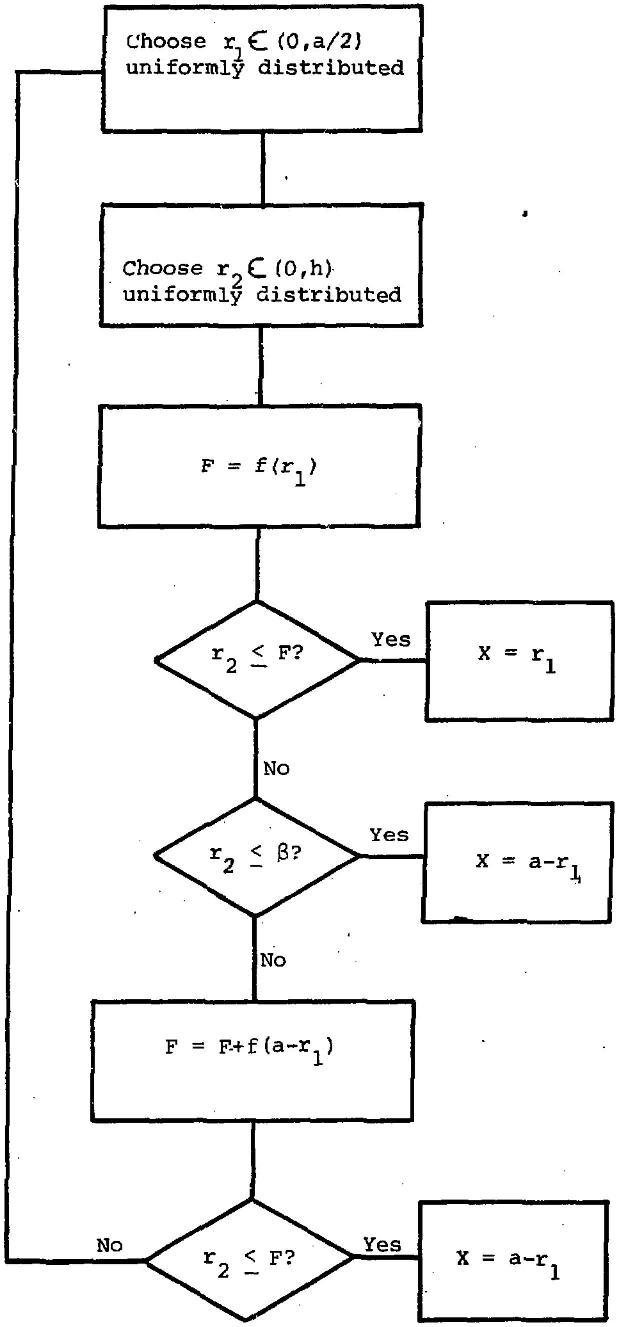

Figure 8
Flowchart for Improved Reflected Acceptance Rejection Technique

A flowchart for the reflected AR technique is shown in Figure 8. A random variable X is to be generated with p.d.f. $f(x)$ $0 \leq x \leq a$. It is also known that $h \geq f(x) + f(a-x)$ for $0 \leq x \leq a$ and that the minimum value for $f(x) + f(a-x)$ is β for $0 \leq x \leq a$.

3. THE NORMAL DISTRIBUTION

A random variable X is said to be normally distributed with mean μ and variance σ^2 if the probability that $X \leq x$ is

$$\Pr(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2}\, dt \qquad (3.1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-z^2/2}\, dz \qquad (3.2)$$

The standard normal defined by (3.2) is sufficient for sampling purposes. If s is a sample from (3.2), then $s\sigma+\mu$ is a sample from (3.1). The p.d.f. for the standard normal is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad (3.3)$$

Sampling from the standard normal distribution is done in a method similar to Algorithm TR described in [1]. First a trapezoid of maximum area is inscribed in the graph of the standard normal (see Figure 9). Its vertices are

$$\begin{array}{ll} (\pm\xi, 0) & \xi = 2.11402\ 80833\ 3742 \\ (\pm X, Y) & X = 0.28972\ 95736 \\ & Y = 0.38254\ 45560\ 42518 \end{array} \qquad (3.4)$$

This divides the area under the standard normal into four regions as shown. Each of these regions is then treated as a separate p.d.f. to be sampled from. The probability that a region is sampled is in proportion to its area. The regions are discussed separately in the next sections and the final section contains a description of the subroutine implementing the sampling algorithm.

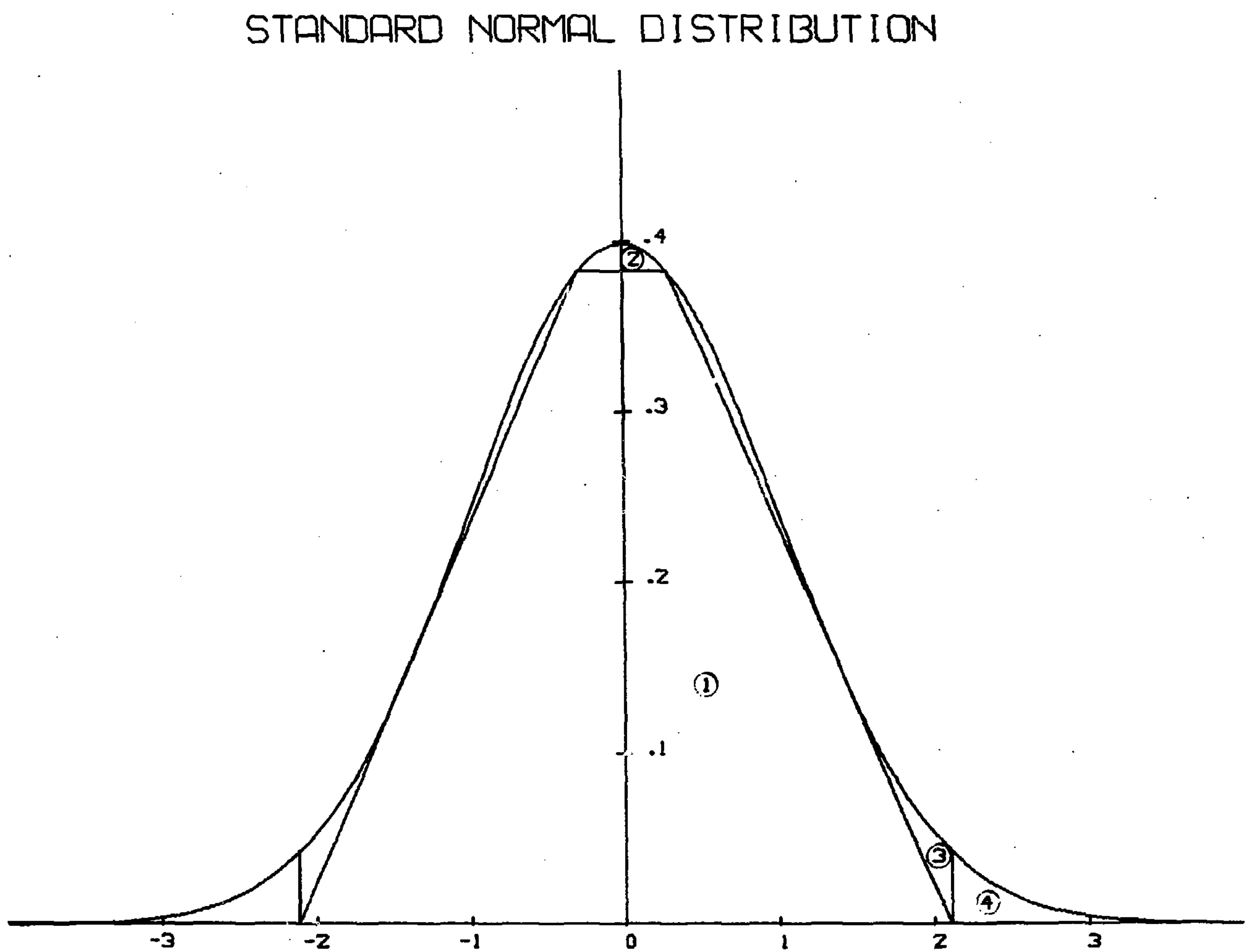

Figure 9

3.1 The Trapezoid

The trapezoid in Figure 9 is not defined so that the values of k and h shown in Figure 1 can be immediately identified.

The variable k may be defined by

$$Y = \frac{1}{2k}$$

$$k = \frac{1}{2Y} \qquad (3.5)$$

The x-axis must now be scaled to the x'-axis with x' = ax so that

$$\xi' = k + h \text{ or } a\xi = \frac{1}{2Y} + h$$

$$X' = k - h \text{ or } aX = \frac{1}{2Y} - h$$

Solving these 2 equations for a and h gives

$$a = \frac{1}{Y(X+\xi)} \qquad (3.6)$$

$$h = \frac{\xi - X}{2Y(X+\xi)} \qquad (3.7)$$

In the (x',y) plane the variable $u' = k(2u_1-1) + h(2u_2-1)$ where u_1 and u_2 are uniformly distributed on (0,1) is a sample from this trapezoid. Substituting for k and h from (3.5) and (3.7),

$$u' = \frac{1}{2Y}(2u_1-1) + \frac{\xi-X}{2Y(X+\xi)}(2u_2-1) \qquad (3.8)$$

In the (x,y) plane this becomes

$$u = u'/a$$

$$= \frac{(X+\xi)}{2}(2u_1-1) + \frac{\xi-X}{2}(2u_2-1)$$

$$= (X+\xi)u_1 + (\xi-X)u_2 - \xi \qquad (3.9)$$

Thus, the value u is a random sample from the trapezoid in Figure 9.

The area of the trapezoid is 0.91954 44057 06926 of the total area under the standard normal. When sampling from the standard normal, therefore, this region should be sampled about 91.95% of the time.

3.2 Region 2

The p.d.f. of the second region of the standard normal distribution is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - Y, \quad -X \leq x \leq X \tag{3.10}$$

where (-X,Y) and (X,Y) are the vertices of the trapezoid. Since this function is symmetric about x=0, it will only be considered on the region [0,X]. A sign will be randomly assigned once the magnitude of the sample value is determined.

It can be seen from Figure 9 that the shape of the p.d.f. for this region is neither symmetric nor anti-symmetric about $x = X/2$. So neither the reflected nor translated AR methods are inappropriate here. The basic AR method is used instead with $g(x) = 1/\sqrt{2\pi} - Y$. Two uniform random numbers u_1 and u_2 on the interval (0,1) are chosen. The value Xu_1 is a sample from f(x) as defined by (3.10) for $0 \leq x \leq X$ if

$$\left(\frac{1}{\sqrt{2\pi}} - Y\right) u_2 \leq \frac{1}{\sqrt{2\pi}} e^{-X^2 u_1^2/2} - Y \tag{3.11}$$

Otherwise, the process must be repeated.

The area of region 2 is 0.00630 79280 00778 of the total area under the standard normal so this region of the standard normal will be sampled about .63% of the time. The area under g(x) $-X \leq x \leq X$ is 0.00950 18113 98. The sampling efficiency in this region, then, is 0.66.

3.3 Region 3

The third region lies between the standard normal and each side of the trapezoid. It consists of two symmetrical regions and only the one for x>0 will be treated. Once a sample value is obtained, the sign will be assigned randomly. The probability density function for this region is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{Y(x-\xi)}{X-\xi}, \quad X \leq x \leq \xi \tag{3.12}$$

and the function is plotted in Figure 10.

Since the region shown in Figure 10 is neither symmetrical nor antisymmetric it is inappropriate for either the reflected or translated AR methods. Ahrens and Dieter [1] divide this region into two smaller regions and use an AR method on each one. Since it was shown in Section 2.4 that the reflected AR method can be much more efficient, it was decided to use it here.

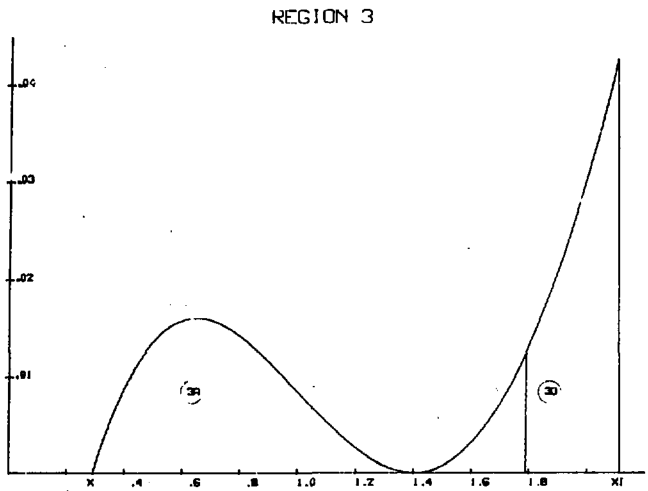

Figure 10

The speed of the reflected AR method depends on two things. Minimizing the largest value of f(x) + f(a-x) over the range of x gives the greatest efficiency. Maximizing the smallest value of f(x) + f(a-x) means fewer function evaluations and so a faster computer subroutine. The break point of x=1.79 was picked so as to give the best possible extreme values in the two subregions.

3.3.1 Subregion 3A

For the subregion 3A, the p.d.f. is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{Y(x-\xi)}{X-\xi}, \quad X \le x \le 1.79 \qquad (3.13)$$

This function is plotted in Figure 11. Since (3.13) is approximately antisymmetric about the midpoint x ≈ 1.04, the reflected AR method is appropriate. The areas under f(x) on the left and right sides of the midpoint are about equal. In this case nothing is gained by using only half the range of x and so the full range will be used. The reflected AR method requires the function

$$f(x) + f(1.79+X-x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{Y(x-\xi)}{X-\xi}$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-(1.79-x+X)^2/2} - \frac{Y(1.79-x+X-\xi)}{X-\xi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(1.79-x+X)^2/2}$$

$$- \frac{Y(1.79+X-2\xi)}{(X-\xi)} \qquad (3.14)$$

The extreme values of (3.14) are needed by the reflected AR algorithm. Differentiating,

$$f'(x) + f'(1.79-x+X) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2}$$

$$+ \frac{(1.79-x+X)}{\sqrt{2\pi}} e^{-(1.79-x+X)^2/2} \qquad (3.15)$$

One extreme value of (3.14) is found at $x^* = (1.79 + X)/2$.

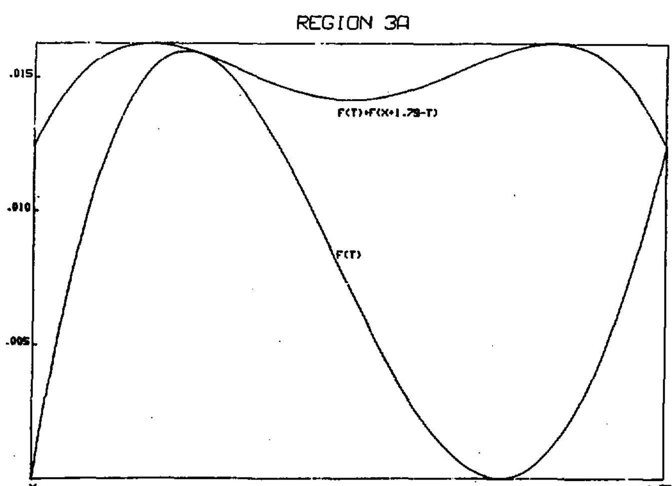

Figure 11

Substituting this into equation (3.14) gives

$$f(x^*) + f(1.79-x^*+X) = 2f(\frac{1.79+X}{2})$$

$$\simeq 0.014173 \qquad (3.16)$$

Other extreme values (found numerically) lie near $x^* = 0.56$ and $x^* = 1.52$ where the value of (3.14) is about 0.016271.

At the end points of the range $x = X$ and $x = 1.79$, the value of (3.14) is (since $f(X) = 0$)

$$f(X) + f(1.79) = f(1.79)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-1.79^2/2} - \frac{Y(1.79-\xi)}{(X-\xi)}$$

$$\simeq 0.012433 \qquad (3.17)$$

The minimum value of (3.14) occurs at $x = X$ and $x = 1.79$ and the maximum value of (3.14) occurs at $x \simeq 0.56$ and $x \simeq 1.52$.

The algorithm for the reflected AR method outlined in Section 2.4 may now be applied using equations (3.13) and (3.14). Also needed are

$$r_1 \in (X,1.79) \qquad (3.18)$$

$$r_2 \in (0,0.01627\ 0801) \qquad (3.19)$$

$$\beta_{3A} = 0.01243\ 33456\ 1586 \qquad (3.20)$$

The sampling efficiency of this algorithm is about 0.93. A sample will be returned after only one function evaluation 76% of the time. The area of region 3A is 0.02271 04144 21382 and so this region will be sampled about 2.27% of the time.

3.3.2 Subregion 3B

For the subregion 3B, the p.d.f. is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{Y(x-\xi)}{X-\xi}, \quad 1.79 \le x \le \xi \qquad (3.21)$$

This function is plotted in Figure 12. Again (3.21) is approximately antisymmetric about the midpoint $x \simeq 1.95$ and so the reflected AR method is appropriate. The function $f(x)$ covers the largest area for $(1.79+\xi)/2 \le x \le \xi$ so only this half of the region will be treated. The reflected AR method requires the function

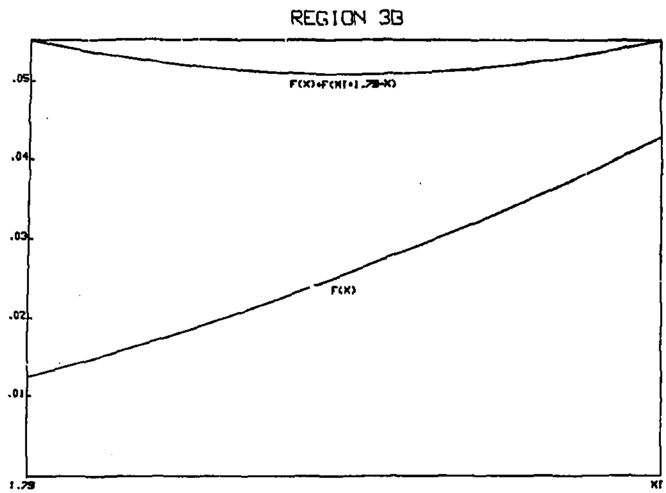

Figure 12

$$f(x) + f(\xi+1.79-x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{Y(x-\xi)}{X-\xi} + \frac{1}{\sqrt{2\pi}} e^{-(\xi+1.79-x)^2/2}$$

$$- \frac{Y(\xi+1.79-x-\xi)}{X-\xi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(\xi+1.79-x)^2/2}$$

$$- \frac{Y(1.79-\xi)}{X-\xi} \qquad (3.22)$$

The extreme values of (3.22) must be found. Differentiating

$$f'(x) + f'(\xi+1.79-x) = - \frac{x\, e^{-x^2/2}}{\sqrt{2\pi}}$$

$$+ \frac{(\xi+1.79-x)}{\sqrt{2\pi}} e^{-(\xi+1.79-x)^2/2} \qquad (3.23)$$

Then one extreme value of (3.22) is found at $x^* = \frac{\xi+1.79}{2}$.

Substituting into (3.22) gives

$$f(x^*) + f(\xi+1.79-x^*) = 2f(\frac{\xi+1.79}{2})$$

$$\simeq 0.050775 \qquad (3.24)$$

At x=1.79 and x=ξ the value of (3.22) is

$$f(1.79) + f(\xi) = \frac{1}{\sqrt{2\pi}} (e^{-1.79^2/2} + e^{-\xi^2/2}) - \frac{Y(1.79-\xi)}{X\xi}$$

$$\simeq 0.055136 \qquad (3.25)$$

The minimum value of (3.22) occurs at $x = (\xi+1.79)/2$ and the maximum value at x=1.79 or x=ξ.

The algorithm for the reflected AR method outlined in Section 2.4 may now be applied using equations (3.21) and (3.22). Also needed are

$$r_1 \in (\frac{1.79+\xi}{2}, \xi) \qquad (3.26)$$

$$r_2 \in (0, 0.05513\ 59272\ 0665) \qquad (3.27)$$

$$\beta_{3B} = 0.05077\ 52236\ 5025 \qquad (3.28)$$

The sampling efficiency of this algorithm is about 0.95. A sample will be returned after only one function evaluation 92% of the time. The area of region 3B is 0.01692 43830 84770 and so this region will be sampled about 1.7% of the time.

3.4 The Tail

The fourth region is the tail of the standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, |x| > \xi \qquad (3.29)$$

Again only the region for x>ξ will be treated and the sign randomly assigned. Marsaglia's Tail Method [3] is used here. The function (3.29) is transformed from the (x,y) plane to the unit square (u_1,u_2). The definition of the transformation is

$$y' = \xi/x \qquad (3.30)$$

$$x' = \frac{\sqrt{2\pi}}{e^{-\xi^2/2}} y \qquad (3.31)$$

Then the definition of the function (3.29) on the unit square is

$$\frac{e^{-\xi^2/2}}{\sqrt{2\pi}} x' = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2y'^2}$$

$$\frac{-\xi^2}{2} + \ell n\ x' = \frac{-\xi^2}{2y'^2}$$

$$y' = \frac{\xi}{\sqrt{\xi^2 - 2\ell n x'}} \qquad (3.32)$$

A plot of this function is shown in Figure 13.

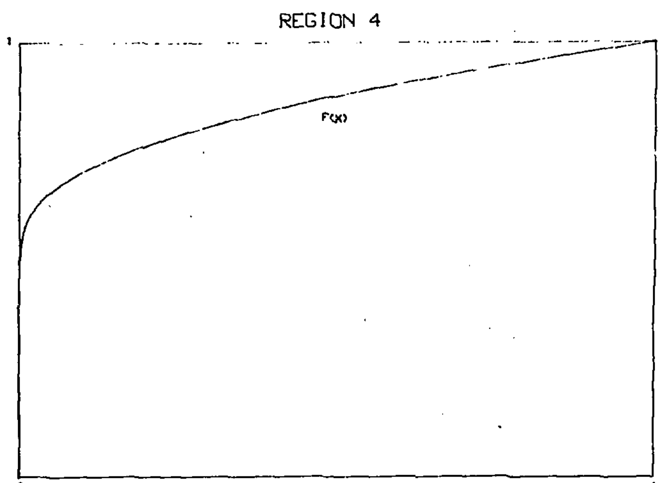

Figure 13

The basic AR method may now be used with (3.32) in the (x',y') plane. Generate two uniform random variables u_1 and u_2 on the range (0,1) if

$$u_2 \leq \frac{\xi}{\sqrt{\xi^2 - 2\ell n u_1}}$$

then accept u_1 as a sample from (3.32). Otherwise, repeat the process. In the original coordinates, it is the value of x corresponding to (u_1,u_2) that is needed. Using (3.30),

$$x = \xi/y'$$
$$= \sqrt{\xi^2-2\ell n u_1} \qquad (3.33)$$

The area of the tail of the standard normal distribution is 0.03451 28687 86142 of the entire area of the standard normal, so it will be sampled about 3.45% of the time. The sampling efficiency of this method is about 0.85.

3.5 The Subroutine RNORM

The methods outlined in the previous sections are incorporated in the subroutine RNORM (see Appendix 1). This subroutine returns a sequence of N normally distributed random numbers.

The uniformly distributed random number $U \in (0,1)$ has two uses. First, it is used to determine which of the regions to sample from. This depends on the area of each region. It is used again for sampling from the trapezoid. It is now known to be uniformly distributed between [0,A] where A is the area of the trapezoid and it replaces u_2 in equation (3.9) thus changing the constants in this equation.

The uniformly distributed random number $U0 \in (0,1)$ also has two uses. It replaces u_1 in equation (3.9) for the sample from the trapezoid. It is also used in each of the other regions to determine the sign of the random variable to be returned.

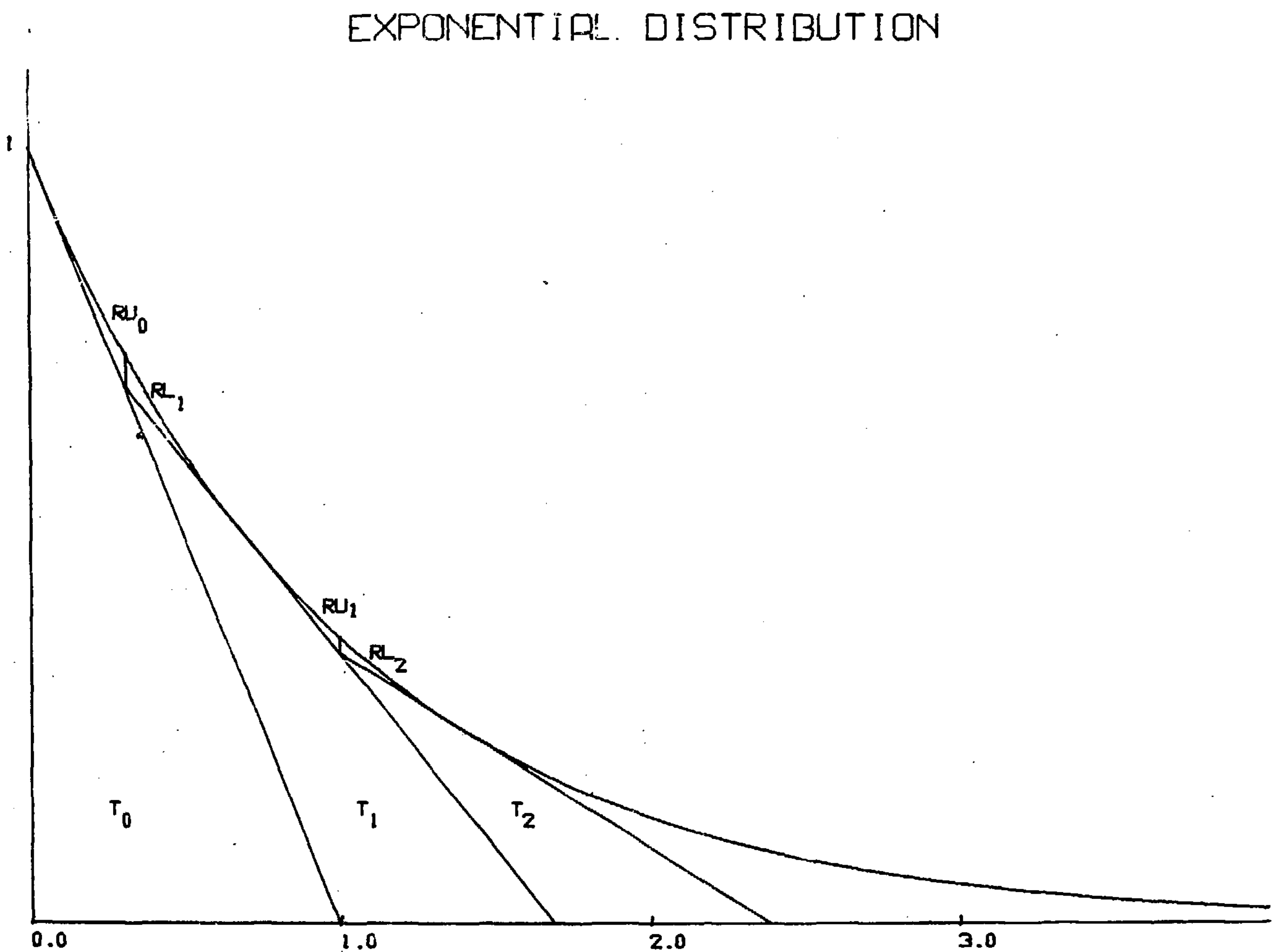
EXPONENTIAL DISTRIBUTION
1
RU0
RL1
RU1
RL2
T0
T1
T2
0.0
1.0
2.0
3.0

Figure 14

4. THE EXPONENTIAL DISTRIBUTION

A random variable X is said to be exponentially distributed with parameter λ if the probability that $X \leq x$ is

$$\Pr(X\leq x) = \begin{cases} 0 & x\leq 0 \\ \lambda \int_0^x e^{-\lambda t}\, dt & x>0 \end{cases} \qquad (4.1)$$

$$= \begin{cases} 0 & x\leq 0 \\ \int_0^{\lambda x} e^{-z}\, dz & x>0,\ z=\lambda t \end{cases} \qquad (4.2)$$

It is sufficient to sample from (4.2) since if s is a sample from (4.2), s/λ is a sample from (4.1). The p.d.f. for the exponential distribution is

$$f(x) = e^{-x} \qquad (4.3)$$

Sampling from the exponential distribution may be done by dividing the area under the function into an infinite number of triangles with special properties. Each triangle is then sampled in proportion to its area. Some of the triangles and other areas are shown in Figure 14.

(1) T_0 is a right-angled triangle with vertices (0,0), (0,1) and (1,0).

(2) T_i, i=1,2,... is an infinite series of triangles.

(3) RL_i, i=1,2,... is an infinite series of regions between the triangles and f(x).

(4) RU_i, i=0,2,... is another infinite series of regions between the triangles and f(x).

The properties of each region and method of sampling from it are discussed below.

4.1 The Triangle T_0

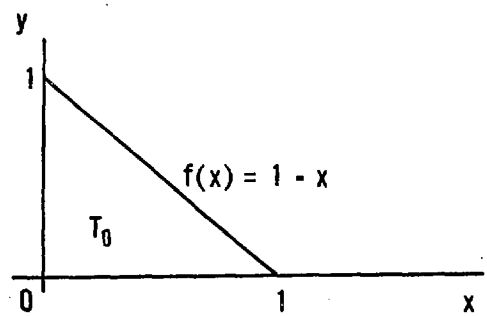

Figure 15
Triangle T_0

Sampling from the triangle T_0 is easy using the reflected AR method. The p.d.f. to be sampled from is

$$f(x) = 1-x \qquad 0 \leq x \leq 1 \qquad (4.4)$$

This is a very special case of the method since

$$f(x) + f(1-x) = 1$$

So, if $g(x) = 1$, then the efficiency of the method is 1. Two uniformly distributed random numbers are generated, u_1 on (0,1), and u_2 on (0,0.5). The method states that if $u_1 \leq 1 - u_2$, then u_2 is to be accepted. This step can be simplified by rewriting the condition as $u_2 \leq 1-u_1$. Since u_1 is uniformly distributed on (0,1), then $1-u_1$ is also uniformly distributed on (0,1). So the condition may be replaced by the equivalent condition if $u_2 \leq u_1$, accept u_2. If u_2 is not accepted, then $1-u_2$ will always be accepted.

The area of triangle T_0 is 0.5. This region will, therefore, be sampled 50% of the time.

4.2 The Triangles T_i, i=1,2,...

The triangles T_i, i=1,2,... are created by satisfying the following rules:

(1) The area of each T_i must be a maximum.

(2) The projection of the long side of each T_i onto the x axis is equal to twice the length of the side which lies along the x axis. In Figure 16, L_i is the long side of T_i and $(x_{H_i} - x_{L_i}) = 2(x_{H_i} - x_{H_{i-1}})$.

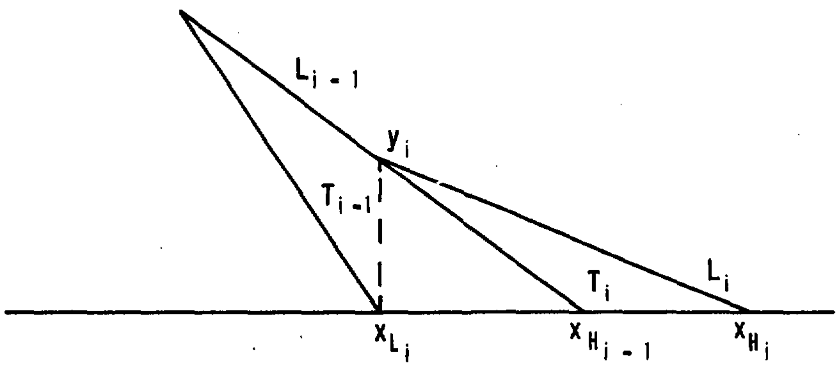

Figure 16
Sample Triangle T_i

Triangle T_0 is a portion of one of the triangles in this series which is distorted because of the definition of the exponential function.

4.2.1 Slope of L_i

From Figure 16, the slope of L_i is

$$S_i = \frac{-y_i}{x_{H_i} - x_{L_i}} \tag{4.5}$$

and the slope of L_{i-1} is

$$S_{i-1} = \frac{-y_i}{x_{H_{i-1}} - x_{L_i}} \tag{4.6}$$

Dividing equation (4.5) by (4.6) gives

$$\frac{S_i}{S_{i-1}} = \frac{x_{H_{i-1}} - x_{L_i}}{x_{H_i} - x_{L_i}} \tag{4.7}$$

However, from rule 2 $(x_{H_i} - x_{L_i}) = 2(x_{H_i} - x_{H_{i-1}})$ so (4.7) becomes

$$S_i = \frac{S_{i-1}}{2}$$

$$= \frac{S_0}{2^i}$$

$$= \frac{-1}{2^i}\ ,\ i=0,1,\ldots \qquad (4.8)$$

4.2.2 Equation of L_i

In order to satisfy rule 1, L_i must be tangent to f(x) at some point (x_{T_i}, y_{T_i}). From (4.3) the derivative of f(x) is

$$f'(x) = -e^{-x} \qquad (4.9)$$

Substituting in (4.9) to find the point of tangency,

$$f'(x_{T_i}) = -e^{-x_{T_i}} = -\frac{1}{2^i}$$

$$x_{T_i} = \ln(2^i)$$

$$= i \ln 2 \qquad (4.10)$$

$$y_{T_i} = e^{-x_{T_i}}$$

$$= e^{-i \ln 2}$$

$$= 2^{-i} \qquad (4.11)$$

Finding the equation of L_i,

$$\frac{y-y_{T_i}}{x-x_{T_i}} = S_i \qquad (4.12)$$

Substituting from (4.10), (4.11) and (4.8),

$$\frac{y-2^{-i}}{x-i \ln 2} = \frac{-1}{2^i}$$

$$y = 2^{-i}(i \ln 2 + 1 - x) \qquad (4.13)$$

4.2.3 The Area of T_i

Referring to Figure 16, we wish to calculate the value of the x_{H_i}, the x intercept of L_i. Substitution in (4.13) gives

$$0 = 2^{-i}(i \ln 2 + 1 - x_{H_i})$$

$$x_{H_i} = i \ln 2 + 1 \qquad (4.14)$$

Thus, the length of the base of T_i is

$$b = x_{H_i} - x_{H_{i-1}}$$

$$= (i \ln 2 + 1) - [(i-1) \ln 2 + 1]$$

$$= \ln 2 \qquad (4.15)$$

Since the projection of L_i on the x-axis is twice the length of the base of T_i,

$$x_{L_i} = (i-2) \ln 2 + 1 \qquad (4.16)$$

The height of T_i is then found by substituting (4.16) into (4.13).

$$h = 2^{-i}(i \ln 2 + 1 - x_{L_i})$$

$$= 2^{-i}(i \ln 2 + 1 - (i-2) \ln 2 - 1)$$

$$= 2^{-i}(2 \ln 2) \qquad (4.17)$$

From equations (4.15) and (4.17), the area of T_i is

$$A(T_i) = \frac{1}{2} bh$$

$$= \frac{1}{2}(\ln 2)(2^{-i})(2 \ln 2)$$

$$= 2^{-i}(\ln 2)^2 \qquad (4.18)$$

4.2.4 Sampling from T_i

Sampling from T_i with vertices

$y_i x_{H_{i-1}} x_{H_i}$

is the same as sampling from the isosceles triangle T_i' with vertices

$y_i' x_{L_i} x_{H_i}$ (see Figure 17)

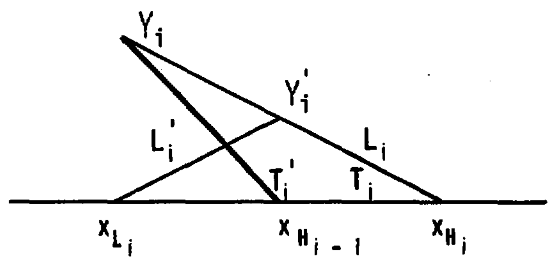

Figure 17
Similar Triangles for Sampling from T_i

To show this we must prove that the portion of the area of T_i between x_{L_i} and x where

$x \le x_{H_{i-1}}$

is equal to the portion of the area in T_i' for the same value of x. It is only necessary to do this for

$x_{L_i} \le x \le x_{H_{i-1}}$

since the triangles are identical for

$x_{H_{i-1}} \le x \le x_{H_{i-1}}$

For T_i, the area lies between the lines L_i and L_{i-1} so, using (4.13),

$$A(x) = \int_{x_{L_i}}^{x} 2^{-i}(i \ln 2 + 1 - t) - 2^{-i+1}((i-1)\ln 2 + 1 - t)dt$$

$$= 2^{-i} \int_{x_{L_i}}^{x} (i \ln 2 + 1 - t - 2(i-1) \ln 2 - 2 + 2t)dt$$

$$= 2^{-i} \int_{x_{L_i}}^{x} ((2-i)\ln 2 - 1 + t)dt \qquad (4.19)$$

The line L_i' has slope 2^{-i} and passes through the point $(x_{L_i}, 0)$ so its equation is

$$y = \frac{1}{2^i}(x - x_{L_i})$$

Substituting from (4.16) gives

$$y = 2^{-i}(x + (2-i)\ln 2 - 1) \qquad (4.20)$$

The area for triangle T_i' is

$$A(x) = \int_{x_{L_i}}^{x} 2^{-i}(t + (2-i) \ln 2 - 1)dt \qquad (4.21)$$

Since the areas (4.19) and (4.21) are the same, it follows that sampling from T_i' is the same as sampling from T_i.

The height of T_i' is one half the height of triangle T_i. From (4.17) this is

$$y_i = 2^{-i} \ln 2 \qquad (4.22)$$

The base of T_i' has length twice that of T_i or $2\ln 2$. In Section 2.2 it was shown how to sample from an isosceles triangle with base and height in a special proportion. The triangle T_i' can be transformed to the (x', y) plane where its base runs from $-2^i/\ln 2$ to $2^i/\ln 2$ by

$$x' = \frac{2^i x + 2^i(1-i) \ln 2 - 2^i}{(\ln 2)^2} \qquad (4.23)$$

In the (x', y) plane

$$u' = \frac{2^i}{\ln 2}(u_1 + u_2 - 1)$$

is a sample from this isosceles triangle where $u_1 \in (0,1)$, $u_2 \in (0,1)$. Transforming back to the (x,y) plane,

$$u = \frac{u'(\ell n\ 2)^2 - 2^i(1-i)\ \ell n\ 2 + 2^i}{2^i}$$

$$= \ell n\ 2\ (u_1 + u_2 - 1) - (1-i)\ \ell n\ 2 + 1$$

$$= \ell n\ 2\ (u_1 + u_2) + (i-2)\ \ell n\ 2 + 1 \qquad (4.24)$$

The value u has p.d.f. T_i.

4.2.5 Area of Sum of All Triangles T_i

The area of the first n triangles is found by summing equation (4.18) over i:

$$TA(n) = \sum_{i=1}^{n} 2^{-i}(\ell n\ 2)^2$$

$$= (\ell n\ 2)^2 \frac{(1-2^{-n})}{2(1-1/2)}$$

$$= (\ell n\ 2)^2 - 2^{-n}(\ell n\ 2)^2 \qquad (4.25)$$

since this is a geometric progression.

There are an infinite number of triangles T_i so the total area is

$$TA(T_i) = \lim_{n\to\infty} TA(n)$$

$$= \lim_{n\to\infty} (\ell n\ 2)^2 - 2^{-n}(\ell n\ 2)^2$$

$$= (\ell n\ 2)^2$$

$$\simeq 0.48045 \qquad (4.26)$$

Sampling from the exponential distribution will then be done about 48.05% of the time from one of these triangles.

4.3 The Regions RL_i, i=1,2,...

The regions RL_i are defined by

$$RL_i(x) = e^{-x} - L_i, \quad x_{L_i} \leq x \leq x_{T_i}, \quad i=1,2,... \tag{4.27}$$

Substituting (4.13) for L_i, (4.16) for x_{L_i}, and (4.10) for x_{T_i}, gives

$$RL_i(x) = e^{-x} - 2^{-i}(i \ell n\, 2 + 1 - x),$$

$$(i-2)\, \ell n\, 2 + 1 \leq x \leq i\, \ell n\, 2 \tag{4.28}$$

These regions are to be sampled using an AR technique. The first task is, therefore, to transform each region so that the range of the independent variable starts at 0. Let

$$t = x - (i-2)\ell n\, 2 - 1 \tag{4.29}$$

then (4.28) becomes

$$RL_i(t) = \exp(-t-(i-2)\ell n2-1) - 2^{-i}(i\ell n2+1-t-(i-2)\ell n2-1)$$

$$= e^{-t-1}\, 2^{2-i} - 2^{-i}(2\ell n2-t)$$

$$= 2^{-i}(4e^{-t-1} - 2\ell n2+t), \quad 0 \leq t \leq 2\ell n2-1 \tag{4.30}$$

4.3.1 The Area of Region RL_i

The area of a region RL_i is found by integrating (4.30) over the entire range of t.

$$A(RL_i) = \int_0^{2\ell n2-1} 2^{-i}(4e^{-t-1} - 2\ell n2+t)dt$$

$$= 2^{-i}(-4e^{-t-1} - 2t\ell n2 + \frac{t^2}{2})\Big|_0^{2\ell n2-1}$$

$$= 2^{-i}[-4(e^{-2\ell n2} - e^{-1}) - 2(2\ell n2-1)\ell n2 + \frac{(2\ell n2-1)^2}{2}]$$

$$= 2^{-i}(-1+4/e-4(\ell n2)^2 + 2\ell n2 + 2(\ell n2)^2 - 2\ell n2 + .5)$$

$$= 2^{-i}(4/e - 2(\ell n2)^2 - .5) \tag{4.31}$$

4.3.2 Total Area of Regions RL_i

The sum of the areas of the first n regions is found by summing (4.31) over i.

$$TA(n) = \sum_{i=1}^{n} 2^{-i}(4/e-2(\ell n2)^2-.5)$$

$$= (4/e - 2(\ell n2)^2-.5)(-2^{-n}+1) \qquad (4.32)$$

Therefore, for all regions RL_i the total area is

$$TA(RL) = \lim_{n\to\infty} TA(n)$$

$$= \lim_{n\to\infty} (4/e - 2(\ell n2)^2-.5)(1-2^{-n})$$

$$= 4/e - 2(\ell n2)^2-.5$$

$$\simeq 0.01061 \qquad (4.33)$$

Sampling of the exponential distribution will be done from one of the regions RL_i about 1.06% of the time.

4.3.3 Sampling from a Region RL_i

We want to sample from the region $RL_i(t)$ using an acceptance rejection technique. To do this, equation (4.30) must first be normalized giving

$$RL(t) = \frac{RL_i(t)}{A(RL_i)}$$

$$= \frac{2^{-i}(4\ e^{-t-1} - 2\ell n2+t)}{2^{-i}(4/e - 2(\ell n2)^2-.5)}$$

$$= \frac{4\ e^{-t-1} - 2\ell n2+t}{4/e - 2(\ell n2)^2-.5}, \quad 0 \le t \le 2\ell n2 - 1 \qquad (4.34)$$

The region to be sampled from is thus the same for each RL_i. The regions are differentiated when the sample t is transformed back to the original coordinates by

$$x = t+(i-2)\ell n2+1 \qquad (4.35)$$

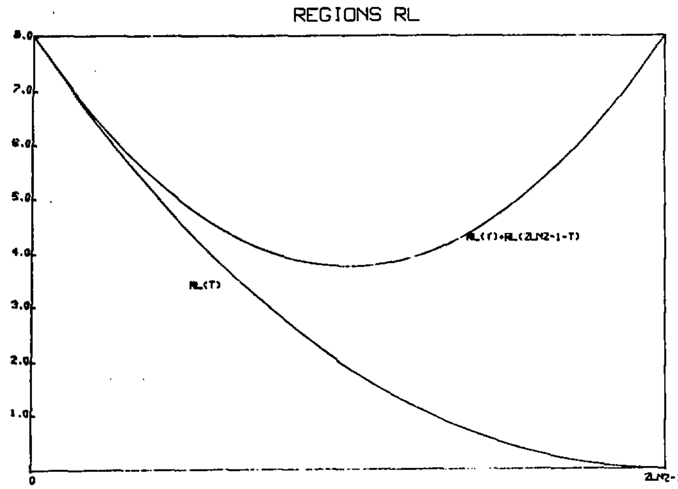

Figure 18

The function (4.34) is plotted in Figure 18 and since it is antisymmetric about its midpoint, the reflected acceptance rejection method is appropriate. For this technique we must also define

$$\overline{RL}(t) = RL(t) + RL(2\ell n2-1-t)$$

$$= \frac{4e^{-t-1} - 2\ell n2+t}{4/e - 2(\ell n2)^2-.5} + \frac{4e^{-2\ell n2+1+t-1} - 2\ell n2 + 2\ell n2-1-t}{4/e - 2(\ell n2)^2-.5}$$

$$= \frac{4e^{-t-1} - 2\ell n2+e^{t}-1}{4/e - 2(\ell n2)^2-.5} \qquad 0 \leq t \leq \ln2-1 \qquad (4.36)$$

The function RL(t) covers the larger area for $0 \leq t \leq \ell n2-.5$ so this half of the region will be treated. The next step is to find the extreme value of $\overline{RL}(t)$ for $0 \leq t \leq \ell n2-.5$.

$$\frac{d\overline{RL}(t)}{dt} = \frac{-4e^{-t-1} + e^{t}}{4/e - 2(\ell n2)^2-.5}$$

Therefore, the only extreme value of (4.36) is at

$$t^* = \ell n2-.5 \qquad (4.37)$$

Substituting (4.37) into (4.36) gives

$$\overline{RL}(t^*) = \frac{4e^{-\ell n2+.5-1} - 2\ell n2 + e^{\ell n2-.5} - 1}{4/e - 2(\ell n2)^2 - .5}$$

$$= \frac{4e^{-.5} - 2\ell n2-1}{4/e - 2(\ell n2)^2-.5}$$

$$\simeq \frac{0.039828}{4/e - 2(\ell n2)^2-.5} \qquad (4.38)$$

Checking the value of $\overline{RL}(t)$ at the other end of the range,

$$\overline{RL}(0) = \frac{4/e - 2\ell n2}{4/e - 2(\ell n2)^2-.5}$$

$$\simeq \frac{0.085223}{4/e - 2(\ell n2)^2-.5} \qquad (4.39)$$

The minimum value of $\overline{RL}(t)$, $0 \leq t \leq \ell n2-.5$ occurs at $t=\ell n2-.5$ and the maximum value at $t=0$.

For the reflected acceptance rejection technique the following definitions are made:

$$\alpha_{RL} = 4/e - 2(\ell n2)^2-.5 \qquad (4.40)$$

$$RL(t) = (4\ e^{-t-1} - 2\ell n2+t)/\alpha_{RL}\ ,\ 0 \leq t \leq \ell n2-.5 \qquad (4.41)$$

$$\overline{RL}(t) = (4\ e^{-t-1} - 2\ell n2+e^{t}-1)/\alpha_{RL},\ 0 \leq t \leq \ell n2-.5 \qquad (4.42)$$

$$g(t) = \overline{RL}(0)$$

$$= 0.08522\ 34035\ 65878/\alpha_{RL} \qquad (4.43)$$

$$\beta_{RL} = \overline{RL}(\ell n2-.5)$$

$$= 0.03982\ 82777\ 30643/\alpha_{RL} \qquad (4.44)$$

The algorithm outlined in Section 2.4 is then followed. Equation (4.35) is used to transform the obtained sample to the appropriate region. The sampling efficiency is about 0.64. A sample will be returned after only one function evaluation about 47% of the time.

4.4 Regions RU_i, i=0,1,2,...

These regions are defined by

$$RU_i(x) = e^{-x} - L_i \ , \ x_{T_i} \le x \le x_{L_{i+1}} \ , \ i=0,1,... \qquad (4.45)$$

Substituting equation (4.13) for L_i, (4.10) for x_{T_i}, and (4.16) for $x_{L_{i+1}}$

$$RU_i(x) = e^{-x} - 2^{-i}(i\ln 2+1-x), \ i\ln 2 \le x \le (i-1)\ln 2+1 \qquad (4.46)$$

These regions are to be sampled using an AR technique. The first task is, therefore, to transform each region so that the range of the independent variable starts at 0. Let

$$t = x-i\ln 2 \qquad (4.47)$$

then (4.46) becomes

$$RU_i(t) = e^{-t-i\ln 2} - 2^{-i}(i\ln 2+1-t-i\ln 2)$$

$$= 2^{-i}(e^{-t}+t-1) \ , \ 0 \le t \le 1-\ln 2 \qquad (4.48)$$

4.4.1 The Area of Region RU_i

The area of a region RU_i is found by integrating (4.48) over the entire range of t.

$$A(RU_i) = \int_0^{1-\ln 2} 2^{-i}(e^{-t}+t-1)dt$$

$$= 2^{-i}(-e^{-t} + t^2/2 - t)\Big|_0^{1-\ln 2}$$

$$= 2^{-i}\left(1-e^{-1+\ln 2} + \frac{(1-\ln 2)^2}{2} - 1+\ln 2\right)$$

$$= 2^{-i}\left(-2/e+ 1/2 - \ln 2 + \frac{(\ln 2)^2}{2} + \ln 2\right)$$

$$= 2^{-i}\left(\frac{(\ln 2)^2}{2} + .5 - 2/e\right) \qquad (4.49)$$

4.4.2 Total Area of Regions RU_i

The sum of the areas of the first n+1 regions is found by summing (4.49) over i.

$$TA(n) = \sum_{i=0}^{n} 2^{-i}\left(\frac{(\ell n2)^2}{2} + .5 - 2/e\right)$$

$$= \left(\frac{(\ell n2)^2}{2} + .5 - 2/e\right)\frac{(1-2^{-n-1})}{(1-1/2)}$$

$$= ((\ell n2)^2 + 1 - 4/e)(1-2^{-n-1}) \qquad (4.50)$$

Therefore, for all regions RU_i the total area is

$$TA(RU) = \lim_{n\to\infty} TA(n)$$

$$= \lim_{n\to\infty} ((\ell n2)^2 + 1 - 4/e)(1 - 2^{-n-1})$$

$$= (\ell n2)^2 + 1 - 4/e$$

$$\simeq 0.008935 \qquad (4.51)$$

When sampli.., from the exponential distribution one of these regions in RU_i will be used about 0.89% of the time.

4.4.3 Sampling from a Region RU_i

The region $RU_i(t)$ is to be sampled using an AR technique. To do this, equation (4.48) must be normalized.

$$RU(t) = \frac{RU_i(t)}{A(RU_i)}$$

$$= \frac{2^{-i}(e^{-t} + t - 1)}{2^{-i}\left(\frac{(\ell n2)^2}{2} + .5 - 2/e\right)}$$

$$= \frac{e^{-t} + t - 1}{\frac{(\ell n2)^2}{2} + .5 - 2/e}, \quad 0 \le t \le 1 - \ell n2 \qquad (4.52)$$

The region to be sampled from is the same for each RU_i and the difference comes when the sample t is transformed back to the original coordinates by

$$x = t + i\ell n2 \qquad (4.53)$$

The function (4.52) is plotted in Figure 19 and since it is antisymmetric about its midpoint, the reflected AR method is appropriate. For this technique we must define

$$\overline{RU}(t) = RU(t) + RU(1-\ell n2-t)$$

$$= \frac{e^{-t} + t - 1}{\frac{(\ell n2)^2}{2} + .5 - 2/e} + \frac{e^{-1+\ell n2+t} + 1-\ell n2-t-1}{\frac{(\ell n2)^2}{2} + .5 - 2/e}$$

$$= \frac{e^{-t} - 1 + 2e^{t-1} - \ell n2}{\frac{(\ell n2)^2}{2} + .5 - 2/e} \qquad 0 \le t \le 1-\ln 2 \qquad (4.54)$$

The function RU(t) covers the larger area for $.5(1-\ell n2) \le t \le 1 - \ell n2$ so this half of the region will be treated. The next step is to find the extreme values of $\overline{RU}(t)$ for $.5(1-\ell n2) \le t \le 1 - \ell n2$.

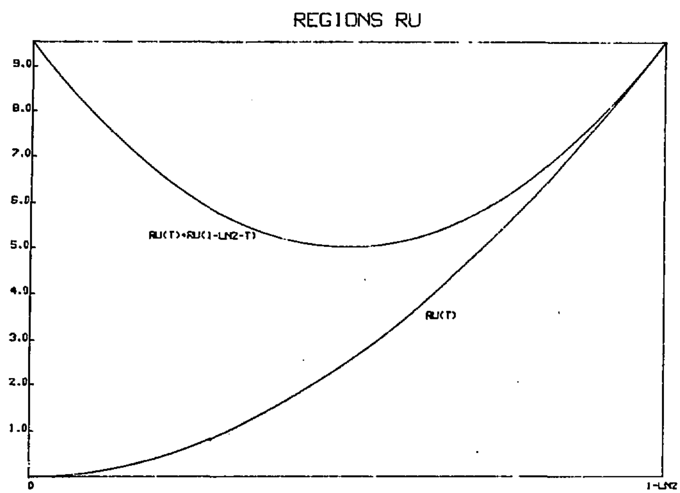

Figure 19

$$\frac{d\overline{RU}(t)}{dt} = \frac{-e^{-t} + 2e^{t-1}}{\frac{(\ell n2)^2}{2} + .5 - 2/e}$$

Therefore, the only extreme value of (4.53) is at

$$t^* = \frac{1-\ell n2}{2} \qquad (4.55)$$

Substituting (4.55) into (4.54) gives

$$\overline{RU}(t^*) = \frac{e^{\frac{\ell n2}{2} - .5} - 1 + 2e^{\frac{-\ell n2}{2} - .5} - \ell n2}{\frac{(\ell n2)^2}{2} + .5 - 2/e}$$

$$= \frac{(8/e)^{1/2} - 1 - \ell n2}{\frac{(\ell n2)^2}{2} + .5 - 2/e}$$

$$\simeq \frac{0.022381}{\frac{(\ell n2)^2}{2} + .5 - 2/e} \qquad (4.56)$$

Checking the value of $\overline{RU}(t)$ at the other end of the range

$$\overline{RU}(1-\ell n2) = \frac{2/e - \ell n2}{\frac{(\ell n2)^2}{2} + .5 - 2/e}$$

$$\simeq \frac{0.042612}{\frac{(\ell n2)^2}{2} + .5 - 2/e} \qquad (4.57)$$

The minimum value of $\overline{RU}(t)$, $.5(1-\ell n2) \leq t \leq 1-\ell n2$ occurs at $t=.5(1-\ell n2)$ and the maximum value at $t = 1-\ell n2$.

For the reflected AR technique the following definitions are made:

$$\alpha_{RU} = \frac{(\ell n2)^2}{2} + .5 - 2/e \qquad (4.58)$$

$$RU(t) = (e^{-t}+t-1)/\alpha_{RU} \ , \ .5(1-\ell n2) \leq t \leq 1 - \ell n2 \qquad (4.59)$$

$$\overline{RU}(t) = (e^{-t}+2e^{t-1}-1-\ell n2)/\alpha_{RU} \qquad .5(1-\ln 2) \leq t \leq 1-\ln 2 \quad (4.60)$$

$$g(t) = \overline{RU}(1-\ln 2)$$

$$= 0.04261\ 17017\ 82939/\alpha_{RU} \qquad (4.61)$$

$$\beta_{RU} = \overline{RU}(.5 - \frac{\ell n2}{2})$$

$$= 0.02238\ 058931\ 61468 \qquad (4.62)$$

The algorithm outlined in Section 2.4 is then followed. Equation (4.53) is used to transform the sample value to the appropriate region RU_i. The sampling efficiency is about 0.68. A sample will be returned after only one function evaluation about 53% of the time.

4.5 The Subroutine REX

The methods outlined in the previous sections are incorporated in the subroutine REX (See Appendix 2). This subroutine returns a sequence of N exponentially distributed random numbers. The only problem that remains is how to pick the region to sample from.

The uniformly distributed random number $U \in (0,1)$ is used for this purpose. If $U \leq .5$, then the triangle T_0 is sampled from. If $U - .5 \leq TA(T_i) = (\ell n2)^2$, then one of the triangles T_i is sampled from. If $U-.5 - (\ell n2)^2 \leq TA(RU) = 4/e - (\ell n2)^2 + 1-4/e$ then one of the regions RU_1 is sampled from. Otherwise a region RL_i is chosen.

Now an appropriate value of i must be chosen. From equations (4.18), (4.31) and (4.49) it is known that

$$A(T_{i-1}) = 2A(T_i) \tag{4.63}$$

$$A(RL_{i-1}) = 2A(RL_i) \tag{4.64}$$

$$A(RU_{i-1}) = 2A(RU_i) \tag{4.65}$$

So we wish to choose region i-2 twice as often as region i-1, region i-1 twice as often as region i, etc. Let u be a uniformly distributed random number on (0,1).

For $2^{-1} \leq u < 2^0$, choose i=1

$2^{-2} \leq u < 2^{-1}$, choose i=2

$2^{-3} \leq u < 2^{-2}$, choose i=3

etc.

In other words, for $2^{-i} \leq u < 2^{-i+1}$, choose the value i to define the appropriate region.

On the CDC 6600 and CYBER 175, the computer representation of u has a fractional base and a biased exponent: $u = a*2^b$.

The 48-bit mantissa has $2^{47} \leq a < 2^{48}$ and the exponent is biased by adding $2^{10} - 1 = 1023$ to the true value.

Rewriting u as a normalized base 2 number gives:

$u = c2^{-i+1}$

where $i \geq 1$ and $2^{-1} \leq c < 1.0$.

The computer representation of u with an unbiased exponent is

$$u = (c*2^{48})*2^{-i+1}*2^{-48}$$

$$= (c*2^{48})*2^{-i-47}$$

and with a biased exponent

$$u = (c*2^{48})*2^{-i+976} \qquad (4.66)$$

Subtracting the exponent of a uniformly distributed random number (which can be obtained by a right shift of 48 bits) from 976 is an efficient way to obtain a value for i.

The algorithm actually calculates i-2 since that is the value needed in equations (4.24) and (4.35). Equation (4.53) is modified to

$$x = t + (i-2)\ell n2 + \ell n2, \; i=1,2 \qquad (4.53a)$$

to speed up the calculations.

TESTING AND TIMING

Both the subroutines RNORM and REX were checked using chi-square tests and Kolmogorov-Smirnov tests. These tested that sequences of numbers generated on part and all of the range were samples from the correct probability distributions. No tests for randomness were performed. It was assumed that RNORM and REX would have the inherent properties of the uniform random number generator RANF. These have previously been tested and found to be satisfactory [4].

The timing tests were done on a CDC 6600/CYBER 170 system. It was found that nothing was gained in this system by implementing RNORM and REX in COMPASS so they were written in FORTRAN. A DO loop was set up to call each subroutine 100,000 times. The total overhead of the loop was found by calling a dummy routine which merely contained an assignment statement. The average overhead for one time through the loop was found to be 7.60 μs.

For the normal distribution the AR method algorithm TR in [1] was programmed, as well as the reflected AR method outlined in Section 3.5. The average time for one call to each routine is shown in Figure 20.

Method	Time (μs)
AR Method	4.67
Reflected AR Method	3.91

Figure 20
Average Time to Calculate One
Normally Distributed Random Number

For the exponential distribution von Neumann's method (Algorithm NE in [1]) and Ahrens' Method (Algorithm SA in [1]) were programmed as well as the method described in Section 4.5. The average time for one call to each routine is shown in Figure 21.

Method	Time (μs)
von Neumann	5.16
Ahrens and Dieter	4.48
REX	4.17

Figure 21
Average Time to Calculate One
Exponentially Distributed Random Number

It can be concluded that the two algorithms presented here are slightly faster than those recommended by Ahrens and Dieter in [1].

6. REFERENCES

[1] Ahrens, J.H. and Dieter, U.: Computer Methods for Sampling from the Exponential and Normal Distributions, CACM, Vol. 15, No. 10 (1972), pp. 873-882.

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[3] Marsaglia, G.: Generating a Variable from the Tail of the Normal Distribution, Technometrics, Vol. 6 (1964), pp. 101-102.

[4] Mulvihill, J.E. and Blair, J.M.: Multiplicative Random Number Generators for 48-Bit Computers, Atomic Energy of Canada Limited, Report AECL-5019, May 1977.

[5] Payne, W.H.: Normal Random Numbers: Using Machine Analysis to Choose the Best Algorithm, ACM TOMS, Vol. 3, No. 4 (1977), pp. 346-358.

APPENDIX 1

LISTING OF RNORM

```
      SUBROUTINE RNORM(Z,N)
      REAL Z(1)
      DATA B/2.1140280833374 /,A/.39894228040143 /,C1/.209694057195486/
     $,C2/.44329912582022/,B2/4.4691147371392/
C
C     U IS USED THROUGHOUT TO CHOOSE WHICH REGION  RNORM IS IN.
C     U IS THEN REUSED IN THE TRAPEZOID.
C     U0 IS USED IN THE TRAPEZOID BUT IT IS ALSO USED TO CHOOSE IF THE
C     VALUE RETURNED IS POSITIVE OR NEGATIVE
C
      DO 100  II=1,N
      U=RANF(A)
      U0=RANF(A)
C
C     CHOOSE FROM THE TRAPEZOID.
C
      IF(U .GE. .91954440570693) GO TO 10
         Z(II)=2.4037576569374 * (U0 + U* .82533928253692) -B
         GO TO 100
C
C     CHOOSE FROM THE TAIL.
C
10    IF(U .LT. .96548713121386 )GO TO 30
20       U1=RANF(A)
         X=B2-2.*ALOG(U1)
         U2=RANF(A)
         IF(X*U2*U2 .GT. B2)GO TO 20
         X=SQRT(X)
         GO TO 80
C
C     CHOOSE FROM REGION 3.
C
30    IF(U .LT. .94856274812909)GO TO 50
40       U1=RANF(A)
         X=B-.1620140416687*U1
         Y=A*EXP(-X*X*.5)-C2+X*C1
         U2=RANF(A)*.05513592720665
         IF(Y .GT. U2)GO TO 80
         X=3.9040280833374-X
         IF(.05077522365025 .GT. U2)GO TO 80
         IF(A*EXP(-X*X*.5)-C2+X*C1 .GT. U2-Y)GO TO 80
         GO TO 40
50    IF(U .LT. .9258523337077)GO TO 70
60       U1=RANF(A)
         X=.2897295736 + 1.5002704264*U1
         Y=A*EXP(-X*X*.5)-C2+X*C1
         U2=RANF(A)*.016270801
         IF(Y .GT. U2)GO TO 80
         X=2.0797295736-X
         IF(.01243334561586 .GT. U2)GO TO 80
         IF(A*EXP(-X*X*.5)-C2+X*C1 .GT. U2-Y)GO TO 80
         GO TO 60
C
C     CHOOSE FROM REGION 2.
C
70    U1=RANF(A)
         X=U1*.2897295736
         U2=RANF(A)
         IF(A*EXP(-X*X*.5)-.38254455604252  .LT. U2*.016397724358915)
     $   GO TO 70
80    Z(II)=X
      IF(U0 .LT. .5)Z(II)=-Z(II)
100   CONTINUE
      RETURN
      END
```

APPENDIX 2

LISTING OF REX

```
      SUBROUTINE REX(X,N)
      REAL X(1)
      DATA TATTO,TARU,XRU,ARU,BRU,XRL,ARL,BRL,CL,E4/.9804530139182,
     ,.00893524923243,.30685281944005,.04261170178294,.02238058936147,
     ,.38629436111989,.08522340356588,.03982827773064,.69314718055994,
     ,1.4715177646858/
C
C     U IS USED THROUGHOUT THE PROGRAM TO PICK THE REQUIRED REGION
C
      DO 50  II=1,N
      U=RANF(A)
      IF(U.GT..5) GOTO 5
C
C     THE MINIMUM OF TWO RANDOM NUMBERS IS CHOSEN HERE
C
         U0=RANF(A)
         X(II)=U
         IF(X(II) .LE. U0)GO TO 50
         X(II)=1.-X(II)
         GO TO 50
    5 U=U-TATTO
C
C     I REFERS TO THE ITH REGION  IN THE INFINITE SERIES OF
C     TRIANGLES  , OR REGIONS CALLED RU(I),RL(I)
C     B PLACES REX IN THE CORRECT AREA OF THE LINE
C
      I=974-SHIFT(RANF(A),-48)
      B=I*CL
      IF(U.GT.0.) GOTO 10
C
C     CHOOSE FROM AN ISOCELES TRIANGLE
C
         X(II)=(RANF(A)+RANF(A))*CL+B+1.
         GO TO 50
   10 U=U-TARU
      IF(U.GT.0.) GOTO 25
C
C     CHOOSE FROM A REGION RU
C
   15    U1=.15342640972002*(1.+RANF(A))
         U2=RANF(A)*ARU
         A=EXP(-U1)
         IF(A+U1-1..GE.U2) GOTO 20
            U1=XRU-U1
            IF(BRU.GE.U2) GOTO 20
               IF(A+.73575888234288/A-1.6931471805599.GE.U2) GOTO 20
                  GOTO 15
20       X(II)=B+CL+U1
         GO TO 50
C
C     CHOOSE FROM A REGION RL
C
   25    U1=.19314718055995*RANF(A)
      U2=RANF(A  )*ARL
      A=EXP(-U1)
      IF(E4*A+U1-1.3862943611199.GE.U2) GOTO 30
         U1=XRL-U1
         IF(BRL.GE.U2) GOTO 30
            IF(E4*A+1./A-2.3862943611199.GE.U2) GOTO 30
               GOTO 25
30    X(II)=1.+B+U1
50    CONTINUE
      RETURN
      END
```

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