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A THEORY OF COHERENT PROPAGATION OF LIGHT WAVE IN SEMICONDUCTORS

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ABSTRACT

In this paper, we suggest a theory to describe the phenomena of coherent propagation of light wave in semiconductors. Basing on two band system and considering the interband and intraband transitions induced by light wave and the interaction between electrons, we obtain the nonlinear equations for the description of interaction between carriers and coherent light wave. We have made use of the equations to analyse the phenomena which arise from the interaction between semiconductors and coherent light, for example, the multiphoton transitions, the saturation of light absorption of exciton, the shift of exciton line in intense light field, and the coherent propagation phenomena such as self-induced transparency etc.

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I. Introduction

When an intence coherent light pulse enters a semiconductor in which resonant transitions can be induced, several processes will occur: (1) creation and recombination of electron-hole pairs induced by the coherent light; (2) collisions of the electrons and holes; (3) interaction of electrons and holes with phonons; (4) recombination of electron-hole pairs through spontaneous emission or at recombination centers. If the intensity of light is high enough, process (1) will predominate. In such a case, the coherence between the excited state of the semiconductor and the light wave becomes important, and a number of specific phenomena of coherent propagation will occur. Thus observations of phenomena, such as self-induced transparency in interband and exciton transitions [1,2], and saturation of absorption [3,4], have been reported. Certain theoretical analyses have been given in the literature [5-9]. The purpose of the present work aims at developing an adequate theory for dealing with such phenomena.

Our work is presented in three parts. In the first part, we neglect the interaction between the electrons and investigate the problem in the framework of single electron band theory. The intense light field induces not only interband transitions, but also intraband transitions of electrons and holes respectively in the conduction and valence bands. Hence our system is different from the ordinary inhomogeneously broadened two-level system. For our treatment we have adopted the "space translation approximation", i.e. the approximate steady states of the carriers moving within their respective bands under the action of the light wave are taken as the base for treating the interband transitions. This is effected by a transformation, which renders the system formally analogous to an inhomogeneously broadened two-level system, but a particular type of multiphoton process will occur here.

In the second part, we take into account the interaction between electrons. From the point of view of the resonant coherent interaction with light, the interaction between electrons can be partitioned into a part that does not change the total momentum of the relevant electron-hole pair and an other part does change the total momentum. In other words, the former is the interaction of the electron with its hole partner which gives rise to the binding of the exciton state. The latter represents collisions between electrons and holes which destroy the coherence of the process. When we are considering the coherent propagation, it is reasonable to take account of the former while neglecting the latter. Because the intense light pulse can generate a high density of excitons, we must treat the electrons and holes, which make up the excitons, as "fermions". This differs from usual treatments representing excitons at low density as "bosons"[6]. We have introduced coherent states of the exciton to describe the process, and obtained a set of non-linear equations which can naturally account for the saturation of light absorption of exciton and the shift of exciton line in intense light field. Within certain approximation, the coherent excitation of discrete exciton lines can be described by a Bloch equation analogous to that for two-level atomic systems; the corresponding density of equivalent "two-level atoms" is determined by features of the wave function of the exciton state.

In the third part, we shall analyse the propagation of coherent light pulse in the system, and derive the Maxwell-Bloch equation describing this process. The equation provides a theoretical basis for treating coherent propagation problems. After making some approximation, we can represent this equation in a standard form which can be solved by the "inverse scattering method".

II. Two-band model

2.1 Basic formula and the space translation approximation

In this part, we shall work out the theory in the single electron approximation. The Hamiltonian of an electron interacting with a light wave can be written as

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{1}{2m}\left(\vec{P}+\frac{e}{c}\vec{A}\right)^2 + V\right]\psi, \tag{2.1}$$

where V is the effective periodic potential for an electron moving in the crystal lattice, and $\vec{A}$ is the vector potential for the light wave, which is assumed to be a plane wave:

$$\vec{A} = \vec{A}_0 \sin(\omega t - \vec{q}\cdot\vec{r} + \varphi). \tag{2.2}$$

As it is weaker than V, we can take the Bloch wave function of the electron as the base to treat this problem. For the sake of simplicity, we consider a simple two-band model:

$$E_c(\vec{k}) = \frac{E_g}{2} + \frac{\hbar^2}{2m_e}\vec{k}^2,$$

$$E_v(\vec{k}) = -\frac{E_g}{2} - \frac{\hbar^2}{2m_h}\vec{k}^2, \tag{2.3}$$

where the indices c and v refer respectively to the conduction and valence bands and m_e, m_h are the electron and hole effective masses, E_g is the forbidden gap width. In the dipole approximation and neglecting spatial variation of the light field, we can write (2.1) in the formalism of second quantization as follows:

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + H_1 + H_2)\psi , \qquad (2.4)$$

where

$$H_0 = \sum_{\vec{K}} E_c(\vec{K}) a^+_{c\vec{K}} a_{c\vec{K}} + E_v(\vec{K})(a^+_{v\vec{K}} a_{v\vec{K}} - 1) . \qquad (2.5)$$

$a_{c\vec{K}}, a^+_{c\vec{K}}$ ($a_{v\vec{K}}, a^+_{v\vec{K}}$) are respectively the annihilation and creation operators for electrons in the conduction band (valence band). By changing their indices, we introduce the annihilation and creation operators respectively for electrons and holes as follows:

$$a^+_{\vec{K}} = a^+_{c\vec{K}} \quad , \quad a_{\vec{K}} = a_{c\vec{K}} \quad ;$$

$$b^+_{-\vec{K}} = a_{v\vec{K}} \quad , \quad b_{-\vec{K}} = a^+_{v\vec{K}} \quad .$$

Thus H_0 becomes

$$H_0 = \sum_{\vec{K}} [E_c(\vec{K}) a^+_{\vec{K}} a_{\vec{K}} - E_v(\vec{K}) b^+_{-\vec{K}} b_{-\vec{K}}] . \qquad (2.6)$$

H_1 in (2.4) represents the part of the interaction with the light wave responsible for interband effects;

$$H_1 = \frac{e}{mc} \vec{A} \cdot \vec{P}_{cv}(\vec{K}) (a^+_{\vec{K}} b^+_{-\vec{K}} + b_{-\vec{K}} a_{\vec{K}}) . \qquad (2.7)$$

where $\vec{P}_{cv}(\vec{K})$ is the interband matrix element of operator $\vec{P}$. By suitable choice of the phases of the Bloch wave functions, $\vec{P}_{cv}(\vec{K})$

can be made real, The two terms in (2.7) correspond to the generation and recombination of an electron-hole pair respectively.

H_2 in (2.4) represents the part of the interaction with the light wave responsible for intraband effects;

$$H_2 = \frac{e}{c} \vec{A} \cdot [a^+_{\vec{K}} a_{\vec{K}} \frac{1}{\hbar} \nabla_{\vec{K}} E_c(\vec{K}) - b^+_{-\vec{K}} b_{-\vec{K}} \frac{1}{\hbar} \nabla_{\vec{K}} E_v(\vec{K})] , \qquad (2.8)$$

where $\frac{1}{\hbar}\nabla_{\vec{K}} E_c(\vec{K})$ and $\frac{1}{\hbar}\nabla_{\vec{K}} E_v(\vec{K})$ are the velocity of electron and hole.

In the dipole approximation, we note that the $\vec{A}^2$ term in (2.1) can be eliminated by introducing a common phase factor which is identical for all states. Hence this term is not included in (2.4). But we should emphesize that the space dependence of $\vec{A}$ has been neglected altogether in (2.4).

By introducing the following canonical transformation:

$$S = \prod_{\vec{K}} S_{\vec{K}}$$

$$S_{\vec{K}} = \exp\{-\frac{i}{\hbar}\int^t [\frac{e}{c}\vec{A}(t')\cdot\frac{1}{\hbar}\nabla_{\vec{K}} E_c(\vec{K})]dt' a^+_{\vec{K}} a_{\vec{K}}$$

$$+ \frac{i}{\hbar}\int^t [\frac{e}{c}\vec{A}(t')\cdot\frac{1}{\hbar}\nabla_{\vec{K}} E_v(\vec{K})]dt' b^+_{-\vec{K}} b_{-\vec{K}} \} , \qquad (2.9)$$

the intraband term can be eleminated from (2.4), giving thus

$$i\hbar \frac{\partial \psi}{\partial t} = H_S \psi$$

$$H_S = \sum_{\vec{K}} [E_c(\vec{K}) a^+_{\vec{K}} a_{\vec{K}} - E_v(\vec{K}) b^+_{-\vec{K}} b_{-\vec{K}}] +$$

$$+\sum_{\vec{K}} \frac{e}{mc} \vec{A} \cdot \vec{P}_{cv} \{ \exp[i(\theta_{\vec{K}} + \varphi_{-\vec{K}})] a^+_{\vec{K}} b^+_{\vec{K}} +$$

$$+ \exp[-i(\theta_{\vec{K}} + \varphi_{-\vec{K}})] b_{-\vec{K}} a_{\vec{K}} \} \quad , \qquad (2.10)$$

where

$$\theta_{\vec{K}} = \frac{1}{\hbar}\int^t \frac{e}{c} \vec{A}(t') \cdot \frac{1}{\hbar} \nabla_{\vec{K}} E_c(\vec{K})\, dt' \ ,$$

$$\varphi_{-\vec{K}} = -\frac{1}{\hbar}\int^t \frac{e}{c} \vec{A}(t') \cdot \frac{1}{\hbar} \nabla_{\vec{K}} E_v(\vec{K})\, dt' \ . \qquad (2.11)$$

The physical meaning of the transformation (2.9) is easily understood. For the electron, it corresponds to a transformation from a base $e^{i\vec{K}\cdot\vec{r}} u_{c\vec{K}}$ to a new base $e^{i\vec{K}\cdot(\vec{r}-\frac{e}{mc}\int^t \vec{A}(t')dt')} u_{c\vec{K}}$. If we neglect the space dependence of $\vec{A}$, it is just the wave function for the steady state motion of electron in the conduction band under the electromagnetic field. For the hole, the situation is similar. Thus we may designate the transformation S as the " space translation approximation" [11].

2.2 Intraband-interband multiphoton transition

The interaction term in the Hamiltonian (2.10) is

$$\frac{e}{mc} \vec{A} \cdot \vec{P}_{cv} \{ \exp[i(\theta_{\vec{K}} + \varphi_{-\vec{K}})] a^+_{\vec{K}} b^+_{\vec{K}} + \exp[-i(\theta_{\vec{K}} + \varphi_{-\vec{K}})] b_{\vec{K}} a_{\vec{K}} \} .$$

By substituting the concrete expression of $\theta_{\vec{K}}$ and $\varphi_{-\vec{K}}$, we have

$$\exp[-i(\theta_{\vec{K}} + \varphi_{\vec{K}})] = \exp[i \frac{e}{\omega c} \vec{A}_0 \cdot (\frac{1}{m_e} + \frac{1}{m_h}) \vec{K} \cdot \cos(\omega t + \theta)] .$$

Introducing

$$\eta = \frac{e}{\omega c} \vec{A}_0 \cdot (\frac{1}{m_e} + \frac{1}{m_h}) \vec{K}$$

we obtain

$$\exp[-i(\theta_{\vec{K}} + \varphi_{-\vec{K}})] = \sum_{m=-\infty}^{\infty} i^m J_m(\eta)\, e^{im(\omega t + \theta)} \quad , \qquad (2.12)$$

where $J_m(\eta)$ is the m-order Bessel function. Obviously, if $E_c(\vec{K}) - E_v(\vec{K}) \approx n\hbar\omega$ (n is an arbitrary integer), corresponding multi-quantum transitions can occur. This is a type of multiphoton transition distinct from the usual the high order interband transitions. This is an intraband-interband multiphoton transition which has been discussed in reference [13-15]. It is to be noted that the probability of this type of process is essentially dependent on the direction and magnitude of $\vec{K}$.

2.3 Bloch equation

We shall assume the wave function of the system to have the following form:

$$\psi = \prod_{\vec{K}} [\alpha_{\vec{K}}(t) + \beta_{\vec{K}}(t) a^+_{\vec{K}} b^+_{-\vec{K}}] \psi_0 \quad , \qquad (2.13)$$

where ψ_0 is the wave function of ground state (the conduction band is empty and the valence band is completely filled). Substituting (2.13) into (2.10) , we obtain

$$i\hbar \frac{d\alpha_{\vec{K}}}{dt} = \frac{e}{mc} (\vec{A} \cdot \vec{P}_{cv}) \exp[-i(\theta_{\vec{K}} + \varphi_{-\vec{K}})] \beta_{\vec{K}}$$

$$i\hbar \frac{d\beta_{\vec{K}}}{dt} = [E_c(\vec{K}) - E_v(\vec{K})]\beta_{\vec{K}} + \frac{e}{mc}(\vec{A}\cdot\vec{P}_{cv})\cdot$$
$$\cdot \exp[i(\theta_{\vec{K}} + \varphi_{-\vec{K}})]\,\alpha_{\vec{K}} \qquad . \qquad (2.14)$$

introducing the notations:

$$\omega_{\vec{K}} = \frac{1}{\hbar}[E_c(\vec{K}) - E_v(\vec{K})] \quad , \quad \Omega_{\vec{K}} = \frac{e}{2mc\hbar}(\vec{A}_0\cdot\vec{P}_{cv}) \quad ,$$

and

$$\tilde{\alpha}_{\vec{K}} = \alpha_{\vec{K}} \qquad , \quad \tilde{\beta}_{\vec{K}} = \beta_{\vec{K}}\exp(i\omega_{\vec{K}}t) \quad ,$$

we have

$$\frac{d\tilde{\alpha}_{\vec{K}}}{dt} = -\Omega_{\vec{K}}\{\exp[i(\omega-\omega_{\vec{K}})t+i\theta\ ]\exp[-i(\theta_{\vec{K}}+\varphi_{-\vec{K}})] -$$
$$-\exp[-i(\omega+\omega_{\vec{K}})t-i\theta\ ]\exp[-i(\theta_{\vec{K}}+\varphi_{-\vec{K}})]\}\,\tilde{\beta}_{\vec{K}} \quad ,$$
$$\frac{d\tilde{\beta}_{\vec{K}}}{dt} = -\Omega_{\vec{K}}\{\exp[i(\omega+\omega_{\vec{K}})t+i\theta\ ]\exp[i(\theta_{\vec{K}}+\varphi_{-\vec{K}})] -$$
$$-\exp[-i(\omega-\omega_{\vec{K}})t-i\theta\ ]\exp[i(\theta_{\vec{K}}+\varphi_{-\vec{K}})]\}\,\tilde{\alpha}_{\vec{K}}$$
$$(2.15)$$

Using (2.12) , the right side of (2.15) becomes a sum of various harmonic terms, thus for example,

$$\frac{d\tilde{\alpha}_{\vec{K}}}{dt} = -\Omega_{\vec{K}}\sum_{m=-\infty}^{\infty} i^m J_m(\eta)\{\exp[i((m+1)\omega-\omega_{\vec{K}})t +$$
$$+ i(m+1)\theta_{\vec{K}}] - \exp[i((m+1)\omega-\omega_{\vec{K}})t + (m-1)\theta\ ]\}$$
$$.$$

If one of the harmonic terms satisfies $(m+1)\omega - \omega_{\vec{K}} \approx 0$, the contribution of this term will clearly be the largest and the rest of terms can neglected approximately, this is equivalent to the rotating wave approximation in magnetic resonance[16]. Physically, this corresponds to the resonant transition with $m+1$ photons. If we introduce in such a case

$$\Delta\omega_{\vec{K}} = (m+1)\omega - \omega_{\vec{K}} \quad , \quad G_m = i^m J_m(\eta) \quad .$$

Then Eq.(2.15) can be written approximately as

$$\frac{d\tilde{\alpha}_{\vec{K}}}{dt} = -\Omega_{\vec{K}}(G_m - G_{m+2})\{\exp[\Delta\omega_{\vec{K}}t + (m+1)\theta\ ]\}\,\tilde{\beta}_{\vec{K}} \ ,$$
$$\frac{d\tilde{\beta}_{\vec{K}}}{dt} = \Omega_{\vec{K}}(G_m^* - G_{m+2}^*)\{\exp[\Delta\omega_{\vec{K}}t + (m+1)\theta\ ]\}\,\tilde{\alpha}_{\vec{K}} .$$
$$(2.16)$$

Eq.(2.16) describes electron-hole pair creation by absorbing $m+1$ photons and electron-hole recombination by emitting $m+1$ photons . Formally, they are entirely analogous the equations for a two level system in a light field, and we can treat the system in analogy with a spin system. thus we introduce

$$m_{\vec{K},x} = \frac{1}{2}(\tilde{\alpha}_{\vec{K}}\tilde{\beta}_{\vec{K}}^* + \tilde{\alpha}_{\vec{K}}^*\tilde{\beta}_{\vec{K}}) \quad ,$$

$$m_{\vec{K},y} = \frac{1}{2i}(\tilde{\alpha}_{\vec{K}}^*\tilde{\beta}_{\vec{K}} - \tilde{\alpha}_{\vec{K}}\tilde{\beta}_{\vec{K}}^*) \quad ,$$

$$m_{\vec{K},z} = \frac{1}{2}(|\tilde{\beta}_{\vec{K}}|^2 - |\tilde{\alpha}_{\vec{K}}|^2) \quad , \qquad (2.17)$$

and

$$\mu_{\vec{K}} = \Omega_{\vec{K}} \left(G_m - G_{m+2} \right) \quad ,$$

$$\psi_{\cdot} = -(m+1)\,\theta .$$

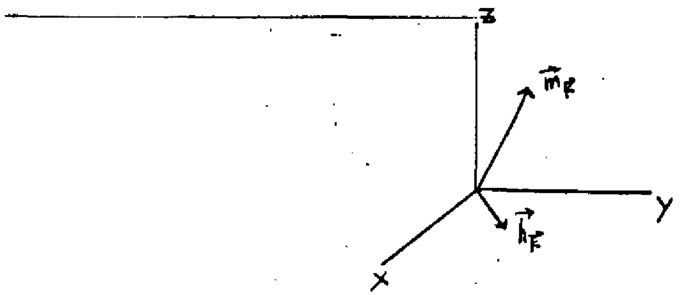

Fig (2.1)

From (2.16), we obtain

$$\frac{d}{dt}\overrightarrow{m_{\vec{K}}} = \overrightarrow{m_{\vec{K}}} \times \overrightarrow{h_{\vec{K}}} \qquad , \qquad (2.18)$$

where $\overrightarrow{m_{\vec{K}}} = (m_{\vec{K},x}, m_{\vec{K},y}, m_{\vec{K},z})$ and $\overrightarrow{h_{\vec{K}}} = (2\mu_{\vec{K}} \sin(\Delta\omega_{\vec{K}} t - \psi_{\cdot}), 2\mu_{\vec{K}} \cos(\Delta\omega_{\vec{K}} t - \psi_{\cdot}), 0)$. This is the equation of motion for a magnetic moment precessing in an external field which is rotating with angular frequency $-(\Delta\omega_{\vec{K}} - \frac{d\psi_{\cdot}}{dt})$. Fig.(2.1) is a diagrammatic representation. If the amplitude A_o and the phase θ of the light wave do not vary with time, the solution of (2.18) is well known [16]. In this case, the motion of $\overrightarrow{m_{\vec{K}}}$ in a coordinate system which rotates with the magnetic field, is just a precession in a constant field. The frequency of precession is

$$\nu_K = \sqrt{(\Delta\omega_{\vec{K}})^2 + (2\mu_{\vec{K}})^2} \qquad . \qquad (2.19)$$

Similar to the two level system,[17] this just represents the frequency of the back and forth interband "transitions" between the two bands. If the intensity of light is $10 \sim 100\ MW/cm^2$, The value of ν_K is approximately $10^{12} \sim 10^{13}$ sec^{-1}. As emphasized in the introduction, when $\nu_K \gg \nu_{p-p}$ and $\nu_{p-ph} > 1/\tau$, coherence with the light becomes important. Here, ν_{p-p} is the frequency of collision between electron-hole pairs, ν_{p-ph} is the frequency of electron-phonon scattering, τ is the life time of electron-hole pair. With ultrashort laser pulses, this condition is easily fulfilled. If the pulse duration is shorter than ν_{p-p}^{-1} and ν_{p-ph}^{-1}, coherent propagating effects will become manifest.

We notice that, because we have neglected the spatial variation of the external field in the above derivation, the transitions induced are vertical and the creation operator for electron-hole pair appearing in (2.13) is accordingly $a_{\vec{K}}^{+} b_{-\vec{K}}^{+}$. If we take account of the momentum of the photon, in the approximation of dipole transition, the creation operator for electron-hole pair by a m-photon absorption should be rewritten as $a_{\vec{K}'}^{+} b_{-\vec{K}}^{+}$, where $\vec{K}' = \vec{K} + m\vec{q}$, $\vec{q}$ being the wave vector of the light wave. Thus the wave function becomes

$$\prod_{\vec{K}} \left(\alpha_{\vec{K}} + \beta_{\vec{K}}\, a_{\vec{K}'}^{+} b_{-\vec{K}}^{+} \right) \psi_o \qquad . \qquad (2.20)$$

The above derivation still remains valid.

III. Effects of interaction between electrons

3.1 Basic formula

The interaction between electrons has important effects for the consideration of interaction between light and semiconductor, particularly, as the interaction is responsible for the appearance of discrete energy levels in the forbidden band—exciton levels[18]. In this part, we shall give a discussion of this problem.

The interaction between the electrons can be written as

$$H_{e-e}=\frac{1}{2}\int d\vec{r}\,d\vec{r}'\,[\hat{\psi}^{+}(\vec{r})\hat{\psi}(\vec{r})V(\vec{r}-\vec{r}')\hat{\psi}^{+}(\vec{r}')\hat{\psi}(\vec{r}')] \quad , \quad (3.1)$$

where $\hat{\psi}$, $\hat{\psi}^{+}$ are the annihilation and creation operators for electron, respectively, V is the Coulomb potential function between electrons. In the two band approximation,

$$\hat{\psi}=\sum_{\vec{K}}(\psi_{c\vec{K}}a_{\vec{K}}+\psi_{v\vec{K}}b^{+}_{-\vec{K}}) \quad ,$$

$$\cancel{\hat{\psi}^{+}=\sum_{\vec{K}}(\psi^{*}_{c\vec{K}}a^{+}_{\vec{K}}+\psi^{*}_{v\vec{K}}b_{-\vec{K}})} \quad . \quad (3.2)$$

where $\psi_{c\vec{K}}$ and $\psi_{v\vec{K}}$ are the Bloch wave functions respectively for the conduction and valence band. Substitution of (3.2) in (3.1) give 16 terms. They represent respectively electron-electron scattering, electron-hole scattering, hole-hole scattering, electron-hole pair creation by electron (or hole) scattering, electron-hole recombination by electron scattering (or hole), etc.

For the problem we are considering, we are mainly interested in the first three processes while suitable account has to be taken of the effects of polarization of medium, as indicated in Fig.(3.1). In Fig.(3.1), the solid line is the electron line, the

dotted line is the hole line, " $\sim\sim$ " represents their interaction, " ⊘ " lumps together various possible polarization effects. If we take account of the polarization effects by the introduction of a "macroscope" dielectric function, the three relevant processes can be represented as follows:

electron-electron interaction

$$\sum_{\vec{q}_1}\sum_{\vec{K}_1}\sum_{\vec{K}_2}v(\vec{q}_1)\,a^{+}_{\vec{K}_1}a_{\vec{K}_1+\vec{q}_1}a^{+}_{\vec{K}_2}a_{\vec{K}_2-\vec{q}_1} \quad , \quad (3.3)$$

hole-hole interaction

$$\sum_{\vec{q}_1}\sum_{\vec{K}_1}\sum_{\vec{K}_2}v(\vec{q}_1)\,b^{+}_{-\vec{K}_1-\vec{q}_1}b_{-\vec{K}_1}b^{+}_{-\vec{K}_2+\vec{q}_1}b_{-\vec{K}_2} \quad , \quad (3.4)$$

electron-hole interaction

$$-2\sum_{\vec{q}_1}\sum_{\vec{K}_1}\sum_{\vec{K}_2}v(\vec{q}_1)\,a^{+}_{\vec{K}_1}a_{\vec{K}_1+\vec{q}_1}b^{+}_{-\vec{K}_2+\vec{q}_1}b_{-\vec{K}_2} \quad . \quad (3.5)$$

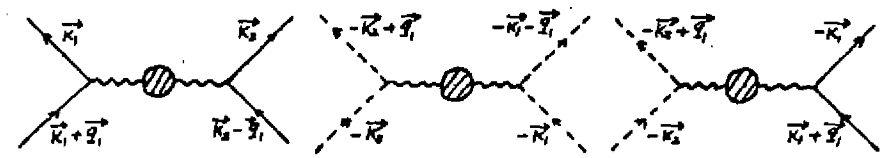

Fig.(3.1)

Here

$$V(\vec{q})=\frac{1}{\epsilon(\vec{q})}\cdot\frac{2\pi e^{2}}{q^{2}} \quad , \quad (3.6)$$

$\frac{4\pi e^2}{q^2}$ is the q-Fourier component of the Coulomb potential function, and $\varepsilon(\vec{q})$ is the dielectric function.

What we want to investigate specifically is the effect of these interactions, directly relevant to the consideration of interaction between coherent light and semiconductors. Thus, for instance, if we are considering a single photon resonant transition, the electron-hole pair which is created or recombined by the light wave will have a total momentum $\hbar\vec{q}$ ($\vec{q}$ is the wave vector of light wave); i.e. only the electron-hole polarization waves with the wave vector $\vec{q}$ are interacting resonantly with the light wave. Hence, for our purpose, we should investigate the effect of the interactions (3.3)—(3.5) on various electron-hole pairs with wave vector $\vec{q}$. Accordingly, we introduce the following creation and annihilation operators for electron-hole pairs:

$$D^+\binom{\vec{K}+\vec{q}}{-\vec{K}} = a^+_{\vec{K}+\vec{q}}\, b^+_{-\vec{K}} ,$$

$$D\binom{\vec{K}+\vec{q}}{-\vec{K}} = b_{-\vec{K}}\, a_{\vec{K}+\vec{q}} \quad . \qquad (3.7)$$

It is readily verified that D and D^+ satisfy the following commutation relations:

$$\left[D^+\binom{\vec{K}_1+\vec{q}_1}{-\vec{K}_1},\ D^+\binom{\vec{K}_2+\vec{q}_2}{-\vec{K}_2}\right] = 0 \ ,$$

$$\left[D\binom{\vec{K}_1+\vec{q}_1}{-\vec{K}_1},\ D\binom{\vec{K}_2+\vec{q}_2}{-\vec{K}_2}\right] = 0 \ ,$$

$$\left[D\binom{\vec{K}_1}{-\vec{K}_1'},\ D^+\binom{\vec{K}_2}{-\vec{K}_2'}\right] = \delta_{\vec{K}_1\vec{K}_2}\,\delta_{\vec{K}_1'\vec{K}_2'} -$$

$$- \delta_{\vec{K}_1'\vec{K}_2'}\, a^+_{\vec{K}_2} a_{\vec{K}_1} - \delta_{\vec{K}_1\vec{K}_2}\, b^+_{-\vec{K}_2'} b_{-\vec{K}_1'} \quad , \qquad (3.8)$$

where

$$[A, B] = AB - BA$$

$$\delta_{\vec{K}_1\vec{K}_2} = 0 \qquad \text{for} \qquad \vec{K}_1 \neq \vec{K}_2$$

$$= 1 \qquad \text{for} \qquad \vec{K}_1 = \vec{K}_2 \ .$$

D^+ and D clearly represent creation and annihilation operators for electron-hole pair with wave vector $\vec{q}$, consisting of an electron $\vec{K}+\vec{q}$ and a hole $-\vec{K}$.

Representing the interaction (3.3)—(3.5) on the basis of such "pair" states, we obtain :

Electron-electron interaction (3.3) :

$$\sum_{\vec{q}_1}\sum_{\vec{K}_1\vec{K}_2}\sum_{\vec{K}_3\vec{K}_4} v(\vec{q}_1)\Big[D^+\binom{\vec{K}_1}{-\vec{K}_3}D^+\binom{\vec{K}_2}{-\vec{K}_4}D\binom{\vec{K}_2-\vec{q}_1}{-\vec{K}_4}D\binom{\vec{K}_1+\vec{q}_1}{-\vec{K}_3} -$$

$$- D^+\binom{\vec{K}_1}{-\vec{K}_4}D^+\binom{\vec{K}_2}{-\vec{K}_3}D\binom{\vec{K}_2-\vec{q}_1}{-\vec{K}_4}D\binom{\vec{K}_1+\vec{q}_1}{-\vec{K}_3}\Big] \quad . \qquad (3.9)$$

It can be described by Fig.(3.2), the second term is the exchange term for the first term.

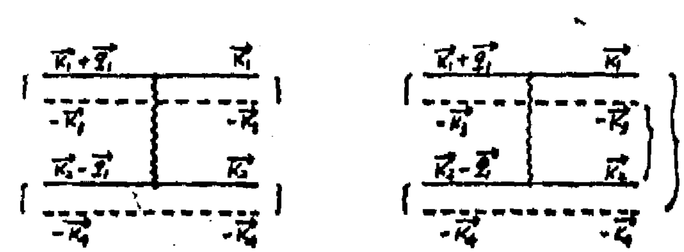

Fig.(3.2)

Hole-hole interaction (3.4) :

$$\sum_{\vec q_1}\sum_{\vec K_1 \vec K_2}\sum_{\vec K_3 \vec K_4} v(\vec q_1)\Big[D^+\binom{\vec K_3}{-\vec K_1-\vec q_1}D^+\binom{\vec K_4}{-\vec K_2+\vec q_1}D\binom{\vec K_4}{-\vec K_2}D\binom{\vec K_3}{-\vec K_1} -$$

$$- D^+\binom{\vec K_4}{-\vec K_1+\vec q_1}D^+\binom{\vec K_3}{-\vec K_2+\vec q_1}D\binom{\vec K_4}{-\vec K_2}D\binom{\vec K_3}{-\vec K_1}\Big] \qquad . \quad (3.10)$$

It can be described by a figure analogous to the Fig.(3.2).

The electron-hole interaction (3.5) :

$$-\sum_{\vec q_1}\sum_{\vec K_1 \vec K_2}\sum_{\vec K_3 \vec K_4} 2v(\vec q_1)\Big[D^+\binom{\vec K_1}{-\vec K_2+\vec q_1}D^+\binom{\vec K_3-\vec q_1}{-\vec K_4}D\binom{\vec K_3}{-\vec K_4}D\binom{\vec K_1}{-\vec K_2} -$$

$$- D^+\binom{\vec K_1}{-\vec K_4}D^+\binom{\vec K_3-\vec q_1}{-\vec K_2+\vec q_1}D\binom{\vec K_3}{-\vec K_4}D\binom{\vec K_1}{-\vec K_2} + D^+\binom{\vec K_1+\vec q_1}{-\vec K_2}D^+\binom{\vec K_3}{-\vec K_4-\vec q_1}D\binom{\vec K_3}{-\vec K_4}D\binom{\vec K_1}{-\vec K_2}$$

$$- D^+\binom{\vec K_1+\vec q_1}{-\vec K_4-\vec q_1}D^+\binom{\vec K_3}{-\vec K_2}D\binom{\vec K_3}{-\vec K_4}D\binom{\vec K_1}{-\vec K_2}\Big] - \sum_{\vec q_1}\sum_{\vec K_1 \vec K_2} 2v(\vec q_1)D^+\binom{\vec K_1+\vec q_1}{-\vec K_2-\vec q_1}D\binom{\vec K_1}{-\vec K_2}. \qquad (3.11)$$

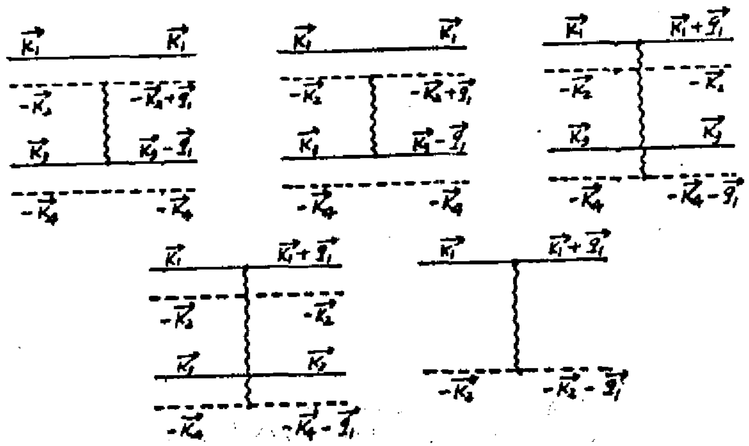

Fig.(3.3)

The five terms are illustrated in Fig.(3.3). Here the second term is the exchange term for the first term, the fourth term is the exchange term for the third term.

In (3.9)—(3.11), the final term of (3.11),

$$\Delta H_{e-h} = -\sum_{\vec q_1}\sum_{\vec K_1 \vec K_2} 2v(\vec q_1)D^+\binom{\vec K_1+\vec q_1}{-\vec K_2-\vec q_1}D\binom{\vec K_1}{-\vec K_2} \qquad , \quad (3.12)$$

is the only term which does not change the wave vector of the electron-hole pair. When we consider the interaction between the light wave and the electron-hole pair generated by the light wave, all terms except (3.12) will act to change the wave vector of the pair and thus remove it from the influence of the light wave. In other word, the effect of these terms is to destroy the coherence between the light wave and the electron-hole pair generated by the light wave. The interaction represented by (3.12) acts quitely differently, it does not destory the coherence. It acts to modify the physical features of the excitation process.

In fact, (3.12) is the interaction responsible for the formation of the exciton states, whereas the other terms describe the scattering between the "excitons" (here "exciton" is used in the broad sense, embracing both the continuous and discret excitonic states). In simple terms, we may describe the interaction term (3.12) as the interaction between the electron and its hole-partner. Therefore it is clear that (3.12) is only term we have incorporate into the coherent interaction between the light wave and the semiconductor, whereas the effects of the other terms are to be described in terms of the collision frequency ν_{p-p} refered to earlier.

There still remains one point to be clarified, namely the effect of the "space translation approximation". After performing the transformation (2.9),(3.3) becomes

$$v(q_1)\exp[i(\theta_{\vec{k}_1}+\theta_{\vec{k}_2}-\theta_{\vec{k}_2+\vec{q}_1}-\theta_{\vec{k}_1-\vec{q}_1})]\,a^+_{\vec{k}_1}a^+_{\vec{k}_2}a_{\vec{k}_2-\vec{q}_1}a_{\vec{k}_1+\vec{q}_1}, \tag{3.3'}$$

Similarly, (3.4) becomes

$$v(q_1)\exp[i(\varphi_{-\vec{k}_1-\vec{q}_1}+\varphi_{\vec{k}_2+\vec{q}_1}-\varphi_{-\vec{k}_1}-\varphi_{-\vec{k}_2})]\,b^+_{-\vec{k}_1-\vec{q}_1}b_{-\vec{k}_1}b^+_{-\vec{k}_2+\vec{q}_1}b_{-\vec{k}_2}, \tag{3.4'}$$

and (3.5) becomes

$$-2v(q_1)\exp[i(\theta_{\vec{k}_1}+\varphi_{\vec{k}_2+\vec{q}_1}-\theta_{\vec{k}_1+\vec{q}_1}-\varphi_{\vec{k}_2})]\,a^+_{\vec{k}_1}a_{\vec{k}_1+\vec{q}_1}b^+_{-\vec{k}_2+\vec{q}_1}b_{-\vec{k}_2}. \tag{3.5'}$$

Substituting the explicit expressions for $\theta_{\vec{k}}$, $\varphi_{-\vec{k}}$ in the above expressions, we find that (3.3') and (3.4') are identical to (3.3) and (3.4), whereas (3.5') differs from (3.5) in that the Fourier component $v(\vec{q})$ is replaced by

$$v'(\vec{q}) = -2v(\vec{q})\exp[-i\,\vec{q}\cdot\frac{e}{m_d c}\int^t \vec{A}(t')\,dt'] \tag{3.13}$$

As function in ordinary space, this means replacing the function $v(\vec{r})$ by

$$v(\vec{r}) = v(\vec{r}-\frac{e}{m_d c}\int^t A(t')\,dt')$$

where $m_d^{-1} = m_e^{-1} + m_h^{-1}$. The physical meaning of these results are self evident.

3.2 The equation of coherent interaction

It follows from the above discussion when the electron-electron interaction is taken into account, the Schrödinger equation describing the coherent interaction between the light wave and a semiconductor can be written approximately as:

$$i\hbar\frac{\partial\psi}{\partial t} = H_c\psi,$$

$$H_c = \sum_{\vec{k}}(E_c(\vec{k})a^+_{\vec{k}}a_{\vec{k}} - E_v(\vec{k})b^+_{-\vec{k}}b_{-\vec{k}}) + \sum_{\vec{q}_1}\sum_{\vec{k}_1} v'(\vec{q}_1)\cdot D^+\binom{\vec{k}_1+\vec{q}_1+\vec{q}}{-\vec{k}_1-\vec{q}_1}\cdot D\binom{\vec{k}_1+\vec{q}}{-\vec{k}_1} + \sum_{\vec{k}_1}\frac{e}{mc}\vec{A}\cdot\vec{P}_{cv}(\exp(i(\theta_{\vec{k}_1+\vec{q}_1}+\varphi_{-\vec{k}_1}))D^+\binom{\vec{k}_1+\vec{q}}{-\vec{k}_1} + \exp(-i(\theta_{\vec{k}_1+\vec{q}_1}+\varphi_{-\vec{k}_1}))D\binom{\vec{k}_1+\vec{q}}{-\vec{k}_1}), \tag{3.14}$$

where the first and third terms are the same as in (2.10), and the second term is electoron-electron interaction (3.12). In analogy with (2.13), we assume the wave function to be of the form:

$$\psi = \prod_{\vec{k}}(\alpha_{\vec{k}} + \beta_{\vec{k}}D^+\binom{\vec{k}+\vec{q}}{-\vec{k}})\psi_0 \tag{3.15}$$

In order to obtain equations governing $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$, we shall first derive the equations of motion for the operators $D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}$, $D\binom{\vec{k}+\vec{q}}{-\vec{k}}$, $a^+_{\vec{k}+\vec{q}}a_{\vec{k}+\vec{q}}$ and $b^+_{-\vec{k}}b_{-\vec{k}}$. In fact, making use of (3.8), we obtain readily:

$$i\hbar\frac{d}{dt}D^+\binom{\vec{k}+\vec{q}}{-\vec{k}} = [D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}, H_c] = -\hbar\omega_{\vec{q}}(\vec{k})D^+\binom{\vec{k}+\vec{q}}{-\vec{k}} - \sum_{\vec{q}_1} v'(\vec{q}_1)D^+\binom{\vec{k}+\vec{q}+\vec{q}_1}{-\vec{k}-\vec{q}_1}(1-a^+_{\vec{k}+\vec{q}}a_{\vec{k}+\vec{q}}-b^+_{-\vec{k}}b_{-\vec{k}}) - \Lambda_-(\vec{k})(1-a^+_{\vec{k}+\vec{q}}a_{\vec{k}+\vec{q}} - b^+_{-\vec{k}}b_{-\vec{k}});$$

$$i\hbar\frac{d}{dt}a^+_{\vec{k}+\vec{q}}a_{\vec{k}+\vec{q}} = \sum_{\vec{q}_1} v'(\vec{q}_1)[D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}D\binom{\vec{k}+\vec{q}-\vec{q}_1}{-\vec{k}+\vec{q}_1} - D^+\binom{\vec{k}+\vec{q}+\vec{q}_1}{-\vec{k}-\vec{q}_1}D\binom{\vec{k}+\vec{q}}{-\vec{k}}) + \Lambda_+D^+\binom{\vec{k}+\vec{q}}{-\vec{k}} - \Lambda_-D\binom{\vec{k}+\vec{q}}{-\vec{k}}; \tag{3.16}$$

where:

$$\hbar\omega_{\vec q}(\vec k) = E_c(\vec k+\vec q) - E_v(\vec k),$$

$$\Lambda_+ = \frac{e}{mc}\,\vec A\cdot\vec P_{cv}\,\exp\left(i\left(\theta_{\vec k+\vec q} + \varphi_{-\vec k}\right)\right),$$

$$\Lambda_- = \frac{e}{mc}\,\vec A\cdot\vec P_{cv}\,\exp\left(-i\left(\theta_{\vec k+\vec q} + \varphi_{-\vec k}\right)\right). \qquad (3.17)$$

Completely analogous equations can be written down for $D\begin{pmatrix}\vec k+\vec q\\ -\vec k\end{pmatrix}$ and $b^+_{-\vec k}\, b_{-\vec k}$.

Next we take the average of these equations over the wave function (3.15). In doing so, we introduce the following notat--ions:

$$\langle\psi|D^+\begin{pmatrix}\vec k+\vec q\\ -\vec k\end{pmatrix}|\psi\rangle = \alpha_{\vec k}\,\beta^*_{\vec k} = d^+_{\vec q}(\vec k),$$

$$\cancel{\langle\psi|D\begin{pmatrix}\vec k+\vec q\\ -\vec k\end{pmatrix}|\psi\rangle = \alpha^*_{\vec k}}\,\beta_{\vec k} = d^-_{\vec q}(\vec k),$$

$$\langle\psi|a^+_{\vec k+\vec q}\,a_{\vec k+\vec q}|\psi\rangle = |\beta_{\vec k}|^2 = m_{\vec q}(\vec k), \qquad (3.18)$$

and make use of the relations:

$$\langle\psi|D^+\begin{pmatrix}\vec k_1+\vec q\\ -\vec k_1\end{pmatrix}a^+_{\vec k_2+\vec q}\,a_{\vec k_2+\vec q}|\psi\rangle = \alpha_{\vec k_1}\beta^*_{\vec k_1}\,|\beta_{\vec k_2}|^2,$$

$$\langle\psi|D^+\begin{pmatrix}\vec k_1+\vec q\\ -\vec k_1\end{pmatrix}b^+_{-\vec k_2}\,b_{-\vec k_2}|\psi\rangle = \alpha_{\vec k_1}\beta^*_{\vec k_1}\,|\beta_{\vec k_2}|^2 \quad ,$$

etc. Then we find

$$i\hbar\frac{d}{dt}m_{\vec q}(\vec k) = \Lambda_+ d^+_{\vec q}(\vec k) - \Lambda_- d^-_{\vec q}(\vec k) + d^+_{\vec q}(\vec k)\sum_{\vec q_1} v'(\vec q_1)$$
$$\cdot\, d^-_{\vec q}(\vec k-\vec q_1) - d^-_{\vec q}(\vec k)\sum_{\vec q_1} v'(\vec q_1)\, d^+_{\vec q}(\vec k+\vec q_1),$$

$$i\hbar\frac{d}{dt}d^+_{\vec q}(\vec k) = -\hbar\omega_{\vec q}(\vec k)\,d^+_{\vec q}(\vec k) - \Lambda_-(1-2m_{\vec q}(\vec k))$$
$$-(1-2m_{\vec q}(\vec k))\sum_{\vec q_1} v'(\vec q_1)\,d^+_{\vec q}(\vec k+\vec q_1),$$

$$i\hbar\frac{d}{dt}d^-_{\vec q}(\vec k) = \hbar\omega_{\vec q}(\vec k)\,d^-_{\vec q}(\vec k) + \Lambda_+(1-2m_{\vec q}(\vec k))$$
$$+(1-2m_{\vec q}(\vec k))\sum_{\vec q_1} v'(\vec q_1)\,d^-_{\vec q}(\vec k-\vec q_1). \qquad (3.19)$$

simplifying notations by introducing

$$\hat V\, d^-_{\vec q}(\vec k) = \sum_{\vec q_1} v'(\vec q_1)\, d^-_{\vec q}(\vec k-\vec q_1),$$

$$\hat V^*\, d^+_{\vec q}(\vec k) = \sum_{\vec q_1} v'(\vec q_1)^*\, d^+_{\vec q}(\vec k-\vec q_1) = \sum_{\vec q_1} v'(\vec q_1)\, d^+_{\vec q}(\vec k+\vec q_1),$$

equations (3.19) can be rewritten as:

$$i\hbar\frac{d}{dt}m_{\vec q}(\vec k) = \Lambda_+ d^+_{\vec q}(\vec k) - \Lambda_- d^-_{\vec q}(\vec k) + d^+_{\vec q}(\vec k)\,\hat V\, d^-_{\vec q}(\vec k) - d^-_{\vec q}(\vec k)\,\hat V^*\, d^+_{q}(\vec k),$$

$$i\hbar\frac{d}{dt}d^+_{q}(\vec k) = -\hbar\omega_{\vec q}(\vec k)\,d^+_{\vec q}(\vec k) - \Lambda_-(1-2m_{\vec q}(\vec k)) - (1-2m_{\vec q}(\vec k))\,\hat V^*\, d^+_{\vec q}(\vec k),$$

$$i\hbar\frac{d}{dt}d^-_{\vec q}(\vec k) = \hbar\omega_{\vec q}(\vec k)\,d^-_{\vec q}(\vec k) + \Lambda_+(1-2m_{\vec q}(\vec k)) + (1-2m_{\vec q}(\vec k))\,\hat V\, d^-_{\vec q}(\vec k).$$

$$(3.19')$$

From (3.19'), we can show that

$$\frac{d}{dt}\left[\left(\frac{1-m_{\vec q}(\vec k)}{2}\right)^2 + d^+_{\vec q}(\vec k)\,d^-_{\vec q}(\vec k)\right] = 0 \quad . \qquad (3.20)$$

Substituting (3.20) into (3.18), we obtain:

$$\frac{d}{dt}\left(|\alpha_{\vec k}|^2 + |\beta_{\vec k}|^2\right) = 0$$

or

$$|\alpha_{\vec k}|^2 + |\beta_{\vec k}|^2 = 1 \qquad (3.21)$$

$$\left[\frac{1}{2}(1-m_{\vec{q}}(\vec{k}))\right]^2 + d^+_{\vec{q}}(\vec{k})\, d^-_{\vec{q}}(\vec{k}) = \frac{1}{4} . \qquad (3.22)$$

(3.19'),(3.21),(3.22) are the equations governing the electronic state excited by the light pulse. We shall proceed with further transformations of these equations so as to render their physical implications more explicit.

3.3 The coherent state of the electron-hole pair

We define

$$\frac{1}{2}\left(1-m_{\vec{q}}(\vec{k})\right) = z_{\vec{k}} ,$$

$$\delta_{\vec{k}} = d^-_{\vec{q}}(\vec{k}) \Big/ \left(\tfrac{1}{2} + z_{\vec{k}}\right) ,$$

$$\delta^*_{\vec{k}} = d^+_{\vec{q}}(\vec{k}) \Big/ \left(\tfrac{1}{2} + z_{\vec{k}}\right) \qquad (3.23)$$

or

$$d^-_{\vec{q}}(\vec{k}) = \frac{\delta_{\vec{k}}}{1+|\delta_{\vec{k}}|^2} , \quad d^+_{\vec{q}}(\vec{k}) = \frac{\delta^*_{\vec{k}}}{1+|\delta_{\vec{k}}|^2} . \qquad (3.24)$$

Using (3.19'), we obtain the following equation for $\delta_{\vec{k}}$:

$$i\hbar\frac{d}{dt}\delta_{\vec{k}} = \hbar\omega_{\vec{q}}(\vec{k})\delta_{\vec{k}} + (\Lambda_+ - \Lambda_- \delta_{\vec{k}}^2) + \hat{V}\frac{\delta_{\vec{k}}}{1+|\delta_{\vec{k}}|^2} - \delta_{\vec{k}}^2\, \hat{V}^* \frac{\delta^*_{\vec{k}}}{1+|\delta_{\vec{k}}|^2} . \qquad (3.25)$$

From (3.8), we have

$$\left[D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}\right]^2 = 0 \qquad (3.26)$$

and

$$\alpha_{\vec{k}} + \beta_{\vec{k}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}} = \alpha_{\vec{k}} \exp\left[\frac{\beta_{\vec{k}}}{\alpha_{\vec{k}}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}\right] . \qquad (3.27)$$

Hence the wave function (3.15) can be rewritten as

$$\psi = \left(\prod_{\vec{k}} \alpha_{\vec{k}}\right) \exp\left[\sum_{\vec{k}} \frac{\beta_{\vec{k}}}{\alpha_{\vec{k}}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}\right] \psi_0 . \qquad (3.28)$$

From (3. 24) and (3.21), we have

$$\delta_{\vec{k}} = \beta_{\vec{k}} / \alpha_{\vec{k}} . \qquad (3.29)$$

Therefore we find for the wave function of the coherently excited system:

$$\psi \sim \exp\left[\sum_{\vec{k}} \delta_{\vec{k}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}\right] \psi_0 \qquad (3.30)$$

the symbol "$\sim$" signifies that ψ not normalized. If we introduce

$$\sum_{\vec{k}} |\delta_{\vec{k}}|^2 = |\lambda|^2$$

and

$$\frac{\delta_{\vec{k}}}{\lambda} = f_{\vec{k}} ,$$

Eq (3.30) can be written as

$$\psi \sim \exp\left[\lambda \sum_{\vec{k}} f_{\vec{k}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}\right] \psi_0 . \qquad (3.31)$$

We define the following pair of opertors:

$$E^+ = \sum_{\vec{k}} f_{\vec{k}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}} ,$$

$$E = \sum_{\vec{k}} f^*_{\vec{k}} D\binom{\vec{k}+\vec{q}}{-\vec{k}} . \qquad (3.32)$$

Using Eq. (3.8), we can show that:

$$[E, E^+] \doteq 1 + \sum_{\vec{k}} |f_{\vec{k}}|^2 (a^+_{\vec{k}+\vec{q}} a_{\vec{k}+\vec{q}} + b^+_{-\vec{k}} b_{-\vec{k}}) \qquad (3.33)$$

and

$$\langle E^+\psi_0 | \sum_{\vec{k}} a^+_{\vec{k}} a_{\vec{k}} | E^+\psi_0 \rangle = 1,$$

$$\langle E^{+2}\psi_0 | \sum_{\vec{k}} a^+_{\vec{k}} a_{\vec{k}} | E^{+2}\psi_0 \rangle = 2 - \sum_{\vec{k}} |f_{\vec{k}}|^4, \qquad (3.34)$$

or, if for the state ψ, $\langle \psi | a^+_{\vec{k}} a_{\vec{k}} | \psi \rangle = n_{\vec{k}}$, then

$$\langle E^+\psi | \sum_{\vec{k}} a^+_{\vec{k}} a_{\vec{k}} | E^+\psi \rangle = 1 + \sum_{\vec{k}} n_{\vec{k}} - \sum_{\vec{k}} |f_{\vec{k}}|^2 n_{\vec{k}} \quad . \qquad (3.35)$$

It follows that when the number of excited particles is sma-ll, the operators E and E^+ satisfy approximately

$$[E, E^+] \approx 1 \quad . \qquad (3.36)$$

Thus E and E^+ can be regard as the annihilation and creat-ion operators for a Bose field. The wave function (3.31) is ju-st of the usual form of a coherent state of a Bose field [19]. λ can be understand as the amplitude of the Bose field, $|\lambda|^2$ is the number of the "Bosons". Indeed, $E^+\psi$ describes the excitation of one electron-hole pair with the wave function $\sum_{\vec{k}} f_{\vec{k}} D^+\binom{\vec{k}+\vec{q}}{-\vec{k}}\psi_0$. this is a electron-hole pair with wave vector $\vec{q}$. If we neglect $\sum_{\vec{k}} |f_{\vec{k}}|^4$ (its order is approximately $\frac{1}{V}|f_{\vec{k}}|^2$, V is the volum-e of the system), $E^{+2}\psi$ represents the excitation of two such electron-hole pairs, and similarly $E^{+n}\psi$ represents approximat-ely the excitation of n such pairs. Hence the wave function (3.31) represents a coherent assembly variously excited states as discribed above and $|\lambda|^2$ gives the average number of excited

pairs. However, as Eq(3.36) is only approximate, the above stat-ement is valid only in the limit of low density of excited pa-irs. As we have seen, E and E^+ are not Bose operators in the rigorous sense. This is an expression of the fact that the electron-hole pairs are constituted of two "fermions" and do not behave as bosons in the strict sense.

We may be discribe the situation as follow: $\delta_{\vec{k}}$ is the amp-litude of the electron - hole wave and Eq(3.25) governs the excitation of the wave. It is a non-linear equation. However when $|\delta_{\vec{k}}|^2 \ll 1$, it can be approximated by the linear eq-uation:

$$i\hbar \frac{d}{dt} \delta_{\vec{k}} = \hbar\omega_{\vec{q}}(\vec{k}) \delta_{\vec{k}} + \hat{V} \delta_{\vec{k}} + \Lambda_+ \quad . \qquad (3.37)$$

It is to the same approximation that we can regard the electron - hole pairs as bosons and equation (3.37) is just the equation of exciton polarization wave introduced in the reference [10]. When $|\delta_{\vec{k}}|$ increases, the non- linear charcter of Eq(3.25) can no longer be neglected, and the non- bose property of the pairs will become manifest. The nonlinearity of Eq(3.25) is in fact a manifestation of the non-bose property of the excited pairs.

It follows from (2.12) that the terms Λ_+ and $\Lambda_- = \Lambda_+^*$ in (3.25) can be expanded into various harmonics of the incident light frequency:

$$\Lambda_+ = \sum_m G_m e^{im\omega t} \quad . \qquad (3.38)$$

As in § 2-3, the dominant term is that with $m\omega \approx \omega_{\vec{q}}(\vec{k})$. Leaving out other terms, Eq(3.25) becomes then an equation des-cribing the electron- hole pairs resonantly excited by m photons.

For the sake of defineteness, let us assume $m=1$. Eq (3.25) then becomes:

$$i\hbar\frac{d}{dt}\delta_{\vec{k}} = \hbar\omega_{\vec{q}}(\vec{k})\delta_{\vec{k}} + \left(G_1 e^{-i\omega t} - G_1^* e^{i\omega t}\delta_{\vec{k}}^2\right) + \hat{V}\left(\frac{\delta_{\vec{k}}}{1+|\delta_{\vec{k}}|^2}\right) - (\delta_{\vec{k}})^2\hat{V}^*\left(\frac{\delta_{\vec{k}}^*}{1+|\delta_{\vec{k}}|^2}\right) \quad . \qquad (3.39)$$

The linear approximation is:

$$i\hbar\frac{d}{dt}\delta_{\vec{k}} = [\hbar\omega_{\vec{q}}(\vec{k}) + \hat{V}]\delta_{\vec{k}} + G_1 e^{-i\omega t} \qquad (3.40)$$

In this equation, $\hbar\omega_{\vec{q}}(\vec{k}) + \hat{V}$ is just the Hamiltonian of the exciton in the Bloch representation. Thus (3.40) can be written as:

$$\left[i\hbar\frac{d}{\partial t} - H_{ex}\right]\delta_{\vec{k}} = G_1 e^{-i\omega t} .$$

This equation can be solved by the method of Coulomb Green function[20]. From the defintion of the Green function, it is not difficult to see that this is equivalent to solving by time dependent perturbation theory.

In contract with the linear approximation, there is another case in which $|\delta_{\vec{k}}|^2 >> 1$. Then let $\delta_{\vec{k}}^{-1} = f_{\vec{k}}$, Eq(3.39) becomes approximately as

$$i\hbar\frac{d}{dt}f_{\vec{k}} = -[\hbar\omega_{\vec{q}}(\vec{k}) - \hat{V}^*]f_{\vec{k}} + G_1^* e^{i\omega t} .$$

It has a form similar to (3.40), but the signs of potential $\hat{V}^*$ are reversal. Physically this is easily understood.

Generally, we can rewrite (3.39) as follows:

$$i\hbar\frac{\partial}{\partial t}\delta_{\vec{k}} = H_{ex}\delta_{\vec{k}} + G_1 e^{-i\omega t} - G_1^* e^{i\omega t}\delta_{\vec{k}}^2 + \Delta H_{ex}\delta_{\vec{k}} ,$$

$$\Delta H_{ex}\delta_{\vec{k}} = -\hat{V}\frac{\delta_{\vec{k}}|\delta_{\vec{k}}|^2}{1+|\delta_{\vec{k}}|^2} - \delta_{\vec{k}}^2\hat{V}^*\frac{\delta_{\vec{k}}}{1+|\delta_{\vec{k}}|^2} . \qquad (3.41)$$

If terms higher than $\delta_{\vec{k}}^2$ are neglected, (3.41) becomes

$$\left[i\hbar\frac{\partial}{\partial t} - H_{ex}\right]\delta_{\vec{k}} = G_1 e^{-i\omega t} - G_1^* e^{i\omega t}\delta_{\vec{k}}^2 . \qquad (3.42)$$

The first term on rght hand side represents "excitation", the second term represents "saturation".

3.4 The coherent excitation of exciton

In the theory of interband transitions induced by light, it is well known that, for transitions with $\hbar\omega > E_g$, consideration of the interaction between the electron and hole makes only a small difference. this difference becomes progressively smaller with increasing value for $(\hbar\omega - E_g)$. But in the case of $\hbar\omega < E_g$, as the interaction between electron and hole can give rise to discrete energy levels in the forbidden gap, the consideration of this interaction will make a qualitative difference. In this section, we shall specifically discuss this case.

In (3.41) let us introduce

$$\delta_{\vec{k}} = \rho_{\vec{k}} e^{i\omega t} , \qquad (3.43)$$

we obtain

$$i\hbar\frac{\partial}{\partial t}\rho_{\vec{k}} = (H_{ex} - \hbar\omega]\rho_{\vec{k}} + (G_1 - G_1^*\rho_{\vec{k}}^2) + \Delta H_{ex}(\rho_{\vec{k}}) . \qquad (3.44)$$

Let $f_{i,\vec{k}}$ be the exciton wave function:

$$H_{ex}f_{i,\vec{k}} = E_{ex,i}f_{i,\vec{k}} . \qquad (3.45)$$

Let us assume that

$$\rho_{\vec{k}} = \sum_i \ell_i f_{i,\vec{k}} , \qquad (3.46)$$

and substitute into (3.44); after using the orthogo-nality of the functions $f_{i,\vec{k}}$, we obtain:

$$i\hbar\frac{\partial}{\partial t}\ell_i = \hbar\Delta\omega_i \ell_i + A_i - \sum_{\vec{k}} f^*_{i,\vec{k}} G^*_1 \Big(\sum_j \ell_j f_{j,\vec{k}}\Big)^2 + \sum_{\vec{k}} f^*_{i,\vec{k}} \Delta H_{ex}\Big(\sum_j \ell_j f_{j,\vec{k}}\Big) , \qquad (3.47)$$

where

$$\Delta\omega_i = \frac{1}{\hbar}\left(E_{ex,i} - \hbar\omega\right),$$

$$A_i = \sum_{\vec{k}} f^*_{i,\vec{k}} G_1 .$$

Suppose that $\hbar\omega$ is very near from the lowest discrete ener-gy level ($|S$ level) of the exciton. As in this case, the proba-bility of exciting other exciton states is very small, we have approximately

$$\rho_{\vec{k}} \approx \ell_1 f_{1,\vec{k}} . \qquad (3.48)$$

Then it follows that

$$i\hbar\frac{d\ell_1}{dt} = \hbar\Delta\omega_1 \ell_1 - A_1 - \Big(\sum_{\vec{k}} G^*_1 |f_{1,\vec{k}}|^2 f_{1,\vec{k}}\Big)\ell_1^2 + \sum_{\vec{k}} f^*_{1,\vec{k}} \Delta H_{ex}(\ell_1 f_{1,\vec{k}}) . \qquad (3.49)$$

From (3.41), we find

$$\sum_{\vec{k}} f^*_{1,\vec{k}} \Delta H_{ex}(\ell_1 f_{1,\vec{k}}) = \alpha(\ell_1)|\ell_1|^2 \ell_1 , \qquad (3.50)$$

where

$$\alpha(\ell_1) = -\sum_{\vec{k}} f^*_{\vec{k}} \hat{v} \frac{|f_{1,\vec{k}}|^2 f_{1,\vec{k}}}{1+|\ell_1|^2|f_{1,\vec{k}}|^2} + |f_{1,\vec{k}}|^2 f_{1,\vec{k}} \hat{v}^* \frac{f^*_{1,\vec{k}}}{1+|\ell_1|^2|f_{1,\vec{k}}|^2} ,$$

$\alpha(\ell_1)$ being a function of $|\ell_1|^2$. Introducing

$$\sigma_1 = \sum_{\vec{k}} \frac{G^*_1 |f_{1,\vec{k}}|^2 f_{1,\vec{k}}}{A_1} , \qquad (3.51)$$

we then have

$$i\hbar\frac{d\ell_1}{dt} = \left(\hbar\Delta\omega_1 + \alpha(\ell_1)|\ell_1|^2\right)\ell_1 + A_1\left(1 - \sigma_1 \ell_1^2\right) . \qquad (3.52)$$

Generally σ_1 is complex, that is $\sigma_1 = |\sigma_1| e^{i\varphi}$. After introdu-cing $\tilde{\ell}_1 = \ell_1 e^{i\frac{\varphi}{2}}$ and $\tilde{A}_1 = A_1 e^{i\frac{\varphi}{2}}$, Eq(3.52) can be rewritten as

$$i\hbar\frac{d\tilde{\ell}_1}{dt} = \left(\hbar\Delta\omega_1 + \alpha(\ell_1)|\tilde{\ell}_1|^2\right)\tilde{\ell}_1 + \tilde{A}_1\left(1 - |\sigma_1|\,\tilde{\ell}_1^2\right) .$$

Introducing a new variable defined by

$$i\sqrt{|\sigma_1|}\,\tilde{\ell}_1 = u , \qquad (3.53)$$

(3.52) becomes a Ricatti equation:

$$\frac{du}{dt} = -i\,\delta\omega_1\, u + \frac{1}{2} h\left(1 + u^2\right) , \qquad (3.54)$$

where

$$h = \frac{2\tilde{A}_1\sqrt{|\sigma_1|}}{\hbar} = 2A_1 e^{i\frac{\varphi}{2}} \frac{\sqrt{|\sigma_1|}}{\hbar}$$

and

$$\delta\omega_1 = \Delta\omega_1 + \frac{1}{\hbar}\alpha(\ell_1)|\ell_1|^2 . \qquad (3.55)$$

Eq(3.54) is related to the equation of motion for a spin in an external field (Bloch equation), and can hence also be related to a two- level system [24]. In fact, on introducing new variables :

$$m_x = \frac{1}{2}\cdot\frac{u + u^*}{1+|u|^2} , \qquad m_y = \frac{1}{2i}\cdot\frac{u - u^*}{1+|u|^2} ,$$

$$m_z = \frac{1}{2}\cdot\frac{1-|u|^2}{1+|u|^2} ; \qquad (3.56)$$

or

$$u = \frac{m_x + i m_y}{\frac{1}{2} + m_z} , \qquad u^* = \frac{m_x - i m_y}{\frac{1}{2} + m_z} , \qquad (3.57)$$

we obtain as equivalent to (3.54) the **following** equations

$$\frac{d\vec{m}}{dt} = \vec{m} \times \vec{H}, \tag{3.58}$$

where

$$\vec{m} = (m_x, m_y, m_z), \quad \vec{H} = (0, -k, \delta\omega_1) \tag{3.59}$$

Now Eq (3.58) is similar to the form of the equation of motion for a two - level atom with the level separation $\hbar(\omega + \delta\omega_1)$ moving in an external field with frequence ω.

From (3.23), we find the average number of excited excitons:

$$N_{ex} = \sum_{\vec{k}} |\beta_{\vec{k}}|^2 = \sum_{\vec{k}} \frac{|\delta_{\vec{k}}|^2}{1+|\delta_{\vec{k}}|^2} = |\ell_1|^2 \sum_{\vec{k}} \frac{|f_{1\vec{k}}|^2}{1+|\ell_1|^2 |f_{1\vec{k}}|^2}. \tag{3.60}$$

To terms of order $|\delta_{\vec{k}}|^2$, we have approximately

$$N_{ex} \approx |\ell_1|^2 \tag{3.61}$$

From (3.53) and (3.56), we obtain

$$|\ell_1|^2 = \frac{1}{|\sigma_1|} \frac{\frac{1}{2} - m_z}{\frac{1}{2} + m_z}. \tag{3.62}$$

and, as we know, $(\frac{1}{2} - m_z)/(\frac{1}{2} + m_z)$ represents the number of excited "spin" in the two - level system.

Hence, the problem of excitation of excitons by light wave is reduced to that of excitation of a "two - level system " with the difference of energy level

$$E_{ef} = E_{ex_1} + \alpha(\ell_1)|\ell_1|^2 \tag{3.63}$$

and density $1/V|\sigma_1|$ (V is the volume of system). From (3.61) the number of excitons approximates to the number of excited " atoms " and the energy level difference is dependent on the number of excitons.

Introducing the wave function of 1s state

$$f_{1\vec{k}} = 2^3 \pi^{1/2} \frac{a^{3/2}}{(1+(ak)^2)^2} \frac{1}{\sqrt{V}}, \tag{3.64}$$

where a is the " Bohr radius" of the exciton. We obtain the "equivalent" density of two-level atom:

$$N_{ex} = \frac{1}{V|\sigma_1|} = \frac{2}{7\pi} \cdot \frac{1}{a^3}. \tag{3.65}$$

If neglecting the dependence of $\vec{P}_{cv}$ on $\vec{k}$, we have

$$k = \sqrt{\frac{7}{2}} \frac{e}{mc\hbar} (\vec{A}_0 \cdot \vec{P}_{cv}) J_0(\eta), \tag{3.66}$$

where η has been defined in §2.2. Substituting (3.64) into (3.60), we have

$$N_{ex} = \frac{64}{7\pi^2} \frac{1}{a^3} |u|^2 \int_0^\infty \frac{x^2 dx}{(1+x^2)^4 + \frac{128}{7}|u|^2},$$

$$|u|^2 = \frac{\frac{1}{2} - m_z}{\frac{1}{2} + m_z}, \tag{3.67}$$

when $\frac{128}{7}|u|^2$ is small, we have approximately

$$N_{ex} \approx \frac{2}{7\pi} \cdot \frac{1}{a^3} \cdot \frac{\frac{1}{2} - m_z}{\frac{1}{2} + m_z}. \tag{3.68}$$

The physical meaning of Eq(3.68) is self-evident.

Let us consider the case of excitons excited by a constant light field. If we take the approximation: $\delta\omega_1 = \Delta\omega_1$, the solution of (3.58) is easily obtained. For the initial condition that $m_z = 1/2$ and $m_x = m_y = 0$ at $t = 0$, $\vec{m}(t)$ is given by

$$\vec{m}(t) = \left(\frac{k}{2\omega_r} \sin\omega_r t, \; \frac{k\,\Delta\omega_1}{2\omega_r^2}(\cos\omega_r t - 1), \; \frac{1}{2\omega_r^2}(k^2 \cos\omega_r t + \Delta\omega_1^2) \right), \tag{3.69}$$

where $\omega_r = \sqrt{k^2 + \Delta\omega_1^2}$ is the "frequency of the Rabi precession ". Introducing $\eta = k/\Delta\omega_1$, we have

$$|u|^2 = \frac{\eta^2 (1 - \cos\omega_r t)}{2 + \eta^2 (1 + \cos\omega_r t)}. \tag{3.70}$$

The time evolution is shown in Fig (3. 5). The figure shows that the system initially absorbs energy from the light wave, and the density of excitons increases; at time $t = \pi/\omega_r$, the density reaches its maxium value η^2, which depends on the off-resonance frequency $\Delta\omega_1$ and the amplitude of the external field; then, the system begins to lose energy to the light wave by stimulated emission, and the density of excitons decreases; at $t = 2\pi/\omega_r$, the system returns to the ground state and the density of exciton reduces to zero. Then a similar cycle begins.

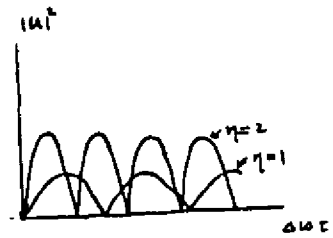

Fig 3.5

From the above discussion, it may be seen that the Rabi frequency for $\Delta\omega_1 = 0$ ($\omega_{r,0} = h$) is a "characteristic" frequency in the sense that if it becomes comparable to the exciton ionization energy (in frequency unit), the discrete exciton energy levels lose their meaning.

IV. The propagation of coherent light pulse

4.1 The equation of coherent propagation

In the first place, we have to establish the relation between the current density $\vec{J}$ of the system and the amplitude $\delta_{\vec{k}}$ of excitation, where $\vec{J}$ is the average value of the operator $\sum_i \frac{e}{m}(\vec{p}_i + \frac{e}{c}\vec{A}(\vec{r}_i))$ (summation is over all electrons) over the wave function of the system. Making use of (3.18) and (3.24), and noting that the energy of photon, $\hbar\omega$, is much greater than the exciton binding energy, we obtain

$$\vec{J} = \vec{J}_t + \vec{J}_p . \tag{4.1}$$

$\vec{J}_t$ is the polarization current corresponding to intraband motion :

$$\vec{J}_t = \frac{e^2}{m_d c}\frac{1}{V}\sum_{\vec{k}}\frac{|\delta_{\vec{k}}|^2}{1+|\delta_{\vec{k}}|^2}\vec{A} , \tag{4.2}$$

where V is the volume of the system, $\frac{1}{m_d} = \frac{1}{m_e} + \frac{1}{m_h}$; $\vec{J}_p$ is polarization current corresponding to interband transition :

$$\vec{J}_p = \frac{e}{mV}\sum_{\vec{k}}\vec{P}_{cv}\frac{\delta_{\vec{k}} + \delta_{\vec{k}}^*}{1+|\delta_{\vec{k}}|^2} . \tag{4.3}$$

Assuming that the light pulses travels in the z - direction

$$\vec{A} = \vec{A}_0 \sin(\omega t - qz + \varphi) , \tag{4.4}$$

where $\vec{A}_0$ and φ are slowly varying functions of z and t. Without loss of generality, we may assume that φ is only a function of z. Substituting (4.4) into the electromagnetic wave equation :

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} , \tag{4.5}$$

and make use of the slowly varying character of A_0 and φ, and that $\omega^2 = c^2 q^2$, we obtain approximately:

$$-\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \approx \left(\frac{\omega}{c^2}\frac{\partial A_0}{\partial t} + \frac{\omega}{c}\frac{\partial A_0}{\partial z} - i\frac{\omega}{c}\frac{\partial \varphi}{\partial z}A_0\right) e^{-i(\omega t - qz + \varphi)}$$

$$+ c.c.$$

Let $\widetilde{A}_0 = A_0 e^{-i\varphi}$, then we have

$$-\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \approx \frac{\omega}{c}\left(\frac{\partial \widetilde{A}_0}{\partial z} + \frac{1}{c}\frac{\partial \widetilde{A}_0}{\partial t}\right)\exp[-i(\omega t - qz)]$$

$$+ c.c. \qquad (4.6)$$

Substituting (4.1) and (4.6) into (4.5), and assuming that the frequencies involved in $\widetilde{A}_0$ is much lower than ω, we have

$$\frac{\partial \widetilde{A}_0}{\partial z} + \frac{1}{c}\frac{\partial \widetilde{A}_0}{\partial t} = i\frac{2\pi e^2}{m_d c\omega}\frac{1}{V}\sum_{\vec{k}}\frac{|\delta_{\vec{k}}|^2}{1+|\delta_{\vec{k}}|^2}\widetilde{A}_0$$
$$+ \frac{4\pi e}{m\omega}\cdot\frac{1}{V}\cdot\sum_{\vec{k}} p_{cv}\frac{\delta_{\vec{k}}}{1+|\delta_{\vec{k}}|^2} e^{-iqz+i\omega t} \qquad (4.7)$$

Now, we shall generalize the equation for $\delta_{\vec{k}}$ so as to take account of space - propagation. rom the physical meaning of $\delta_{\vec{k}}$ and (3.25), (3.39), it can be seen that, with a monochromatic light of wave vector $\vec{q}$, $\delta_{\vec{k}}$ will vary with z as e^{-iqz}. In (3.39), letting $\delta_{\vec{k}} = \rho_{\vec{k}} e^{-i\omega t}$ and using the concrete expression of $\vec{A}$, we obtain:

$$\frac{\partial \rho_{\vec{k}}}{\partial t} = -i\Delta\omega_q(\vec{k})\rho_{\vec{k}} - i\mathcal{E} + i\mathcal{E}^*\rho_{\vec{k}}^2 + \left(-i\frac{1}{\hbar}\hat{V}\left(\frac{\rho_{\vec{k}}}{1+|\rho_{\vec{k}}|^2}\right)\right.$$
$$\left. + i\frac{1}{\hbar}\rho_{\vec{k}}^2\hat{V}^*\left(\frac{\rho_{\vec{k}}^*}{1+|\rho_{\vec{k}}|^2}\right)\right), \qquad (4.8)$$

where (assuming $\vec{A}_0 /\!/ X$ axis)

$$\mathcal{E} = i\frac{e}{2mc\hbar}(p_{cv})_x \widetilde{A}_0 e^{iqz} J_0(\eta), \qquad (4.9)$$

$$\Delta\omega_q(\vec{k}) = \omega_{q}(\vec{k}) - \omega \quad . \qquad (4.10)$$

It is clear that $\rho_{\vec{k}} \sim e^{iqz}$. Next we consider

$$\Delta\omega_q(\vec{k}) = \frac{1}{\hbar}\left(E_g + \frac{\hbar^2}{2m_e}(\vec{k}+\vec{q})^2 + \frac{\hbar^2}{2m_h}\vec{k}^2 - \hbar\omega\right)$$
$$= \frac{1}{\hbar}\left(E_g + \frac{\hbar^2}{2m_d}\vec{B}^2 - \hbar\omega\right) + \frac{\hbar}{2M}\vec{q}^2, \qquad (4.11)$$

where $\vec{B} = \vec{k} + \frac{m_h}{m_e+m_h}\vec{q}$, $M = m_e + m_h$, and $\frac{1}{m_d} = \frac{1}{m_e} + \frac{1}{m_h}$. Clearly $\frac{\hbar^2}{2M}\vec{q}^2$ represents the kinetic energy of the electron - hole pair, $\frac{\hbar^2}{2m_d}\vec{B}^2$ represents the kinetic energy for the relative motion of electron and hole. As eventually we have to sum $\vec{k}$ over the whole $\vec{k}$ - space, therefore, we can repeal $\vec{k}$ by $\vec{B}$ and obtain

$$\Delta\omega_q(\vec{k}) = \Delta\omega(\vec{B},\vec{q}) = \frac{1}{\hbar}\left(E_g - \hbar\omega + \frac{\hbar^2}{2m_d}\vec{B}^2\right) + \frac{\hbar}{2M}\vec{q}^2.$$

Accordingly , (4.8) can be rewritten as

$$\frac{\partial \rho_{\vec{k}}}{\partial t} = -i\Delta\omega(\vec{B},-i\vec{\nabla})\rho_{\vec{k}} - i\mathcal{E} + i\mathcal{E}^*\rho_{\vec{k}}^2 + \left(-i\frac{1}{\hbar}\hat{V}\frac{\rho_{\vec{k}}}{1+|\rho_{\vec{k}}|^2}\right.$$
$$\left. + i\frac{1}{\hbar}\rho_{\vec{k}}^2\hat{V}^*\frac{\rho_{\vec{k}}^*}{1+|\rho_{\vec{k}}|^2}\right), \qquad (4.12)$$

where $-i\vec{\nabla} = \left(-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z}\right)$. Now considering the case that A_0 is a slowly varying function of z and t, we may assume

$$\rho_{\vec{k}} = \rho_{0\vec{k}} e^{iqz} \qquad (4.13)$$

where $\rho_{0\vec{k}}$ is also aslowly varying function of z and t. Substituting into (4.12) and taking account of the slowly varying condition, we obtain

$$\frac{\partial \rho_{0\vec{k}}}{\partial t} + u_{\vec{q}} \frac{\partial \rho_{0\vec{k}}}{\partial z} = -i\Delta\omega(\vec{k},\vec{q})\rho_{0\vec{k}} - i\mathcal{E}_0 + i\mathcal{E}_0^* \rho_{0\vec{k}}^2$$

$$+ \left(-i\frac{1}{\hbar}\hat{V}\frac{\rho_{0\vec{k}}}{1+|\rho_{0\vec{k}}|^2} + i\frac{1}{\hbar}\rho_{0\vec{k}}^2 \hat{V}^* \frac{\rho_{0\vec{k}}^*}{1+|\rho_{0\vec{k}}|^2}\right), \qquad (4.14)$$

where

$$\mathcal{E}_0 = \mathcal{E} e^{-iqz} J_0 = i\frac{e}{2mc\hbar}\vec{p}_{cv}\cdot\tilde{\vec{A}}_0 J_0$$

and

$$u_{\vec{q}} = \frac{\hbar}{M}\vec{q}, \qquad (4.15)$$

which is the propagation velocity of the electron - hole pair. written in term of $\rho_{0\vec{k}}$ instead of $\delta_{\vec{k}}$, Eq(4.7) become:

$$\frac{\partial \tilde{\vec{A}}_0}{\partial z} + \frac{1}{c}\frac{\partial \tilde{\vec{A}}_0}{\partial t} = i\frac{2\pi e^2}{m_d c\omega}\frac{1}{V}\sum_{\vec{k}}\frac{|\rho_{0\vec{k}}|^2}{1+|\rho_{0\vec{k}}|^2}\tilde{\vec{A}}_0 + \frac{4\pi e}{m\omega}\cdot\frac{1}{V}\cdot$$

$$\sum_{\vec{k}}\vec{p}_{cv}\frac{\rho_{0\vec{k}}}{1+|\rho_{0\vec{k}}|^2}, \qquad (4.16)$$

The equation (4.14) and (4.16) are the equation of propagation for a coherent light pulse. They correspond to the usual Maxwell - Bloch equation for a two - level system, but are muon complex.

4.2 Self - induced transparency of the exciton line

Let us consider the case discussed in § 3.4 that the photon energy of light wave is close to the lowest energy level of the exciton, and $\hbar\omega_{r_0}$ is smaller than the separation of this level from other energy level. For such case, we have already reduced the problem to a form completely analogous to a two - level atom. One therefore expects the phenomenon of self - induced transparency to occur. The Fig(3.5) already explicitly indicates how this can occur. Namely if the first half of a light pulse is just such as to raise the density of exciton from zero to saturation, and during the second half of the pulse the density of exciton is reduced by stimulated emission exactly to zero. So that the energy of the pulse will not suffer any change and its propagation is thus lossless. However , owing to the up and down shuttling in the state of excitation , the velocity of propagation of the pulse will be reduced considerably.

Following the method used in § 3.4 for (4.14), we introduce

$$\rho_{0\vec{k}} = \ell_1 f_{1\vec{k}} \qquad (4.17)$$

and define the following quantities:

$$r = \frac{\sum_{\vec{k}}(p_{cv})_x J_0(q)|f_{1\vec{k}}|^2 f_{1\vec{k}}}{\sum_{\vec{k}}(p_{cv})_x J_0(q) f_{1\vec{k}}^*} \approx \frac{\sum_{\vec{k}}|f_{1\vec{k}}|^2 f_{1\vec{k}}}{\sum_{\vec{k}} f_{1\vec{k}}^*},$$

$$\frac{h}{2} = \sqrt{r}\,\frac{e}{2mc\hbar}\sum_{\vec{k}} J_0(q)(p_{cv})_x f_{1\vec{k}}^* A_0$$

$$\approx \frac{e}{2mc\hbar}(p_{cv})_x \sqrt{\sum_{\vec{k}} f_{1\vec{k}}^* \sum_{\vec{k}} |f_{1\vec{k}}|^2 f_{1\vec{k}}}\cdot A_0, \qquad (4.18)$$

Besides, we introduce

$$\delta\omega(1,\vec{q}) = \delta\omega_1 - u_{\vec{q}}\,\partial\varphi/\partial z,$$

$$u = \sqrt{r}\,\ell_1 e^{i\varphi} \qquad (4.19)$$

and the operator

$$\hat{D}_t = \frac{\partial}{\partial t} + u_{\vec{q}}\frac{\partial}{\partial z}, \qquad (4.20)$$

$\delta\omega_1$ and φ in the above formulas have the same meaning as defined earlier in (3.55) and (4.4).

In term of these notation, we obtain $\hat{D}_t u = -i\delta\omega(1,\vec{q})u + \frac{h}{2}(1+u^2)$, (4.21) which can be related to an equation of Bloch type (see § 3.4):

$$D_t \vec{m} = \vec{m} \times \vec{H} \quad ,$$

$$\vec{H} = (0, -h, \delta\omega(1,\vec{q})) \quad . \qquad (4.22)$$

In general, the exciton energy level $E_{ex,1}$ may form a distribution, accordingly $\Delta\omega_1 = \frac{1}{\hbar}E_{ex,1} - \omega$ corresponds to a distribution function $\alpha(\Delta\omega_1)$, which satisfies

$$\int \alpha(\Delta\omega_1)\, d\Delta\omega_1 = 1 \quad . \qquad (4.23)$$

As the solution of Eq(4.20) and (4.21) are function of $\Delta\omega_1$, they should be averaged over $\alpha(\Delta\omega_1)$ and given the following notation:

$$\langle f \rangle = \int f(\Delta\omega_1)\, \alpha(\Delta\omega_1)\, d\Delta\omega_1 \quad . \qquad (4.24)$$

Hence Eq(4.16) can be written as

$$\frac{\partial \tilde{A}_0}{\partial z} + \frac{1}{c}\frac{\partial \tilde{A}_0}{\partial t} = i\frac{2\pi e^2}{m_d c\omega}\cdot\frac{1}{Vr}\left\langle |u|^2 \sum_{\vec{k}} \frac{|f_{1\vec{k}}|^2}{1+\frac{1}{r}|u|^2|f_{1\vec{k}}|^2}\right\rangle \tilde{A}_0$$

$$+\frac{4\pi e}{m\omega}(P_{cv})_x \frac{1}{\sqrt{Vr}}\, e^{-i\varphi}\left\langle u \sum_{\vec{k}} \frac{\frac{1}{\sqrt{V}} f_{1\vec{k}}}{1+\frac{1}{r}|u|^2|f_{1\vec{k}}|^2}\right\rangle \quad . \qquad (4.25)$$

In § 3.4, we have shown that $\frac{1}{Vr} \sim \frac{1}{a^3}$, i. e. $\sim$ (volume of exciton)$^{-1}$, then we shall introduce the notation

$$\frac{1}{V_0} = \frac{1}{Vr} \qquad (4.26)$$

and write

$$\psi_1(0) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} f_{1\vec{k}} \quad . \qquad (4.27)$$

The latter in fact represents the value of the wave function of the exciton evaluated at its origin. the Eq(4.25), (4.21) and (4.22) are the equation governing the coherent propagation.

For the sake of simplicity, we shall use the notation:

$$\chi_1 = \frac{2\pi e^2}{m_d c\omega}\,\frac{1}{V_0} \quad ,$$

$$\chi_2 = \frac{4\pi e}{m\omega}(P_{cv})_x \frac{1}{\sqrt{V_0}}\,\psi_1(0) \quad .$$

The Eq (4.25) can be rewritten as

$$\left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right]\tilde{A}_0 = i\chi_1 \left\langle |u|^2 \sum_{\vec{k}} \frac{|f_{1\vec{k}}|^2}{1+\frac{1}{r}|u|^2|f_{1\vec{k}}|^2}\right\rangle \tilde{A}_0$$

$$+\frac{\chi_2}{\psi_1(0)}\left\langle u \sum_{\vec{k}} \frac{1}{\sqrt{V}} f_{1\vec{k}} \Big/ \left(1+\tfrac{1}{r}|u|^2|f_{1\vec{k}}|^2\right)\right\rangle e^{-i\varphi}$$

~~The first term on the~~ right side of (4.25) represents the change of the refractive index due to the excitation of the electron and hole carriers. It is expected to be smaller than the second term. If the first term is neglected, (4.25) becomes

$$\left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right]\tilde{A}_0 = \frac{\chi_2}{\psi_1(0)}\left\langle u \sum_{\vec{k}} \frac{\frac{1}{\sqrt{V}} f_{1\vec{k}}}{1+\frac{|u|^2}{r}|f_{1\vec{k}}|^2}\right\rangle e^{-i\varphi} \qquad (4.28)$$

Letting

$$M_+ = (m_x + i m_y)\, e^{-i\varphi} \quad , \qquad M_- = (m_x - i m_y)\, e^{i\varphi} \quad ,$$

$$M_z = m_z \qquad (4.29)$$

we obtain (4.22) in the following form

$$\hat{D}_t M_z = -\chi_3(\tilde{A}_0 M_+ + \tilde{A}_0^* M_-) \quad ,$$

$$D_t M_+ = -i\delta\omega_1 M_+ + \chi_3 \tilde{A}_0 M_z \quad ,$$

$$D_t M_- = i\delta\omega_1 M_- + \chi_3 \tilde{A}_0^* M_z \quad , \qquad (4.30)$$

where

$$\chi_3 = \frac{e}{mc\hbar}\sqrt{v_0}\,(P_{cv})_x\,\psi_1^*(0) = \frac{e}{mc\hbar}\sqrt{v_0}\,(P_{cv})_x\,\psi_1(0) \tag{4.31}$$

The equation (4.30) and (4.28) are very similar to the equation of self - transparency in the case of a two - level system. It follows from the definition of $\psi_1(0)$ and Υ that the following should be an acceptable approximation:

$$\left\langle u\,\frac{1}{\psi_1(0)}\sum_{\vec{k}}\frac{\frac{1}{\sqrt{v}}\,f_{1\vec{k}}}{1+\frac{|u|^2}{F}|f_{1\vec{k}}|^2}\right\rangle \approx \left\langle \frac{u}{1+|u|^2}\right\rangle . \tag{4.32}$$

we assume further that $\delta\omega_1 = \Delta\omega_1$, and make use of the rela -tion:

$$\frac{u}{1+|u|^2} = m_x + i m_y ,$$

then we obtain Eqs (4.28) and (4.30) in the following form:

$$\left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right]\tilde{A}_0 = \chi_2 \langle M_+\rangle ,$$

$$\left[\frac{\partial}{\partial t} + u_z\frac{\partial}{\partial z}\right]M_+ = -i\Delta\omega_1 M_+ + \chi_3\tilde{A}_0 M_z ,$$

$$\left[\frac{\partial}{\partial t} + u_z\frac{\partial}{\partial z}\right]M_z = -\frac{\chi_3}{2}\left(\tilde{A}_0 M_+ + \tilde{A}_0^* M_-\right) . \tag{4.33}$$

In term of the new variables

$$Z = z - u_z t$$

and

$$\tau = \frac{t - \frac{z}{c}}{1 - \frac{u_z}{c}} , \tag{4.34}$$

(4.33) becomes

$$\frac{\partial}{\partial Z}\tilde{A}_0 = \chi_2\langle M_+\rangle\frac{1}{1-\frac{u_z}{c}} ,$$

$$\frac{\partial}{\partial \tau}M_+ = -i\Delta\omega_1 M_+ + \chi_3\tilde{A}_0 M_z ,$$

$$\frac{\partial}{\partial \tau}M_z = -\frac{\chi_3}{2}\left(\tilde{A}_0 M_+ + \tilde{A}_0^* M_-\right) \tag{4.35}$$

These equations have exactly the form of the equations for self - induced transparency in a two - level system. the solutions of these equations can be obtained by the socalled "inverse scattering method", a detailed description of which may be found in the literature [21, 23, 25] .

It follow from an analysis by the inverse scattering meth-od that the coherent propagation of light in such a system can take up two different modes, which have been disignated as "radiation" and "soliton" modes. For weak light pulses, i.e. the "area" of pulse is small, only the "radiation" mode exists and it is essentially the solution obtained from the linear approximation of the non - linear equation. The propagation of the "radiation" mode is attenuated, with the medium absorbing energy from the mode. In fact this mode is essentially a coupled wave of the exciton polaron type [10] . When the intensity of the light pulse (its "area") exceeds a certain critical value , a new propagation mode -- soliton will emerge . In such a case , the light pulse will start by continuously changing its shape; this process is in fact a manifestation of the decay of the "radiation" mode involved. Thus the light pulse evolves asymptoticly into a "soliton" mode, which retains its shape during propagation and is lossless. So the light pulse has become a self-induced transparency pulse. If the intensity of the original pulse is sufficiently high, two or even more solitons will emerge.

As an example, using the standard method of inverse scattering transformation [23], we obtain from (4.33) and (4.35) the solution representing a single soliton as

$$A_0(z,t) = 4a\exp(i(\xi_1 Z - 2bT - \theta))\,\mathrm{sech}(\xi_2 Z - 2aT + \varphi), \qquad (4.37)$$

where θ and φ are constant, and the parameters a, b, ξ_1 and ξ_2 are related by the following relations:

$$\xi_1 = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{b-x}{(b-x)^2+a^2}\,\alpha'(x)\,dx,$$

$$\xi_2 = \frac{1}{2}\int_{-\infty}^{\infty}\frac{a}{(b-x)^2+a^2}\,\alpha'(x)\,dx, \qquad (4.38)$$

where $\alpha'(x) = 2\chi_3\alpha(2\chi_3 x)$ and α is the line shape function defined in (4.23). the variable Z and T in (4.37) are defined as follows :

$$Z = \chi_2 z = \chi_2(z - u_g t),$$

$$T = \chi_3\tau = \frac{\chi_3}{1-\frac{u_g}{c}}\left(t - \frac{z}{c}\right). \qquad (4.39)$$

This solution represents pulse with the following properties:

pulse height = 4 a;

pulse duration = $\frac{1}{c}\,\frac{1}{\frac{2a}{c-u_g}\chi_3 + \frac{u_g}{c}\chi_2\xi_2}$

$$\approx \frac{\chi_3}{2a}\left(1 + \frac{\chi_2\xi_2}{\frac{2a}{c}\chi_3}\,\frac{u_g}{c}\right)^{-1}$$

$$\approx \chi_3/2a \, ;$$

and the velocity of propagation ;

$$V = \frac{1}{\frac{2a\chi_3}{c} + \xi_2\chi_2}\left(\frac{2a\chi_3}{c} + \xi_2\chi_2\frac{u_g}{c}\right)c ,$$

$$\approx \left(1 + \frac{\xi_2\chi_2}{2a\chi_3/c}\right)^{-1} c$$

We notice that : the height, duration and propagation velocity of the soliton (self - induce transparency pulse) are closely related. As we know, this is a general feature with the propagation of non-linear waves. Substituting appropriate numerical values, we find that the velocity of coherent propagation is smaller compared with the velocity of light in the vacuum. This is a characteristic feature with self-induced transparency.

4 . 3 Self - induced transparency with interband transitions

When $\hbar\omega > E_g$, the electron - hole pair excited by light will be in the continuous spectrum of the exciton. For a discussion of this case, we first perform a transformation on the basic equation (4.14). Following the standard technique for solving the Ricatti equation, we introduce a new variable $w_{\vec{k}}$, which is related to $\rho_{0\vec{k}}$ by

$$\rho_{0\vec{k}} = \frac{i}{\varepsilon_0^*}\,\frac{\hat{D}_t w_{\vec{k}} - \frac{i}{2}\Delta\omega(\vec{k},\vec{\xi})\,w_{\vec{k}}}{w_{\vec{k}}} . \qquad (4.40)$$

Then the equation (4.14) can be rewritten as

$$[\hat{D}_t - (\hat{D}_t\varepsilon^*) + i\tfrac{1}{2}\Delta\omega(\vec{k},\vec{\xi})][\hat{D}_t - \tfrac{i}{2}\Delta\omega(\vec{k},\vec{\xi})]\,w_{\vec{k}}$$

$$= -|\varepsilon|^2 - i\varepsilon^* w_{\vec{k}}\,\hat{U}(\rho_{0\vec{k}}) , \qquad (4.41)$$

Where

$$\hat U(\rho_{0\vec k}) = -\frac{i}{\hbar}\hat V \frac{\rho_{0\vec k}}{1+|\rho_{0\vec k}|^2} + \frac{i}{\hbar}\rho_{0\vec k}^* \hat V^* \frac{\rho_{0\vec k}^*}{1+|\rho_{0\vec k}|^2},$$

$$(\hat D_t \mathcal{E}^*) = \frac{\partial \mathcal{E}^*}{\partial t} + u_s\frac{\partial \mathcal{E}^*}{\partial \zeta} .$$

Introducing further the new variables:

$$v_{2\vec k} = w_{\vec k},$$

$$v_{1\vec k} = -\frac{1}{\mathcal{E}^*}\left(\hat D_t - \frac{i}{2}\Delta\omega(\vec k,\vec s)\right) v_{2\vec k}, \qquad (4.42)$$

Eq(4.41) can be recast into the following pair of equations:

$$\left(\hat D_t + \frac{i}{2}\Delta\omega(\vec k,\vec s)\right) v_{1\vec k} = \mathcal{E} v_{2\vec k} + i v_{1\vec k}\hat U\left(-i \frac{v_{1\vec k}}{v_{2\vec k}}\right),$$

$$\left(\hat D_t - \frac{i}{2}\Delta\omega(\vec k,\vec s)\right) v_{2\vec k} = -\mathcal{E}^* v_{1\vec k} .$$

If one introduces the following notation:

$$\hat\Pi(x_{\vec k}) = -\frac{i}{\hbar}\frac{1}{x_{\vec k}}\hat V\frac{x_{\vec k}}{1+|x_{\vec k}|^2} + \frac{i}{\hbar} x_{\vec k}\hat V^* \frac{x_{\vec k}^*}{1+|x_{\vec k}|^2}, \qquad (4.43)$$

and effects the following transformations:

$$u_{1\vec k} = \exp\left(i\int^t \Pi\left(\frac{v_{1\vec k}}{v_{2\vec k}}\right)dt\right) v_{1\vec k},$$

$$u_{2\vec k} = \exp\left(i\int^t \Pi\left(\frac{v_{1\vec k}}{v_{2\vec k}}\right)dt\right) v_{2\vec k},$$

one obtain the equations in the very symmetric form:

$$\cancel{\left[\hat D_t + \frac{i}{2}\right(}\Delta\omega(\vec k,\vec s) - \hat\Pi\left(\frac{u_{1\vec k}}{u_{2\vec k}}\right)\Big)\Big] u_{1\vec k} = \mathcal{E} u_{2\vec k},$$

$$\left[\hat D_t - \frac{i}{2}\left(\Delta\omega(\vec k,\vec s) - \hat\Pi\left(\frac{u_{1\vec k}}{u_{2\vec k}}\right)\right)\right] u_{2\vec k} = -\mathcal{E}^* u_{1\vec k} \qquad (4.44)$$

and that

$$\rho_{0\vec k} = -i\, u_{1\vec k}/u_{2\vec k} . \qquad (4.45)$$

(4.44) (4.45) and (4.46) are the equations governing the coherent propagation; clearly to solve them in general will be difficult.

However, as we know , in usual problems of absorption of light, if ($\hbar\omega - E_g$) is not too small, the effect of the elect~~ron -- hole interaction is~~ not important. In the present case, as $V \approx V^*$, it follows from (4.43) that

$$\hat\Pi(x_{\vec k}) = -\Pi\left(\frac{1}{x_{\vec k}}\right) . \qquad (4.46)$$

Hence for very strong excitation ($|\rho_{0\vec k}| >> 1$) and very weak excitation ($|\rho_{0\vec k}| << 1$), for the equation (4.44) the magnitude of $\hat\Pi$ is similar only with its sign reversed. but when $\hbar\omega - E_g > 0$, for states near $\hbar\omega = \hbar\omega_{\vec k}(\vec k)$,the off - resonance frequency $\Delta\omega(\vec k,\vec s)$ is much smaller than the Rabi frequency $\omega_{r.0}$ for the case of resonance. It follow from §2.3 or §3.4 that these states can be strongly excited. As they should be similar to the case of weak excit~~ø~~ion, we can neglect the $\hat\Pi$ term and get

$$\left(D_t + \frac{i}{2}\Delta\omega(\vec k,\vec s)\right) u_{1\vec k} = \mathcal{E} u_{2\vec k}$$

$$\left[D_t - \frac{i}{2}\Delta\omega(\vec k,\vec s)\right] u_{1\vec k} = -\mathcal{E}^* u_{1\vec k} \qquad (4.47)$$

These equation together with (4.16) are completely analogous to the equations for an inhomogeneously broadened two-level system. The innomogeneously broadened line shape function corresponds in the present case to the reduced density of state for energy band. The inverse scattering prob~~l~~lem for this type of equation has been solved in the literature[24]. Follo-

-wing the method in the papers [11,13,14], one can then obtain the solution for coherent propagation. Naturally one finds likewise the phenomena of saturation, self-induced transparency and etc.

V. Conclusion

In the paper, for the sake of clearity and simplicity, we have adopted a simple two band model. It is fairly obvious that the main conclusions will not be qualitatively different for more realisitic band structures. We have considered in detail only the case of single-photon resonance; but the generalization to the case of multi-photon resonance is not difficult.

It follows from our analysis that the characteristic phen-omena of coherent propagation in usual two - level atomic systems should also occur in semiconductors. the equations we have obtained in §3.4 §4.2 and §4.3 will be basic equation for treating such phenomena.

If we want to take account of the collision between ele-ctron - hole pairs and between electron and phonons, phenomenologically it will be analogous to introduce relaxation terms in two-level systems. There should be no difficulty in principle.

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