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ABSTRACT

Low frequency magnetohydrodynamic (MHD) waves are studied in both rectangular slab and cylindrical geometry cavities containing low β plasmas. The plasma density distribution is modelled by an inner region of constant density surrounded by an outer region of lower density and a conducting boundary. The wave frequencies and fields are obtained as functions of the density distribution and the wavenumber components $k_{\|}$ and $k_{\perp}$, the subscripts meaning parallel and perpendicular to the ambient magnetic field. The lowest frequency wave mode is a surface wave in which the wave fields decrease in magnitude with distance from the interface between the two plasma densities. It has the properties of a shear wave when $k_{\perp} / k_{\|}$ is either small or large but is compressive when $k_{\perp} \simeq k_{\|}$. The surface wave does not exist when $k_{\perp} = 0$. Higher frequency modes have the properties of fast magnetosonic waves, at least in the inner density region.

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HYDROMAGNETIC WAVES; CYLINDRICAL CONFIGURATION; RECTANGULAR CONFIGURATION; LOW-BETA PLASMA; DISPERSION RELATIONS; FREQUENCY RANGE; EIGENFREQUENCY; ALFVEN WAVES; DIFFERENTIAL EQUATIONS

CONTENTS

(Continued)

1. INTRODUCTION

The analysis of magnetohydrodynamic (MHD) waves in tokamaks is difficult because of the toroidal geometry and the variation with position of both the magnetic fields and the plasma density. Consequently, most authors use a simplified model of the plasma which includes sufficient detail to allow specific effects to be analysed. For example, Swanson [1974] assumed a uniform plasma density in his investigation of the effects of toroidal field variation on the properties of ion cyclotron and fast waves in a toroidal cavity, whereas Hasegawa and Chen [1976] studied the resonant mode conversion of the compressional Alfvén wave to the kinetic Alfvén wave in the plasma surface by assuming a plasma density which varies linearly with position.

In the present report, a simple density distribution is used to explore the density dependence of the frequency and fields of MHD waves with frequencies well below the ion cyclotron frequency Ω_i. The analysis is first carried out in rectangular slab geometry because this model is often used [for example, Ott et al. 1978] and because the results are easily interpreted. Circular cylindrical geometry is a better approximation to toroidal geometry so the analysis is repeated. The density is described by a step function with a plasma of density ρ_1 surrounded by a plasma of lower density ρ_2. A zero pressure approximation is used, so the slow magnetosonic wave is not considered. The motivation for treating a step function density distribution is both its simplicity and the fact that the plasmas in tokamaks are often approximately of this shape [Mercier et al. 1979].

The simplest model of a bounded plasma assumes a uniform density, and the properties of MHD waves in such a plasma have been extensively investigated [for example, Woods 1962]. The effects of a varying plasma density have been treated by Pneuman [1964] for axisymmetric waves in a parabolic density distribution surrounded by a vacuum. Here, both axisymmetric and non-axisymmetric waves are considered for plasma density distributions which range from a uniform plasma with conducting boundary conditions to a plasma column surrounded by vacuum.

In the non-axisymmetric case, the variation in plasma density allows the existence of a surface wave in which the wave fields decrease with distance from the density discontinuity. The properties of these waves have been analysed for unbounded plasmas in slab geometry by Wentzel [1979a], and in cylindrical geometry by Wentzel [1979b] and Uberoi and Somasundaram [1980].

Uberoi and Somasundaram have assumed that the wave does not compress the plasma ($\nabla \cdot v = 0$), and the validity of this approximation is assessed here. Particular emphasis is given to the surface wave since it can have a frequency considerably below the cut-off frequency of the fast magnetosonic wave even though it possesses some of its properties such as magnetic field compression.

2. RECTANGULAR GEOMETRY

A rectangular slab geometry model of the plasma is used as it allows the simplest analysis of waves in a plasma with spatially varying density and also because a clear differentiation between surface waves and body waves is obtained. The plasma density is

$$\rho_0(x) = \rho_1 \qquad |x| \le a$$

$$= \rho_2 \qquad a < |x| \le e \quad ,$$

with $\rho_1 \ge \rho_2$. A perfectly conducting wall at $|x| = e$ surrounds the plasma and the ambient magnetic field $B_0\hat{z}$ is constant.

A more realistic description would allow a certain surface thickness δ, as shown by the dashed line in Figure 1. Wentzel [1979c] has studied the surface wave for two homogeneous plasmas separated by a region in which the density varies linearly. The finite surface width introduces imaginary terms into the dispersion relation. They arise because there is a loss of wave energy at the position in the surface where the Alfvén shear wave dispersion relation ($\omega = v_A k_z$) is satisfied. The wave amplitude is large at this point and energy is dissipated through viscous and resistive heating and/or mode conversion to the kinetic Alfvén wave [Hasegawa and Chen 1976]. However, as long as the surface width is not too large, Wentzel obtains a similar dispersion relation to that found using the step function density discontinuity considered here.

2.1 The Wave Equations

The particle pressure is ignored in the following equations since, for low β plasmas, it has little effect on the wave modes considered here. The displacement current is included but it is of little importance unless plasma densities are very low. Neglecting dissipation, the relevant MHD and

Maxwell's equations are:

$$\rho \frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{j} \times \underset{\sim}{B} \quad ,$$

$$\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0 \quad ,$$

$$\mu_0 \underset{\sim}{j} = \underset{\sim}{\nabla} \times \underset{\sim}{B} - \frac{1}{c^2} \frac{\partial \underset{\sim}{E}}{\partial t} \quad ,$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \frac{\partial \underset{\sim}{B}}{\partial t} \quad ,$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \quad .$$

The equilibrium configuration has a uniform magnetic field $B_0 \hat{\underset{\sim}{z}}$, a position dependent density $\rho_0(x)$, and zero current and velocity. A perturbation expansion of the fields in the form

$$\underset{\sim}{B} = \underset{\sim}{B}_0 + \varepsilon \underset{\sim}{b} \quad ,$$

where ε is a small parameter, leads to the linearised perturbation equations

$$\rho_0(x) \frac{\partial \underset{\sim}{v}}{\partial t} = \underset{\sim}{j} \times \underset{\sim}{B}_0 \quad , \tag{1}$$

$$\frac{\partial \underset{\sim}{b}}{\partial t} = \underset{\sim}{\nabla} \times (\underset{\sim}{v} \times \underset{\sim}{B}_0) \quad , \tag{2}$$

$$\mu_0 j = \underset{\sim}{\nabla} \times \underset{\sim}{b} - \frac{1}{c^2} \frac{\partial}{\partial t} (\underset{\sim}{v} \times \underset{\sim}{B}_0) \quad , \tag{3}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{b} = 0 \quad . \tag{4}$$

The wave fields vary as

$$\underset{\sim}{b}(\underset{\sim}{r},t) = \underset{\sim}{b}(x) \exp \{ i(k_y y + k_z z - \omega t)\} \quad . \tag{5}$$

In this report the equations are solved for the eigenvalue ω, assuming that k_y and k_z are fixed and $k_z \neq 0$. Alternatively, the eigenvalue k_z, given ω and k_y, can be obtained in a similar fashion.

Substitution of (5) into Equations (1) to (4) gives

$$\psi b_x = -\frac{d}{dx}\left(\frac{db_x}{dx} + ik_y b_y\right) = ik_z \frac{db_z}{dx} \quad , \qquad (6)$$

$$\psi b_y = -ik_y\left(\frac{db_x}{dx} + ik_y b_y\right) = -k_y k_z b_z \quad , \qquad (7)$$

$$\psi \frac{d}{dx}\left(\frac{1}{\psi}\frac{db_z}{dx}\right) + (\psi - k_y^2)b_z = 0 \quad , \qquad (8)$$

where b_x, b_y and b_z are functions of x,

$$\psi = \omega^2/v_\alpha^2 - k_z^2 \quad ,$$

$$v_\alpha^2 = v_A^2/(1 + v_A^2/c^2) \quad \text{and}$$

$$v_A = B_0/[\mu_0\rho_0(x)]^{\frac{1}{2}}$$

is the Alfvén speed. Define

$$v_{\alpha 1} = v_\alpha(x) \quad , \quad \psi_1 = \psi(x) \qquad |x| \leq a \quad ,$$

$$v_{\alpha 2} = v_\alpha(x) \quad , \quad \psi_2 = \psi(x) \qquad a < |x| \leq e \quad ,$$

and

$$R_\rho = v_{\alpha 1}^2/v_{\alpha 2}^2 \simeq \rho_2/\rho_1 \quad .$$

In each homogeneous region, the x-dependence of the wave fields takes the form

$$\underset{\sim}{b}(x) = \underset{\sim}{b} \exp\{ik_x x\} \quad , \qquad (9)$$

Equations (6) and (7) reduce to the dispersion relations

$$\omega^2 = v_\alpha^2(k_x^2 + k_y^2 + k_z^2) \qquad (10)$$

or

$$\omega^2 = v_\alpha^2 k_z^2 \quad , \tag{11}$$

and Equation (8) gives the dispersion relation in Equation (10) or

$$b_z = 0 \quad . \tag{12}$$

The dispersion relation in Equation (10) pertains to the fast magnetosonic wave (a compressional wave herinafter labelled F), whereas both the dispersion relation in Equation (11) and the zero b_z field condition in Equation (12) pertain to the Alfvén wave (a shear wave hereinafter labelled A).

2.2 MHD Waves in a Uniform Plasma

Before solving the dispersion relation for waves in the two-region plasma, it is useful to summarise the properties of the MHD waves which exist in a uniform plasma of density ρ_1 with a conducting boundary at $|x| = a$. These results are well known and can easily be deduced from the properties of MHD waves in a homogeneous plasma [Akhiezer et al. 1975].

2.2.1 The Alfvén wave

The Alfvén wave dispersion relation is given by Equation (11), and the magnetic field components are given by the relations

$$k_x b_x + k_y b_y = 0 \quad \text{and} \quad b_z = 0 \quad .$$

The conducting boundary imposes the condition

$$b_x(|x| = a) = 0 \quad .$$

This condition is satisfied for any $k_x \neq 0$ when $k_y = 0$, and the non-zero field component b_y then has an arbitrary functional dependence on x. When $k_y \neq 0$, the Fourier components of the b_x field are of the form

$$b_x(x) \propto \sin(k_{ns}x) \quad \text{or} \quad \cos(k_{na}x) \quad , \quad n = 1,2,...$$

with

$$k_{ns} = \frac{n\pi}{a} \quad \text{and} \quad k_{na} = \frac{(2n-1)\pi}{2a} \quad .$$

As the Alfvén wave dispersion relation is independent of k_x and k_y, each of these Fourier modes has the same phase velocity, so b_x is an arbitrary function of x, except at the boundary.

2.2.2 The fast wave

The fast wave dispersion relation is given by Equation (10), and the magnetic field components are related by Equations (6) to (8). From Equation (6), the conducting boundary imposes the condition

$$\left.\frac{db_z}{dx}\right|_{|x|=a} = 0 \quad .$$

The symmetric Fourier components of the b_z field are

$$b_z \propto \cos(k_{ns}x) \qquad n = 0,1,2,\ldots$$

with frequency

$$\Omega_{ns}^2 = v_\alpha^2\left(\left(\frac{n\pi}{a}\right)^2 + k_y^2 + k_z^2\right) \quad . \tag{13}$$

When $k_x = k_y = 0$, the fast wave is identical to the Alfvén wave.

The antisymmetric Fourier components of the b_z field are

$$b_z \propto \sin(k_{na}x) \qquad n = 1,2,\ldots$$

with frequency

$$\Omega_{na}^2 = v_\alpha^2\left(\left(\frac{(2n-1)\pi}{2a}\right)^2 + k_y^2 + k_z^2\right) \quad . \tag{14}$$

Using Equations (6) to (8), the relations $b_y/b_x = k_y/k_x$ and $b_z/b_x = -(k_x^2 + k_y^2)/k_xk_z$ can be deduced. In the two-region plasma, some waves are evanescent in the x-direction, that is,

$$k_x^2 = -\kappa_x^2 < 0 \quad ;$$

the above relations then become

$$b_y/b_x \;=\; -\,i\,k_y/\kappa_x \quad \text{and} \quad b_z/b_x \;=\; -\,i(\kappa_x^2 - k_y^2)/\kappa_x k_z \quad .$$

Comparison with the appropriate relations for the Alfvén wave, namely

$$b_y/b_x \;=\; -\,i\,\kappa_x/k_y \quad \text{and} \quad b_z \;=\; 0 \quad ,$$

shows that the fields associated with a fast wave, which is evanescent in the x-direction, are the same as those of an Alfvén wave as $\kappa_x \rightarrow k_y$.

2.3 Dispersion Relations for the Two-region Plasma

Because of symmetry, the analysis can be confined to the positive x region. The wave fields are expressed in the form

$$\begin{aligned} \underset{\sim}{b}(x) &= \underset{\sim}{b}_1 \exp(ik_1x) \qquad 0 \leq x \leq a \\ &= \underset{\sim}{b}_2 \exp(ik_2x) \qquad a < x \leq e \end{aligned} \tag{15}$$

and, in each homogeneous region, the wave properties correspond to either those of a fast wave or those of an Alfvén wave. The conducting boundary at $|x| = e$ imposes the conditions

$$b_x(e) \;=\; 0 \quad \text{and} \quad \left.\frac{db_z}{dx}\right|_e \;=\; 0 \quad .$$

The dispersion relation is obtained from the continuity conditions which apply to the wave fields at x = a. Stix [1962] presented a detailed discussion of the continuity conditions applying at step function discontinuities in plasmas. Woods [1971] has pointed out that care must be taken that the surface width is indeed small when compared with the length scales associated with relevant plasma behaviour.

For low frequency waves, the appropriate conditions are obtained by integrating Equations (7) and (6) across a small interval δx around x = a. This gives

$$b_x(a^-) \;=\; b_x(a^+) \tag{16}$$

and

$$b_z(a^-) = b_z(a^+) \quad . \tag{17}$$

The dependence of $\underset{\sim}{b}$ on y, z and t is implicit in Equations (16) and (17). If the wave properties on both sides of the interface are obtained using the fast wave equations, then (16) is equivalent to

$$\frac{1}{\psi}\frac{db_z}{dx}\bigg|_{a^-} = \frac{1}{\psi}\frac{db_z}{dx}\bigg|_{a^+} \quad . \tag{18}$$

The dispersion relation depends on whether an Alfvén wave or a fast wave in the inner region is coupled to an Alfvén wave or a fast wave in the outer region.

2.3.1 Alfvén-Alfvén wave coupling

Since Alfvén waves propagate at the local Alfvén speed, independent of k_x and k_y, condition (16) requires that $b_x(a^-) = b_x(a^+) = 0$. This is satisfied automatically when $k_y = 0$ since $b_x(x) = 0$. When $k_y \neq 0$, following the argument in Section 2.2.1, the b_x field can have an arbitrary x dependence apart from being zero at x = a and x = e. Given these conditions, the waves in each region propagate independently of one another with phase speeds equal to the local Alfvén speed. Sy [1978] has shown that this phenomenon of Alfvén wave propagation at local Alfvén speed occurs when plasma resistivity is negligible. As resistivity increases, adjacent layers of plasma become coupled and the phase speed tends to a constant value.

2.3.2 Alfvén-fast wave coupling

For this coupling to occur, the fast wave fields must satisfy the boundary conditions.

$$b_z(a^+) = 0 \quad \text{and} \quad \frac{db_z}{dx}\bigg|_e = 0 \quad .$$

If $k_y \neq 0$ and $b_x(a) \neq 0$, the dispersion relation is $\omega^2 = v_{\alpha 1}^2 k_z^2$, the fast wave fields are evanescent in the x-direction and these conditions cannot be satisfied. If $k_y = 0$ or $b_x(a) = 0$, the additional condition $b_x(a^+) = 0$ is imposed and once again no solution is possible. Thus there is no coupling of an Alfvén wave to a fast wave.

2.3.3 Fast-Alfvén wave coupling

If $k_y = 0$ there is no coupling as $b_x(a^-)$ and $b_z(a^-)$ cannot both be zero. When $k_y \neq 0$ the boundary conditions are:

$$b_x(a^-) = b_x(a^+) \neq 0, \quad b_z(a^-) = b_z(a^+) = 0 \text{ and } b_x(e) = 0 \quad .$$

As discussed in Section 2.2.1, the Alfvén wave fields have arbitrary x dependence and can satisfy the appropriate conditions with the proviso that the frequency is given by $\omega^2 = v_{\alpha 2}^2 k_z^2$. A symmetric b_z field of the form

$$b_z = B_0 \cos\left(\frac{(2n-1)\pi x}{2a}\right) \qquad |x| \leq a$$

leads to a solution if the equations

$$\Omega_{ns}^2 = v_{\alpha 1}^2 \left\{\left(\frac{(2n-1)\pi}{2a}\right)^2 + k_y^2 + k_z^2\right\}, \qquad n = 1,2,...$$

$$= v_{\alpha 2}^2 k_z^2$$

can be satisfied. An antisymmetric b_z field of the form

$$b_z = B_0 \sin\left(\frac{n\pi x}{a}\right) \qquad |x| \leq a$$

leads to a solution if the equations

$$\Omega_{na}^2 = v_{\alpha 1}^2 \left(\left(\frac{n\pi}{a}\right)^2 + k_y^2 + k_z^2\right), \qquad n = 1,2,...$$

$$= v_{\alpha 2}^2 k_z^2$$

can be satisfied. It is apparent that given four of the parameters $v_{\alpha 1}$, $v_{\alpha 2}$, a, k_y and k_z, a solution is only possible for specific values of the fifth parameter. This F-A coupling occurs as a special case of the F-F coupling considered below.

2.3.4 Fast-fast wave coupling

The rest of this report concentrates on the properties of waves which result from the coupling of two fast waves. The x-component of the wavenumber in regions 1 and 2 is then given by

$$k_j^2 = \omega^2/v_{\alpha j}^2 - k_y^2 - k_z^2 \quad , \quad j = 1,2 \quad . \tag{19}$$

If k_j^2 is negative, k_j is replaced by $i\kappa_j$.

Symmetric and antisymmetric fields are possible and, applying the conditions

$$b_z(a^-) = b_z(a^+) \quad \text{and} \quad \left.\frac{db_z}{dx}\right|_e = 0 \quad ,$$

the symmetric field is

$$\begin{aligned} b_z(x) &= B_0 \frac{\cos k_1 x}{\cos k_1 a} & 0 \leq x \leq a \\ &= B_0 \frac{\cos k_2(e-x)}{\cos k_2(e-a)} & a < x \leq e \quad , \end{aligned} \tag{20}$$

and the antisymmetric field is

$$\begin{aligned} b_z(x) &= B_0 \frac{\sin k_1 x}{\sin k_1 a} & 0 \leq x \leq a \\ &= B_0 \frac{\cos k_2(e-x)}{\cos k_2(e-a)} & a < x \leq e \quad . \end{aligned} \tag{21}$$

The condition (18) gives the symmetric dispersion relation

$$-\frac{k_1 \tan k_1 a}{\psi_1} = \frac{k_2 \tan k_2 d}{\psi_2} \tag{22}$$

and the antisymmetric dispersion relation

$$\frac{k_1 \cot k_1 a}{\psi_1} = \frac{k_2 \tan k_2 d}{\psi_2} \quad , \tag{23}$$

where $d = e - a$.

There are an infinite number of eigenfrequencies ($\Omega_{s\ell}$, $\Omega_{a\ell}$, $\ell = 1,2,\ldots$) which satisfy Equations (22) and (23), but only those meeting the condition $\Omega_\ell \leq 0.1\ \Omega_i$ are acceptable.

2.4 Bounds on the Eigenfrequencies

The solutions of the dispersion relations (22) and (23) must be obtained numerically. However, the following analysis of the zeros and poles of the functions in Equations (22) and (23) allows estimates of the first few eigenfrequencies.

Define the frequencies

$$\omega_1 = v_{\alpha 1}\, k_z \quad , \quad \omega_3 = v_{\alpha 1}(k_y^2 + k_z^2)^{\frac{1}{2}} \quad ,$$

$$\omega_2 = v_{\alpha 2}\, k_z \quad , \quad \omega_4 = v_{\alpha 2}(k_y^2 + k_z^2)^{\frac{1}{2}} \quad .$$

Then

$$k_1^2 = (\omega^2 - \omega_3^2)/v_{\alpha 1}^2 \quad \text{and} \quad k_2^2 = (\omega^2 - \omega_4^2)/v_{\alpha 2}^2 \quad .$$

For $\omega < \omega_3$ the wave is evanescent in region 1, and for $\omega < \omega_4$ it is evanescent in region 2. A given eigenmode with frequency Ω is characterised as an E-E wave if $\Omega < \omega_3$, a P-E wave if $\omega_3 \leq \Omega < \omega_4$ and a P-P wave if $\Omega \geq \omega_4$. The E-E waves are commonly called surface waves since their amplitudes decrease exponentially with distance from the surface.

Define $LS(\omega)$ and $LA(\omega)$ to be the left hand sides of (22) and (23) respectively and $R(\omega)$ to be the right hand side.

LS is a monotonically decreasing function of ω with poles at the frequencies

$$PLS(1) = \omega_1, \quad PLS(n+1) = v_{\alpha 1}\left\{\left(\frac{(2n-1)\pi}{2a}\right)^2 + k_y^2 + k_z^2\right\}^{\frac{1}{2}} \quad n = 1,2,\ldots$$

and zeros at the frequencies

$$ZLS(n) = v_{\alpha 1}\left\{\left(\frac{(n-1)\pi}{a}\right)^2 + k_y^2 + k_z^2\right\}^{\frac{1}{2}} \quad n = 1,2,\ldots$$

LA is monotonically decreasing function of ω with poles at the frequencies

$$PLA(1) = \omega_1, \quad PLA(n+1) = v_{\alpha 1}\left((\tfrac{n\pi}{a})^2 + k_y^2 + k_z^2\right)^{\frac{1}{2}} \quad n = 1,2,\ldots$$

and zeros at the frequencies

$$ZLA(n) = v_{\alpha 1}\left\{\left(\frac{(2n-1)\pi}{2a}\right)^2 + k_y^2 + k_z^2\right\}^{\frac{1}{2}} \qquad n = 1,2,...$$

R is a monotonically increasing function of ω with poles at the frequencies

$$PR(1) = \omega_2, \quad PR(n+1) = v_{\alpha 2}\left\{\left(\frac{(2n-1)\pi}{2d^2}\right) + k_y^2 + k_z^2\right\}^{\frac{1}{2}} \qquad n = 1,2,...$$

and zeros at the frequencies

$$ZR(n) = v_{\alpha 2}\left\{\left(\frac{(n-1)\pi}{d}\right)^2 + k_y^2 + k_z^2\right\}^{\frac{1}{2}} \qquad n = 1,2,...$$

If k_y is identically zero, the first pole and first zero of LS coalesce and LS is continuous at ω_1. Similarly, R is continuous at ω_2.

A solution of the symmetric dispersion relation (LS - R = 0) lies between each pair of adjacent poles of the set {PLS $\cup$ PR}, and a solution of the antisymmetric dispersion relation (LA - R = 0) lies between each pair of adjacent poles of the set {PLA $\cup$ PR}. Stronger bounds hold for those eigenfrequencies which lie below the frequency of the first pole of R. In this case

$$PLS(n) < \Omega_{ns} < ZLS(n) \quad \text{and} \quad PLA(n) < \Omega_{na} < ZLA(n)$$

for Ω_n < PR(1) when $k_y \neq 0$, or Ω_n < PR(2) when $k_y = 0$.

The functions LS and R are shown in Figure 2 for $k_y/k_z = 1$, and in Figure 3 for $k_y/k_z = 0$, with $k_z a = 1$, $d/a = 0.5$ and $R_\rho = 0.1$. It is apparent that one eigenfrequency exists between each adjacent pair of poles. The function LA is similar to LS except that it retains the pole at ω_1 when $k_y = 0$.

2.5 Wave Properties

In this section, the wave properties are discussed as a function of the parameters ρ_1, ρ_2, a, d, k_y and k_z. First, the relationships between the various wave field components are summarised.

Given the eigenfrequency, the appropriate b_z field is obtained from Equations (20) or (21). Equations (6) and (7) may then be used to find b_x and b_y. Equations (1) and (2) give

$$v_x = -(\omega/k_z)\, b_x/B_0 \quad , \quad v_y = -(\omega/k_z)\, b_y/B_0$$

$$\text{and} \quad v_z = 0 \quad . \tag{24}$$

The conditions (16) and (17) indicate that b_x and b_z are continuous at a, but it is apparent from (7) that b_y is discontinuous unless $k_y = 0$, in which case $b_y = 0$. The discontinuity in b_y implies the presence of a current sheet j_z^* at the surface. From (24), it is seen that v_x is continuous and v_y is discontinuous at the surface.

The wave properties are now discussed for various parameter values.

(i) a uniform plasma of density ρ_1 with conducting boundaries at $|x| = a$ has eigenfrequencies given by LS = 0 and LA = 0, in agreement with the frequencies given by Equations (13) and (14).

(ii) A plasma surrounded by a vacuum has low order eigenfrequencies given by

$$\text{LS,LA} = [(k_y^2 + k_z^2)^{\frac{1}{2}} / k_z^2] \tanh [(k_y^2 + k_z^2)^{\frac{1}{2}} d] \quad .$$

As d increases the eigenfrequencies decrease. Figure 4 shows the dependence of the first four eigenfrequencies on d for $k_z a = 1$, $R_\rho = 0.0001$ and for k_y/k_z equal to 0.1 and 3. It is apparent that the larger k_y is, the smaller is the value of d at which the frequencies become constant. This occurs because the wave modes are evanescent in region 2, with a relaxation length that shortens as k_y increases.

(iii) Figure 5 shows the variation of the first four eigenfrequencies as the plasma density ρ_2 is increased from zero to ρ_1 for $k_z a = 1$ and d = 0.5a. The frequencies decrease as ρ_2 increases, with the higher order, low k_y modes decreasing most rapidly when ρ_2 is small. As $\rho_2 \to \rho_1$, Ω_{1s}, Ω_{1a} and Ω_{2s} tend to ω_1 for $k_y/k_z = 0.1$, but Ω_{1s} and Ω_{1a} tend to ω_1 for $k_y/k_z = 3$. The associated waves resemble Alfvén waves, with small b_z. For $k_y/k_z \gtrsim 1$, the lowest fast mode frequencies that are expected are given by the values Ω_{2s} and Ω_{2a}, and the waves associated with Ω_{1s} and Ω_{1a} possess a significant b_z component when $\rho_2 < 0.2\rho_1$. The plasma surface therefore introduces two low frequency waves which have a b_z field which is appreciable when $k_y \doteq k_z$ and $\rho_2 < 0.2\rho_1$.

(iv) The symmetric modes have the following properties. For $k_y \neq 0$, the lowest frequency mode is always an E-E wave, the second mode is always a P-E wave and higher order modes are either P-E or P-P waves depending on the parameters R_ρ, a, d, k_y and k_z. As $k_y \to 0$, the first eigenfrequency $\Omega_{1s} \to \omega_1$ and all the magnetic field components of this wave, apart from b_y in region 1, tend to zero; this wave corresponds to an Alfvén wave, with $k_y = 0$, propagating in region 1. Similarly, as $k_y \to 0$, some higher order eigenfrequency $\Omega_{\ell s}$ tends to ω_2 and this wave corresponds to an Alfvén wave, with $k_y = 0$, propagating in region 2. These two waves are not predicted by the dispersion relation (22) when k_y is set equal to zero because the first pole and first zero coalesce in both LS and R. All other modes behave continuously as $k_y \to 0$, with $b_y \to 0$ and $j_z^* \to 0$. When $k_y = 0$, the eigenmodes may be P-E or P-P waves depending on the parameters.

(v) The antisymmetric modes have the following properties. For $k_y \neq 0$, the lowest frequency mode is an E-E wave if R_ρ, a, d or k_y are sufficiently large, otherwise it is a P-E wave. As $k_y \to 0$, one of the eigenfrequencies $\Omega_{\ell s}$ tends to ω_2; this wave corresponds to an Alfvén wave, with $k_y = 0$, propagating in region 2. It is not predicted by the dispersion relation (23) when k_y is set to zero. When $k_y = 0$, $\Omega_{1a} < \Omega_{1s}$.

2.6 Surface Waves

It is apparent from Equations (22) and (23), and from Figures 4 and 5, that as a, d and k_y increase, the frequencies of the two E-E modes, Ω_{1s} and Ω_{1a}, become independent of a and d. They tend to have the same value, Ω_s, which is given by the equation

$$\Omega_s^2/v_{\alpha 1}^2 k_z^2 \;=\; [(1+R_\rho)\;(1+k_y^2/k_z^2) - \{(1+R_\rho)^2(1+k_y^2/k_z^2)^2 -$$
$$- 4R_\rho(1+2k_y^2/k_z^2)\}^{\frac{1}{2}}\,]/2R_\rho \quad . \tag{25}$$

This is the dispersion relation for a surface wave propagating along the interface between two semi-infinite plasmas. It is plotted as a function of k_y/k_z for $R_\rho = 0.1$ and 0.5 in Figure 6. The surface wave frequency is an increasing function of k_y/k_z and lies within the limits

$$v_{\alpha 1}^2 k_z^2 < \Omega_s^2 < \frac{2v_{\alpha 1}^2 k_z^2}{1 + R_\rho} \;\simeq\; \frac{B_0^2 k_z^2}{\mu_0(\rho_1+\rho_2)/2} \quad . \tag{26}$$

The inverse attenuation lengths are shown as a function of k_y/k_z in Figure 7. The relation $\kappa_1\kappa_2 = k_y^2$ holds with $\kappa_1 < k_y$ and $\kappa_2 > k_y$.

For a plasma-vacuum system, the dispersion relation simplifies to

$$\Omega_s^2/v_{\alpha 1}^2 k_z^2 \;=\; (1 + 2k_y^2/k_z^2)/(1 + k_y^2/k_z^2) \quad , \tag{27}$$

and the inverse attenuation lengths are given by

$$\kappa_1^2 \;=\; k_y^4/(k_y^2 + k_z^2) \quad \text{and} \quad \kappa_2^2 \;=\; k_y^2 + k_z^2 \quad . \tag{28}$$

In the following discussion of the surface wave, the plasma interface is shifted to $x = 0$ and the b_x field is normalised to unity at $x = 0$. The magnetic field components are:

$$b_x \;=\; e^{\kappa_1 x} \qquad x \leq 0$$
$$\;=\; e^{-\kappa_2 x} \qquad x < 0 \quad ,$$
$$b_y \;=\; i(k_y/\kappa_1)\; e^{\kappa_1 x} \qquad x \leq 0$$
$$\;=\; -\, i(k_y/\kappa_2)\; e^{-\kappa_2 x} \qquad x > 0 \quad ,$$
$$b_z \;=\; -\, i((k_y^2 - \kappa_1^2)/\kappa_1 k_z)\; e^{\kappa_1 x} \qquad x \leq 0$$

$$= - i((k_y^2 - \kappa_1^2)/\kappa_1 k_z)\, e^{-\kappa_2 x} \qquad x > 0 \quad .$$

It is apparent that $|b_y(0^-)| \times |b_y(0^+)| = |b_x(0)|^2$. The magnitudes of these fields at $x = 0$ are shown in Figure 8 as a function of k_y/k_z for $R_\rho = 0.1$. For small k_y, the major field component in region 1 is b_y, whereas b_x and b_z dominate in region 2. As R_ρ increases, b_z becomes smaller. For large k_y, both $b_y(0^-)$ and $b_y(0^+)$ are similar in magnitude to $b_x(0)$, although they differ from one another in sign. It should be noted that as $k_y \to 0$, the b_y field in region 1 becomes dominant and the surface wave becomes an Alfvén wave propagating only in region 1 with $k_y = 0$.

Tataronis [1975] and Uberoi and Somasundaram [1980] have investigated surface wave phenomena using the incompressibility approximation ($\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$). The validity of this approximation can be assessed by expressing

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \left(1 - \frac{k_y^2}{\kappa(x)^2}\right) \frac{dv_x}{dx} \quad .$$

The condition $|\underset{\sim}{\nabla} \cdot v| \ll |\frac{dv_x}{dx}|$ requires $\kappa(x) \simeq k_y$ which is only satisfied when $k_y \gg k_z$. In the limit $k_y \gg k_z$, the surface wave dispersion relation is given by the upper limit in (26) which is the same as the expression obtained using the incompressibility approximation.

When $k_y \gg k_z$ the wave fields resemble those of Alfvén waves, even though the dispersion relation has been derived using fast wave relations. The reason for this is given in Section 2.2.2.

2.7 Discussion

The properties of the low frequency waves in a rectangular slab waveguide have been investigated using a step function plasma density distribution to approximate the effects of a varying plasma density. Simple expressions have been obtained for the upper and lower limits on each eigenfrequency, and the properties of the low frequency modes have been outlined.

When $k_y \neq 0$, apart from the Alfvén wave, the lowest frequency mode is a surface wave. When $k_y \simeq k_z$ and $\rho_2 < 0.2\rho_1$, this wave has an appreciable b_z component and a frequency lower than expected for fast waves. It can be used for transit time magnetic heating [Dawson and Uman 1965] of a collisionless plasma.

When $k_y \neq 0$, all wave modes have a discontinuity in b_y and v_y and a surface current j_z^* at the interface between the two plasma regions. Therefore resistive and viscous damping will be particularly important in the surface region for low mode number waves in collisional plasmas.

As $k_y \to 0$ the surface wave frequency $\Omega_{1s} \to \omega_1$, and the wave resembles an Alfvén wave confined to the high density plasma region. The b_y and v_y fields associated with the other wave modes tend to zero as $k_y \to 0$.

If an eigenmode has a frequency in the range $v_{\alpha 1}k_z < \Omega < v_{\alpha 2}k_z$ then, for a surface region of finite width, there is a position at which the Alfvén resonance condition, $\Omega = v_\alpha(x)k_z$, is satisfied. When $k_y \neq 0$, the wave loses energy at this position by resistive and viscous damping or by mode conversion. The surface wave always satisfies this condition.

3. CYLINDRICAL GEOMETRY

Most experimental investigations of MHD waves have been performed using cylindrical waveguides [see, for example, Cross and Lehane 1967] and the theory of such waves, for a homogeneous plasma, has been developed by Newcomb [1957]. Assuming the conditions detailed in Section 2, and the plasma density distribution shown in Figure 1 with x replaced by r, the wave modes that can exist in a cylindrical waveguide are summarised here.

3.1 The Dispersion Relation

The wave fields vary as

$$\underset{\sim}{b}(\underset{\sim}{r},t) = \underset{\sim}{b}(r)\exp(m\theta + k_z z - \omega t) \quad . \tag{29}$$

From Equations (1) to (4) the following equations can be derived:

$$\psi b_r = -\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rb_r) + \frac{im}{r}b_\theta\right) = ik_z\frac{db_z}{dr} \quad , \tag{30}$$

$$\psi b_\theta = -\frac{im}{r}\left(\frac{1}{r}\frac{d}{dr}(rb_r) + \frac{im}{r}b_\theta\right) = -\frac{m}{r}k_z b_z \quad , \tag{31}$$

$$\frac{\psi}{r}\frac{d}{dr}\left(\frac{r}{\psi}\frac{db_z}{dr}\right) + \left(\psi - \frac{m^2}{r^2}\right)b_z = 0 \quad , \tag{32}$$

where b_r, b_θ and b_z are functions of r, and ψ has been defined in Section 2.1.

The conducting boundary at $r = e$ imposes the conditions

$$b_r(e) = 0 \quad \text{and} \quad \left.\frac{db_z}{dr}\right|_e = 0 \quad .$$

Integration of (30) and (31) across the interface shows that

$$b_r,\ b_z \text{ and } \frac{1}{\psi}\frac{db_z}{dr} \text{ are continuous at } r = a \quad .$$

The properties of A-A, A-F and F-A waves in cylindrical geometry are easily derived by analogy with the corresponding waves in rectangular geometry, so only F-F waves are considered.

It is convenient to analyse the wave modes occurring in the two frequency ranges $\omega_1 < \omega < \omega_2$ and $\omega > \omega_2$ separately. No wave has a frequency less than ω_1.

When $\omega_1 < \omega < \omega_2$, the b_z field can be written in the form

$$b_z(r) = B_0 \frac{J_m(k_1 r)}{J_m(k_1 a)} \qquad 0 \le r \le a$$

$$= B_0 \frac{I_m(\kappa_2 r)K_m'(\kappa_2 e) - K_m(\kappa_2 r)\, I_m'(\kappa_2 e)}{I_m(\kappa_2 a)\, K_m'(\kappa_2 e) - K_m(\kappa_2 a)\, I_m'(\kappa_2 e))} \qquad a \le r \le e \quad (33)$$

where

$$k_1^2 = \omega^2/v_{\alpha 1}^2 - k_z^2 \quad \text{and} \quad \kappa_2^2 = k_z^2 - \omega^2/v_{\alpha 2}^2 \quad .$$

J_m, Y_m, I_m and K_m are the Bessel functions defined by Abramowitz and Stegun [1965], and $I_m'(r) = dI_m(r)/dr$. The dispersion relation is obtained by substituting (33) into the expression

$$\frac{1}{k_1}\left.\frac{db_z}{d(k_1 r)}\right|_{a^-} = -\frac{1}{\kappa_2}\left.\frac{db_z}{d(\kappa_2 r)}\right|_{a^+} \quad . \qquad (34)$$

When $\omega > \omega_2$, the b_z field can be written in the form

$$b_z(r) = B_0 \frac{J_m(k_1 r)}{J_m(k_1 a)} \qquad 0 \leq r \leq a$$

$$= B_0 \frac{J_m(k_2 r)\, Y_m'(k_2 e) - Y_m(k_2 r)\, J_m'(k_2 e)}{J_m(k_2 a)\, Y_m'(k_2 e) - Y_m(k_2 a)\, J_m'(k_2 e)} \qquad a \leq r \leq e \qquad (35)$$

where

$$k_2^2 = \omega^2/v_{\alpha 2}^2 - k_z^2 \quad .$$

The dispersion relation is found by substituting (35) into the expression

$$\frac{1}{k_1} \frac{db_z}{d(k_1 r)}\bigg|_{a^-} = \frac{1}{k_2} \frac{db_z}{d(k_2 r)}\bigg|_{a^+} \quad . \qquad (36)$$

The left hand side (LC) and right hand side (RC) of the dispersion relation are functions of ω similar in form to LS and R in the rectangular geometry case. LC has poles at ω_1 and at the zeros of $J_m(k_1 a)$, with interlaced zeros at the zeros of $J_m'(k_1 a)$. RC has poles at ω_2 and at zeros of $[J_m(k_2 a)\, Y_m'(k_2 e) - Y_m(k_2 a)\, J_m'(k_2 e)]$, with interlaced zeros at the zeros of $[J_m'(k_2 a)\, Y_m'(k_2 e) - Y_m'(k_2 a)\, J_m'(k_2 e)]$. When $m = 0$, the LC pole at ω_1 and the RC pole at ω_2 are no longer present. In the limit of small or large k_1, or when LC is positive, it has been shown that LC is a monotonically decreasing function of ω. Also, in the limit of small or large k_2, RC is a monotonically increasing function of ω. It is conjectured, by analogy with the rectangular geometry analysis, that this behaviour holds for all frequencies, in which case each eigenfrequency is bounded by adjacent poles of the dispersion relation functions LC and RC.

3.2 Wave Properties

The eigenmodes in cylindrical geometry depend on the parameters ρ_1, ρ_2, a, e, m and k_z in a similar way to the analogous modes in rectangular geometry, with m/a corresponding to k_y and even (odd) m modes corresponding to symmetric (antisymmetric) modes.

(i) For $m \neq 0$ and $\rho_2 << \rho_1$, the low order modes, with $\Omega < \omega_2$, have frequencies in the range

$$v_{\alpha 1}\left(\left(\frac{j_{m,n-1}}{a}\right)^2 + k_z^2\right)^{\frac{1}{2}} < \Omega_n < v_{\alpha 1}\left(\left(\frac{j'_{m,n}}{a}\right)^2 + k_z^2\right)^{\frac{1}{2}}$$

where $j_{m,n}$ and $j'_{m,n}$ are the nth zeros of J_m and J'_m, excluding the origin, and $j_{m,o} = 0$. The lowest order mode is a surface wave in which b_θ and b_z decrease with distance from a; the b_r field decreases for $r > a$ but has its maximum amplitude at some point $r \leq a$ which depends on the values of a, e and m. The b_z field of the nth mode is decreasing for $r > a$ and has n-1 nodes for $r < a$, excluding the origin. The description of the $n > 1$ modes as surface waves by Wentzel [1979b] seems inappropriate as they are similar, for $r < a$, to body waves in a homogeneous cylindrical plasma.

(ii) For $m = 0$, the surface wave does not occur and the b_z field of the lowest order mode has a node for $r < a$.

(iii) The following comments are valid for $k_z(e-a) > 1$. The surface wave has a maximum b_z field when $m/a \simeq k_z$. For $m/a << k_z$, the b field in region 1 is dominant and $\Omega_s \simeq \omega_1$. When $m/a >> k_z$, the dispersion relation for the surface wave is given by the upper limit in Equation (26) and the associated wave fields have magnitudes $|b_r| \simeq |b_\theta| >> b_z$. The same behaviour is found for the surface wave in rectangular geometry assuming that m/a is equivalent to k_y.

(iv) Finally it is emphasised that the cylindrical geometry wave modes are damped and undergo mode conversion in a similar way to the comparable modes in rectangular geometry, as summarised in Section 2.7.

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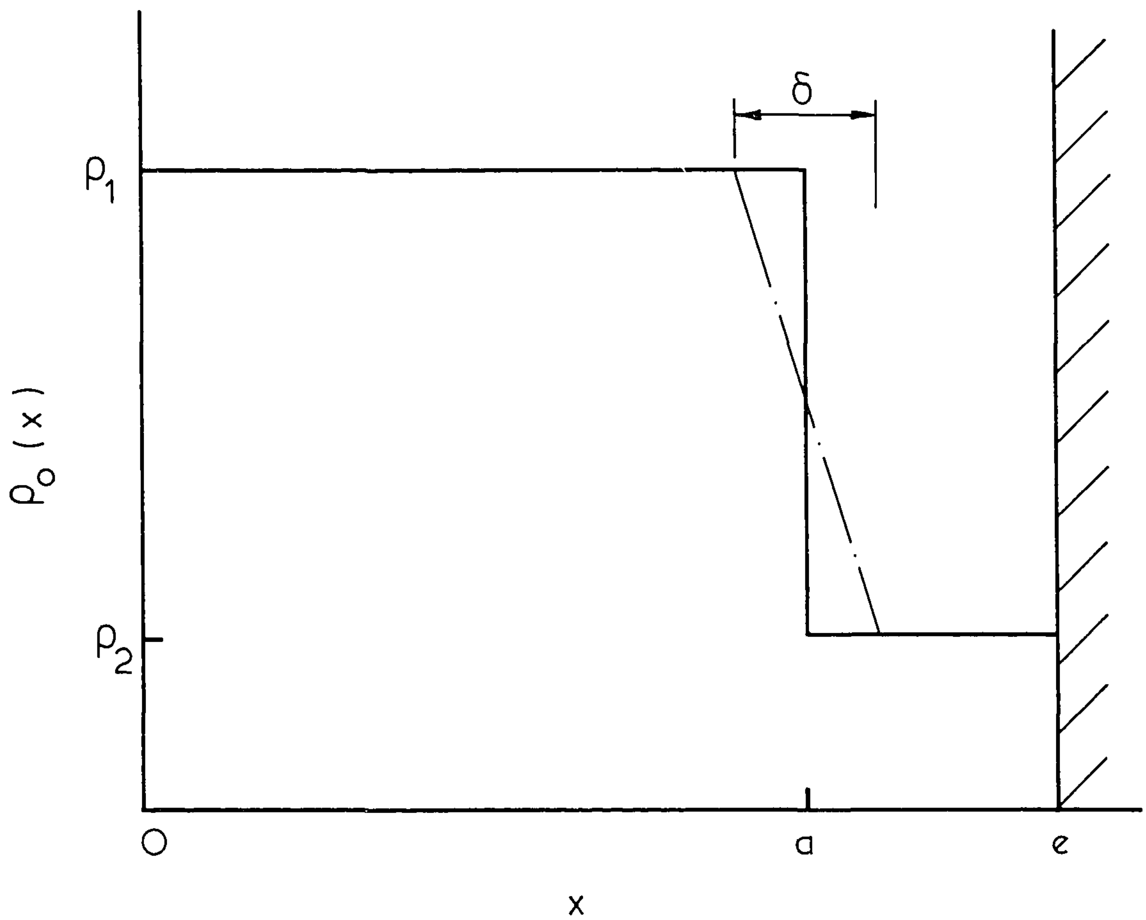

FIGURE 1. THE PLASMA DENSITY DISTRIBUTION. THE DASHED LINE INDICATES A MORE REALISTIC SURFACE DISTRIBUTION

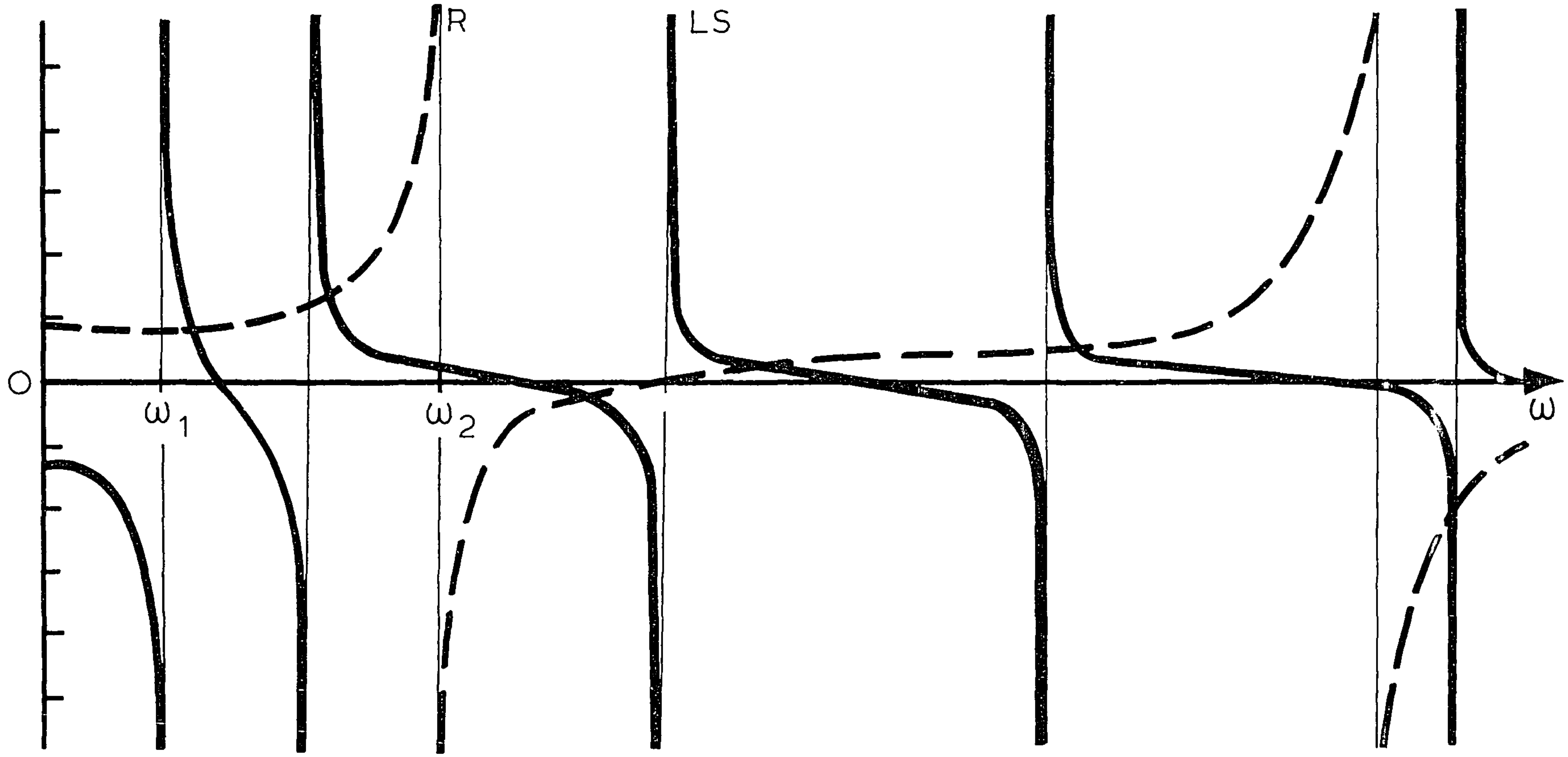

FIGURE 2. THE FUNCTIONS $LS(\omega)$ AND $R(\omega)$ WHEN $R_\rho = 0.1$, $k_y/k_z = 1$, $k_z a = 1$ AND $d = 0.5a$

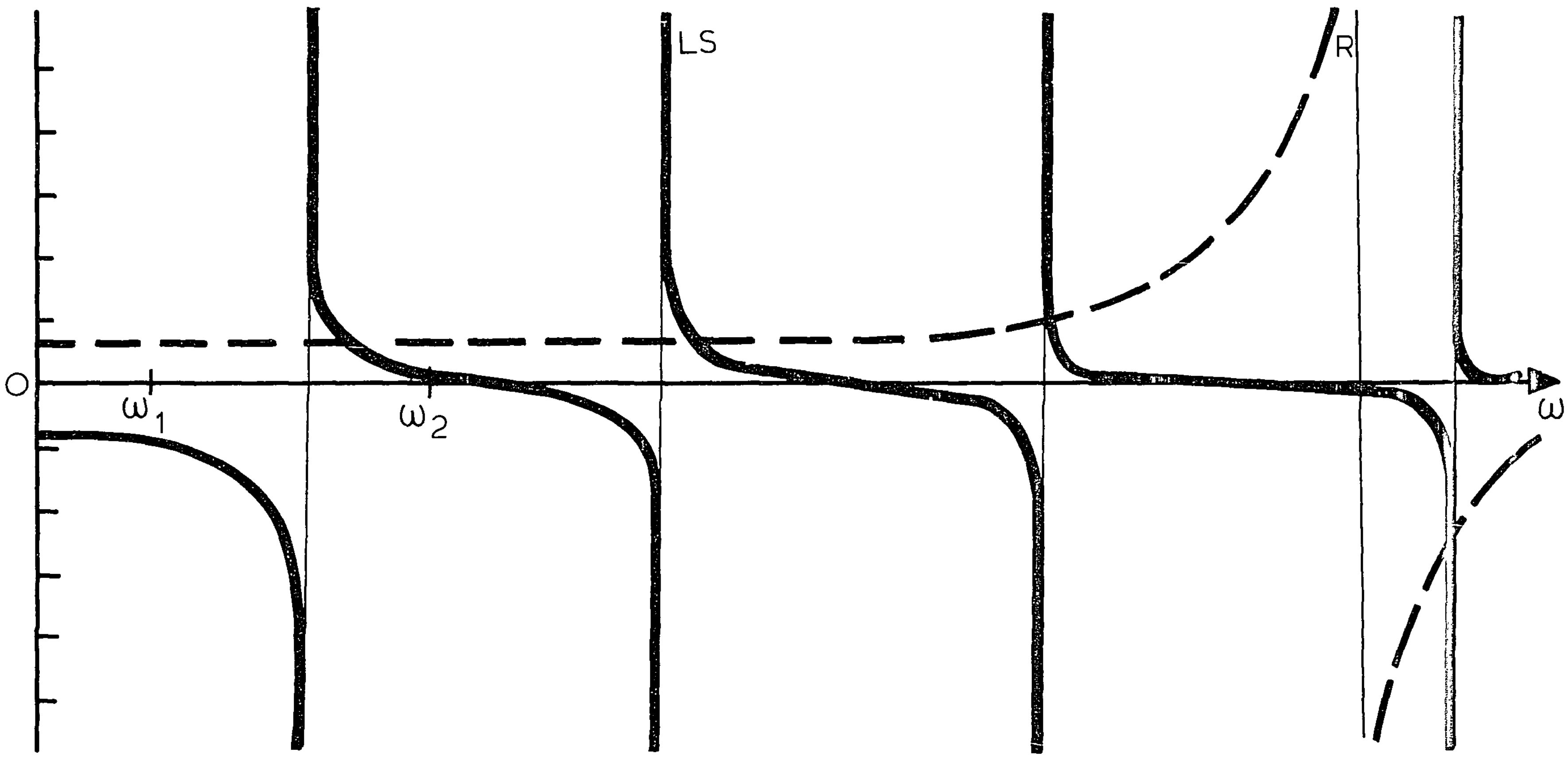

FIGURE 3. THE FUNCTIONS $LS(\omega)$ AND $R(\omega)$ WHEN $R_\rho = 0.1$, $k_y = 0$, $k_z a = 1$ AND $d = 0.5a$

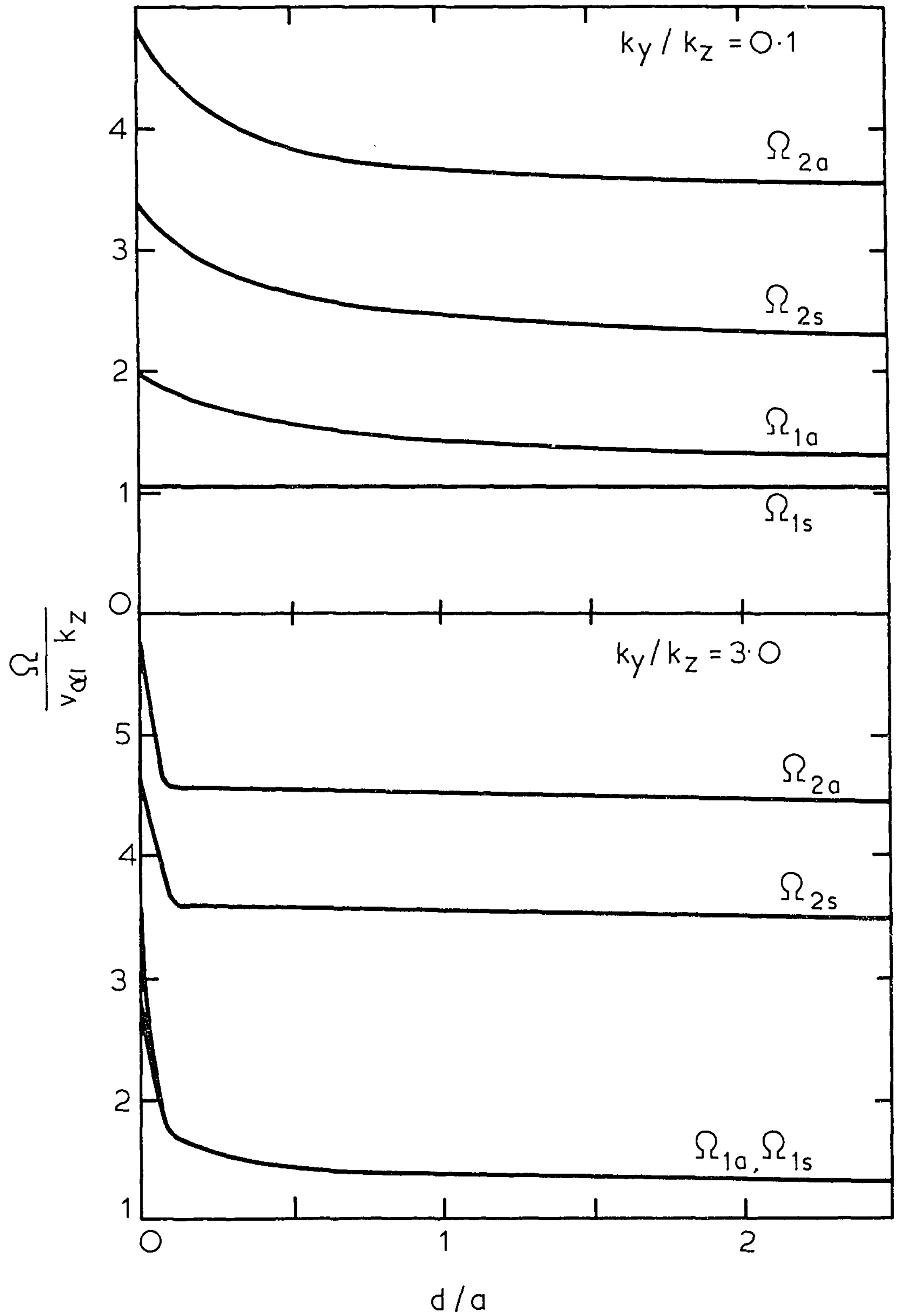

FIGURE 4. THE FIRST FOUR EIGENFREQUENCIES AS A FUNCTION OF d WHEN R_ρ = 0.0001 AND $k_z a$ = 1

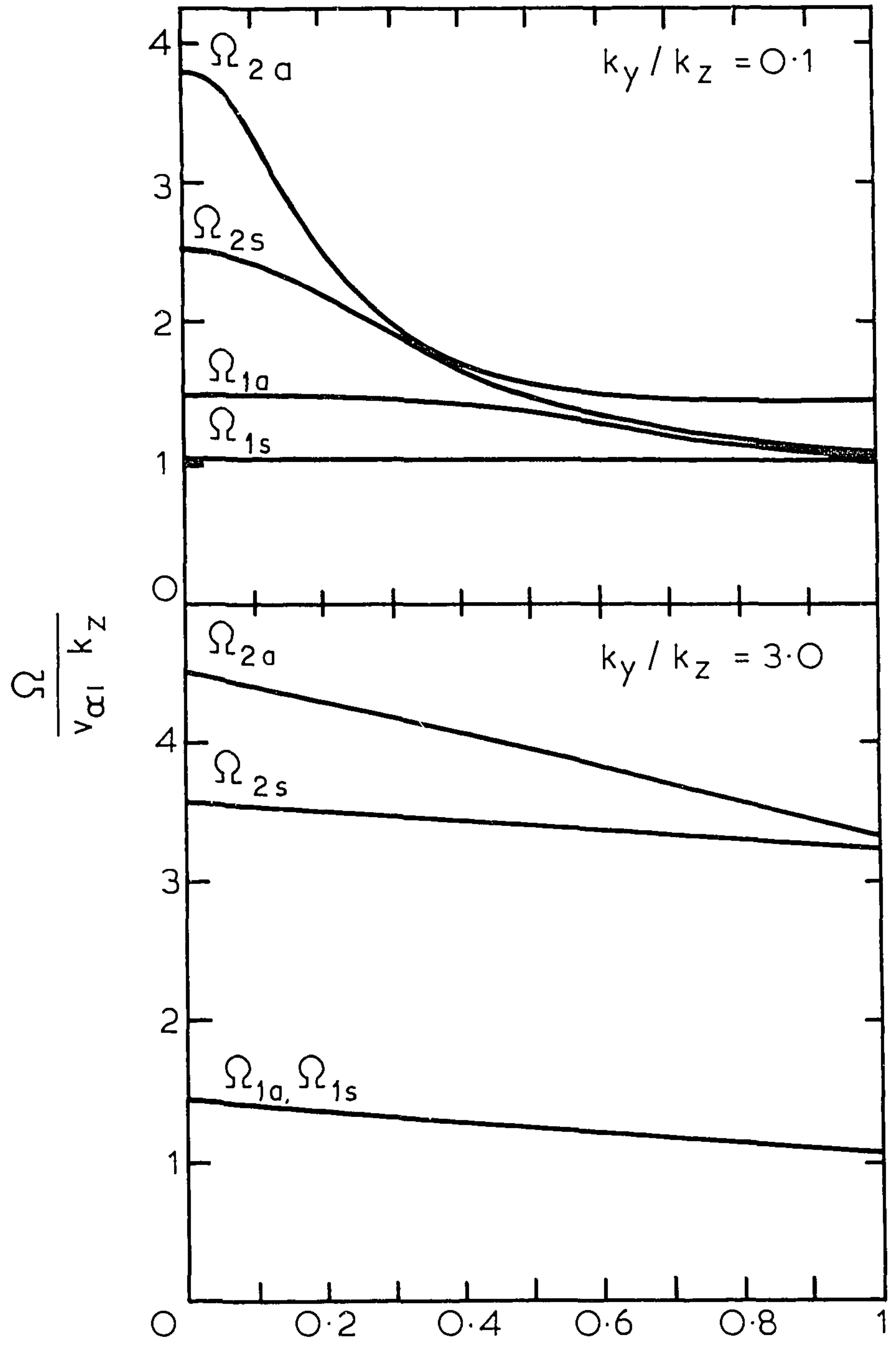

FIGURE 5. THE FIRST FOUR EIGENFREQUENCIES AS A FUNCTION OF ρ_2 WHEN $k_z a = 1$ AND $d = 0.5a$

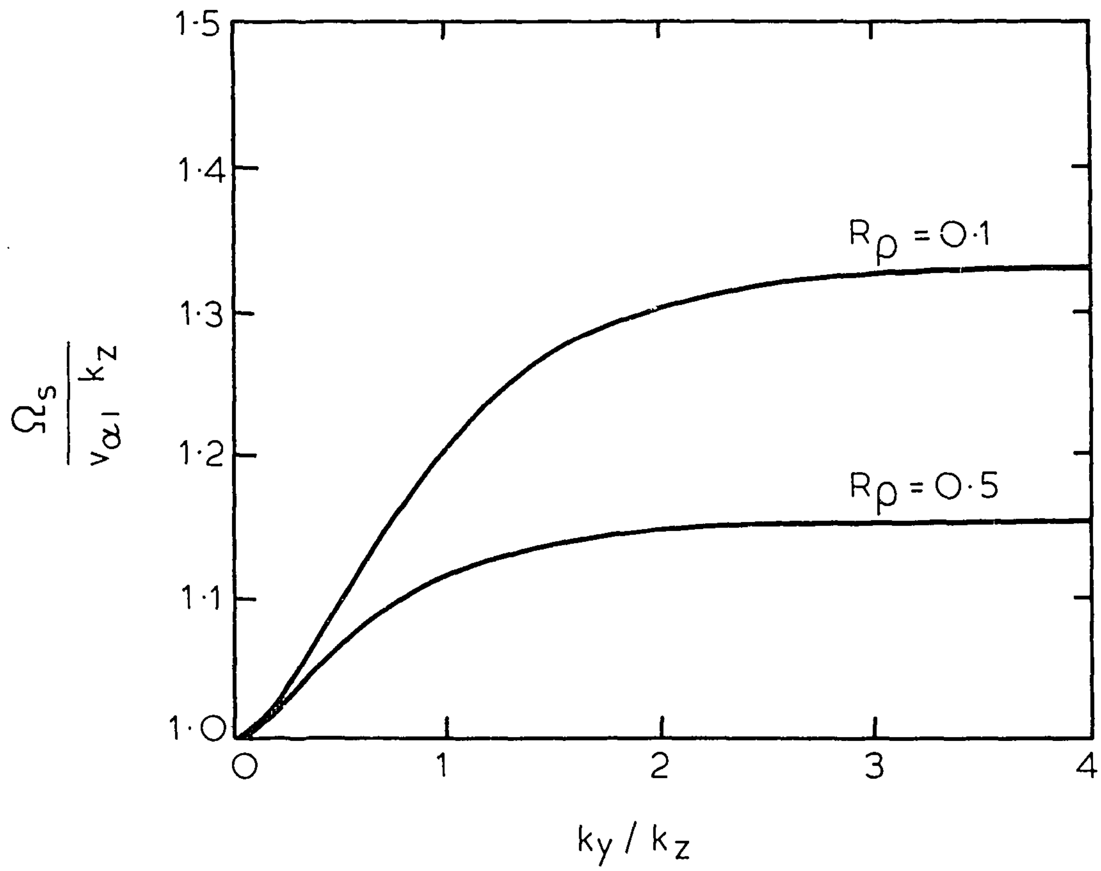

FIGURE 6. THE SURFACE WAVE FREQUENCY AS A FUNCTION OF k_y/k_z WHEN R_ρ = 0.1 AND R_ρ = 0.5

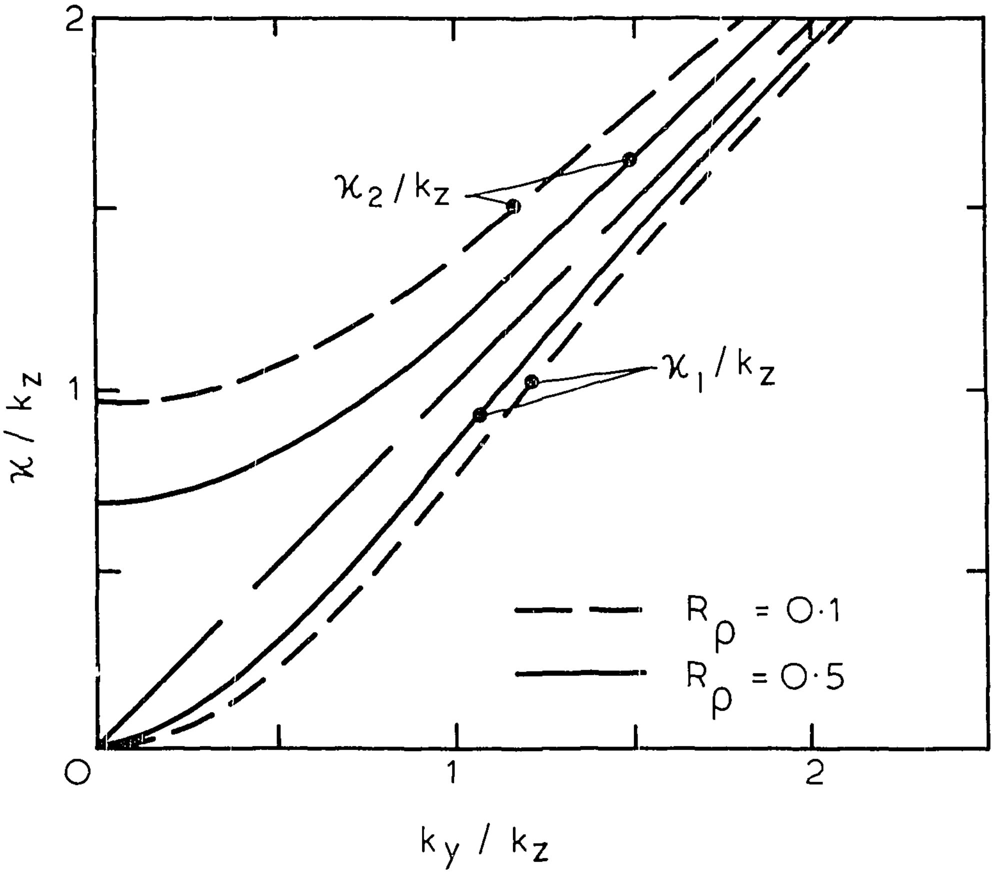

FIGURE 7. κ_1 AND κ_2 AS FUNCTIONS OF k_y/k_z WHEN $R_\rho = 0.1$ AND $R_\rho = 0.5$

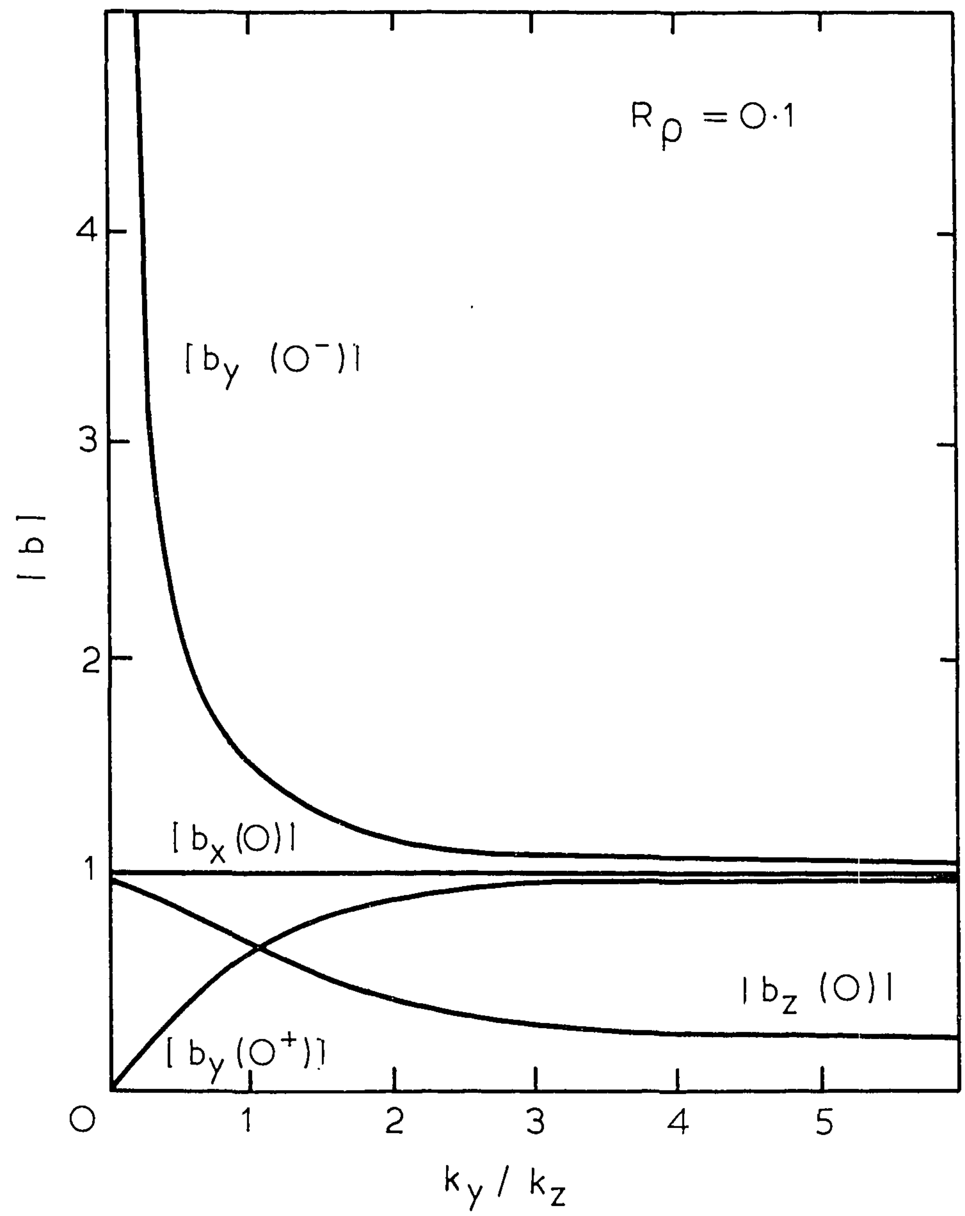

FIGURE 8. THE SURFACE WAVE MAGNETIC FIELD COMPONENTS AS A FUNCTION OF k_y/k_z. NOTE: $b_y(0^-)$ AND $b_y(0^+)$ HAVE OPPOSITE SIGN