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UNCERTAINTY RELATIONS AND SEMI-GROUPS IN B-ALGEBRAS *

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ABSTRACT

Starting from a B-algebra which satisfies the conditions of a structure theorem, we obtain directly a Lie algebra for which the Lie ring satisfies automatically the Heisenberg uncertainty relations.

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The analytical theory of semi-groups of bounded linear operators in a Banach space (B-space) deals with the exponential functions in infinite-dimensional function spaces.

It is concerned with the problem of determining the most general bounded linear operator-valued function $T(t)$, $t \geqslant 0$, which satisfies the equations

$$T(t+s) = T(t)\,T(s), \quad T(0) = 1 \quad .$$

The introduction of the infinitesimal generator A of $T(t)$ by E. Hille and K. Yosida defined by

$$A = s - \lim \frac{1}{t}\,[T(t) - I] \tag{1}$$

is very important from the phyisical point of view.

The basic result of the semi-group theory may be considered as a natural generalization of the theorem of Stone [1] on one parameter group of unitary operators in a Hilbert space.

Applications of the theory were made in quantum statistical mechanics [2] to the integration of the equations of evolution, which include wave equations and Schrödinger equations. Also in processes such as laser action, spin relaxation, etc. of Markovian master equations, where the Liouville operator is just the generator of the dynamical semi-group, have lead some authors to introduce the semi-group law as the fundamental dynamical postulate for open (non-Hamiltonian system)[3)-6)].

From the theory of the symplectic manifold [7] there exists on this manifold a closed non-degenerate differential 2-form D. The form D is defined on T^*N where T^*N is the cotangent bundle (symplectic manifold) over an arbitrary manifold N.

The collection $(q_1,\dots,q_k\ ,\ p_1,\dots,p_k)$ will be a local co-ordinate system in the region $U \subset T^*N$ which is the inverse image of $\widetilde{U}$ under the natural projection $p : T^*N \longrightarrow N$. We define a differential 1-form ϕ_v in U, setting

$$\phi_v = \sum_{i=1}^{k} p_i\, dq_i \quad . \tag{2}$$

We set $D = d\phi$. Then D is a closed form, since $dD = d^2\phi = 0$. Furthermore, the form D is non-degenerate, since in a local system of co-ordinates U it has the form

$$D_V = \sum_{i=1}^{k} dp_i \wedge dq_i \quad . \tag{3}$$

A vector field z on a symplectic manifold M with a form D is called Hamiltonian if

$$L_z D = 0 \quad , \tag{4}$$

where L_z is the Lie operator.

Let F and G be two smooth real functions on M. Then we have the equalities

$$z_F G = 2D\,(z_F, z_G) = -z_G F \quad . \tag{5}$$

The common value of the tree expressions entering in (5) is called the Poisson bracket of the functions F and G and is denoted by $\{F,G\}$. If the form D has the form (3) in a certain local co-ordinate system, then the Poisson bracket is defined by the equality

$$\{F,G\} = \sum_{k=1}^{k} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) \quad . \tag{6}$$

The space of $C^\infty(M,R)$ of smooth real functions on M forms an infinite-dimensional Lie algebra under the Poisson bracket. In fact, it can be shown that the Jacobi identity

$$\{F\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0 \tag{7}$$

holds.

The fundamental object in classical Hamiltonian mechanics is the phase space, which is a smooth symplectic manifold M.

Physical quantities are real functions on M, and a state of the system is a point on M. The change of the system with time is described by a strictly Hamiltonian vector field, the generating function of which is called the energy of the system and is denoted by H. Thus the equation that describes the change with time of a quantity F has the form

$$\dot{F} = \{H,F\} \quad . \tag{8}$$

In quantum mechanics, the role of the phase space is assumed by a projective space $P(V)$, where V is a certain Hilbert space. Physical quantities are self-adjoint operators on V. The value of the quantity A at the state defined by a unit vector $z \in V$ is a random variable with distribution function $p(t) = (E_t z, z)$ where E_t is the spectral projection measure for the operator A.

A change of the system with time is given by a group of unitary operators in V, which has the form

$$U(t) = e^{i\hbar tH} \quad .$$

Here h is Planck's constant, $\hbar = \frac{h}{2\pi}$ and H is a certain self-adjoint operator, called the energy operator. A change of the quantity F with time is described by the equation

$$\dot{F} = i\hbar\,[H,F] \quad . \tag{9}$$

Quantization is the process of constructing from a given classical system a quantum system that corresponds to it. Classical mechanics is in a certain sense an idealization of quantum mechanics - it follows by taking the limit as $h \to 0$.

Consider the physical quantities associated with a system. Among these we single out a certain set of <u>primary quantities</u> forming a Lie algebra under Poisson brackets. When we go over to quantum mechanics, the commutation relations among primary quantities are preserved in the following sense. Let h be a Planck constant and $\hat{F}$ the quantum-mechanical operator corresponding to the primary classical quantity F. Then the following relation must be satisfied:

$$\widehat{\{F_1, F_2\}} = i\hbar\,[\hat{F}_1 , \hat{F}_2] \quad . \tag{10}$$

This means that the correspondence $F \to = \frac{1}{i\hbar} \hat{F}$ is an operator representation of the Lie algebra of primary quantities. Ordinary constants are included among the primary quantities and one requires that the relation

$$\hat{1} = 1 \text{ (identity operator)} \tag{11}$$

holds.

For a system of cotangent sheaf one takes as primary quantities the set of linear functions of the co-ordinates $p_1,\dots,p_k$ and arbitrary functions of $q_1,\dots,q_k$.

To construct a representation of the Lie algebra of primary quantities it is natural to use the fact that the correspondence $F \to z_F$ is a representation of the Lie algebra $C^{\infty}(M)$ of all smooth functions on M (under the Poisson brackets). Hence setting $\hat{F} = i\hbar z_F$ we satisfy the relation (10).

The relevant aspect here is that the algebras characterized by brackets (8) and (9) coincide as abstract algebras. Thus the theory considered here can be called the "Lie algebra approach to quantum-mechanical systems".

In the present paper we give an application to the Heisenberg uncertainty relation. It is an extension of part of my doctoral thesis whose topic was suggested by Professor P. Tsilimigras.

It is known from quantum mechanics that if $\hat{F}_1$ and $\hat{F}_2$ are two hermitian operators defined on a Hilbert space which do not commute, the physical quantities $\hat{F}_1$ and $\hat{F}_2$ cannot both be sharply defined simultaneously. The degree to which an inevitable lack of precision in $\hat{F}_1$ and $\hat{F}_2$ must be admitted is measured by their commutator

$$\frac{1}{i}[\hat{F}_1, \hat{F}_2] = \hat{G} \quad . \tag{12}$$

The positive square roots of the variances $\Delta\hat{F}_1$ and $\Delta\hat{F}_2$ are called the uncertainties in $\hat{F}_1$ and $\hat{F}_2$. For these quantities it holds

$$\Delta\hat{F}_1 \, \Delta\hat{F}_2 \geq \left|\tfrac{1}{2}\langle \hat{G} \rangle\right| \quad . \tag{13}$$

As pointed out in Ref.8 the commutation relation (12) for quantum-mechanical observables $\hat{F}_1$, $\hat{F}_2$ and $\hat{G}$ by itself does not imply the uncertainty relation (13). The problem which arises for the uncertainty relation, the orbital angular momentum component L_z and the corresponding angle [9)-11)] ϕ as well as between the Hamiltonian H of the harmonic oscillator and the angle [12)] ensures that (8) is not valid.

The very important concepts of angle and phase operators have been a great challenge to understanding the fundamentals of quantum mechanics. In recent years these concepts attracted much attention in connection with coherent states, in laser physics, superconductivity and also in connection with the energy time uncertainty relation.

The main difficulty in treating the angle and phase co-ordinates is connected with their representation by a linear operator.

Kraus [13)] derived the uncertainty relation (13), assuming that $\hat{F}_1$, $\hat{F}_2$ and $\hat{G}$ were generators of a unitary representation of a suitable Lie group such that (12) is implied by the group structure.

In the following we give a sufficient condition in order to get the Heinsberg uncertainty relation (13).

Let B be a complex B-algebra with unit element e. We recall that the regular elements of B form a group, the maximal group of B which we shall now denote by G(B), while $G_1(B)$ stands for its principal component. Certain subgroups of $G_1(B)$ have a fairly simple structure which may be determined. The decisive step in this connection was taken by von Neumann [14),15)].

Let B_m be the algebra of m-rowed square matrices $M = (m_{ij})$ where the m_{ij} are complex numbers. This is a complex B-algebra under the norm

$$\|M\| = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |m_{ij}|^2 \right\}^{1/2} \quad .$$

Supposing that G is a group of regular matrices in B_m which is connected and closed in $G(B_m)$, von Neumann showed that G is a Lie group. More explicitly stated, there exists a set of infinitesimal generators R the so-called Lie ring of G with the following properties:

(i) every element of R belongs to B_m and R is closed,
(ii) if $U, V \in R$, so do $U + V$, $UV - VU$ and αU, α real,
(iii) R has a finite basis,
(iv) if $x = \exp(U)$, $U \in R$, then $X \in G$ and there is a neighbourhood of E in which all elements of G are of this form,
(v) every element of G is of the form

$$\exp(U_1)\, \exp(U_2) \ldots \exp(U_k), \quad U_i \in R \quad .$$

Here (iii) is implied by (i). It is clear that R cannot contain more than $2m^2$ elements that are linearly independent with respect to real numbers. The properties (i)-(v) hold with B_m replaced by B. In this case (iii) is no longer implied by (i).

Now we shall present a generalization of the above statement in an abstract B algebra.

Definition 1. Let S be a set in B. S is said to be differentiable at $x = x_0$, $x_0 \in S$, if every sequence $\{x_n\}$ in S with $x_n \to x_0$ contains a subsequence $\{x_{n_k}\}$ such that there is a sequence of real numbers $\{\epsilon_k\}$, $\epsilon_k \to 0$ and

$$\lim_{k \to \infty} \frac{1}{\epsilon_k} (x_{n_k} - x_0) = u \neq 0$$

exists as an element of B. We call u a differential coefficient of S at x_0.

It is clear from the definition that if u is a differential coefficient so is αu for every real α. We shall state a construction theorem which gives us the ability that when we start by a semi-group in a B-algebra we can construct a Lie ring [16].

Theorem 1. Let T be a semi-group in B with the following properties:

(1) T contains at least two elements of G(B),

(2) T is closed under inversion: if $x \in T \cap G(B)$, then $x^{-1} \in T$,

(3) $G = T \cap G(B)$ is closed in G(B), locally compact and connected,

(4) T is locally a division ring: if $x, y \in T$ and $\|x-y\| < \delta_x$ then there exists a $g \in G$ such that $x = yg$ and $g \to e$ when $y \to x$.

Then

(i) T is differentiable at $x = e$ and hence everywhere.

(ii) T is a Lie semi-group where the set of differential coefficients of T at $x = e$, form a Lie ring R having properties (i) to (v).

With the above state of affairs given a B-algebra we can always define one-parameter semi-groups, e.g. in a given B algebra we can define one-parameter semi-group as follows:

Let $C(\mathfrak{X})$ be a complex Banach algebra of bounded linear transformations on $\mathfrak{X}$ to itself and let $f(z)$ be a function on positive numbers to $C(\mathfrak{X})$, such that

$$f(z_1 + z_2)\, x = f(z_1)\,[f(z_2)x], \quad 0 < z_1, z_2 < \infty$$

for all $x \in \mathfrak{X}$. Thus $T = [f(z)]$ is a semi-group of operators in $C(\mathfrak{X})$. Therefore we conclude that when in a given B-algebra we can define a semi-group such that the conditions of the theorem are fulfilled, then the generators of the maximal group G of the given B-algebra $\mathfrak{X}$ satisfies the relation (13)

$$\Delta\hat{F}_1 \, \Delta\hat{F}_2 \geq \left| \frac{1}{2} \langle \hat{G} \rangle \right| \quad .$$

The dimensions of the Lie ring are two or three, or any finite number of dimensions.

Thus with the help of the aforementioned construction theorem starting from a B-algebra (which satisfies the conditions of the theorem) we obtain directly a Lie algebra for which the Lie ring satisfies automatically the uncertainty relation.

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REFERENCES

1) M.H. Stone, "On one-parameter unitary groups in Hilbert space", Ann. Math. 2, 33 (1932).

2) E.B. Davis, Quantum Theory of Open Systems (Acdemic Press, 1976).

3) G. Lindbland, "On the generators of quantum dynamical semigroups", Commun. Math. Phys. 48, 119 (1976).

4) J. Mehra and E.C.G. Sudarshan, Nuovo Cimento 11B, 215 (1972).

5) A. Kossakowski, Rep. Math. Phys. 3, 247 (1972).

6) S.R. Ingarden and A. Kossakowski, Ann. Phys. 89, 451 (1975).

7) A.A. Kirilov, Elements of the Theory of Representations (Springer-Verlag, 1975).

8) K. Kraus, Z. Physik 188, 374 (1965).

9) D. Judge, Phys. Letters 5, 189 (1963).

10) D. Judge, Nuovo Cimento 31, 332 (1964).

11) L.C. Papaloucas, Lettere al Nuovo Cimento 14, No.4, 113 (1975).

12) P. Tsilimigras and L.C. Papaloucas, Phys. Letters 34A, 7 (1971).

13) K. Kraus, Z. Physik 201, 134 (1966).

14) J. von Neumann, Zur Theorie der Darstellungen Kontinuierlicher Gruppen. Sitzungsber Preuss. Akad. Wiss. 76 (1927).

15) J. von Neumann, "Uber die analytischen Eigenschaften von Gruppen linearer Transformationen und ihrer Darsteblungen, Math. Z. 30, 3 (1929).

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