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ABSTRACT

A nonlinear field model in three space-dimensions with nontrivial Hopf invariant is studied in an intermediate S^3-manifold. Energy-spectrum of the system for different values of topological charge is calculated numerically, which with fair accuracy coincides with $\frac{n(n+1)}{2} E_1$ where n = topological charge.

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1. Introduction

Nonlinear classical field models with nontrivial topological charges received much attention in recent years. The dependence of energies of such systems on their topological charges Q is, however, still an open question. Only in few cases when the systems allow self-dual solutions with $Q = \pm n$, e.g. for nonlinear O-model in two-dimensional and SU(2) Yang-Mills field model in four-dimensional Euclidean spaces, actions (energy) of the systems depend linearly on topological charges. In other cases, particularly for models defined in a real space, it is difficult to find the exact solutions and hence the energy dependence on topological charges. In the present note we investigate a nonlinear model in three space dimensions and through numerical calculations find the topological soliton solutions as well as the energy-spectrum of the system for different values of the topological charge.

2. The Model

The model is given in (3+1) space-time and the field function carries a topological charge called the Hopf index /1/ of the mapping $S^3 \to S^2$ which may be given by

$$Q = -(8\pi)^{-2} \int \underline{a} \cdot \underline{b} \; d^3x \quad , \qquad (1)$$

where $\underline{b} = \text{rot}\,\underline{a}$ and

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu = 2 \epsilon^{abc} \partial_\mu n_a \partial_\nu n_b n_c \qquad (2)$$

$n_a(\underline{x}, t): R^3 \times R \to S^2$ being the field function satisfying the conditions

$$\sum_{a=1}^{3} n_a n_a = 1 \quad \text{and} \quad \lim_{\underline{x} \to \infty} n^a = \delta^a_3$$

Since two forms are not exact on S^2 vector $\underline{a}$ in (1) is not globally defined on it /2/. Hence $\underline{a}$ is not a local function of the field $n^a(x)$. On the other hand, since on S^3 all closed forms are exact, one may overcome the above difficulty by pulling back the two form ω_2 from S^2 to S^3_I /2/. Thus we have $\omega_2(S^2) \to h^*\omega_2(S^3_I)$ where h is the induced map of the Hopf mapping $h: S^3_I \to S^2$ given by

$$h: n^a = \Psi^+ \sigma^a \Psi \quad , \tag{3}$$

where σ^a Pauli matrices and Ψ is an isospinor given on S^3_I by

$$\Psi = \begin{pmatrix} X \\ Y \end{pmatrix} \quad , \qquad \Psi^+\Psi = 1 \tag{4}$$

with $X = \varphi_1 + i\varphi_2$, $Y = \varphi_3 + i\varphi_4$ where $\varphi_m \in S^3_I$, i.e. $\varphi_m\varphi_m = 1$, $m = 1,2,3,4$. In S^3_I-coordinates $f_{\mu\nu}$ takes the form (denote it by $F_{\mu\nu}$)

$$F_{\mu\nu} = -4i\left(\partial_\mu\Psi^+\partial_\nu\Psi - \partial_\nu\Psi^+\partial_\mu\Psi\right) \tag{5}$$

which may be easily checked by using Hopf mapping (3). From (5) one immediately gets vector a as a regular function in S^3_I-coordinates (denote it by A_μ):

$$A_\mu = -4i\,\Psi^+\partial_\mu\Psi \quad , \tag{6}$$

where the relation $\partial_\mu\Psi^+\Psi = -\Psi^+\partial_\mu\Psi$ has been used. Hence the Hopf index may be expressed /2/ as

$$Q = -\frac{\epsilon^{ijk}}{2(8\pi)^2}\int_{S^3} A_i\, F_{jk}\, d^3x = -(8\pi)^{-2}\int \underline{A}\cdot\underline{B}\; d^3x \tag{7}$$

with $\underline{B}$ = rot $\underline{A}$. It is interesting to note that the Hopf index (7)

may also be expressed as

$$Q = -\frac{\epsilon^{ijk}}{4(8\pi)^2}\int C_i\, V^*_{jk}\, d^3x = -\frac{\epsilon^{ijk}}{4(8\pi)^2}\int C^*_i\, V_{jk}\, d^3x \quad , \tag{8}$$

where

$$C_\mu = -4(\,Y\partial_\mu X - X\partial_\mu Y), \qquad V_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \tag{9}$$

are also defined in S^3_I.

We investigate the model given by the Lagrangian density /3/

$$\mathcal{L} = \frac{1}{16}\left\{\frac{m^2}{2}\left(A_\mu A^\mu + C^*_\mu C^\mu\right) - \frac{1}{4e^2}\left(F_{\mu\nu}F^{\mu\nu} + V^*_{\mu\nu}V^{\mu\nu}\right) - \gamma^2\left(1-(\varkappa eX)^2\right)\right\} \tag{10}$$

The last term in (10) is added to ensure the exponential vanishing of the field solutions at space-infinities. It can be easily shown that the Hamiltonian of the above model is estimated from below through the topological charge (7) or (8):

$$H \geqslant H_{st} = (m/16e)\int\left\{\left(\underline{A}^2 + \underline{C}^*\underline{C}\right) + \frac{1}{4}\left(\underline{B}^2 + \underline{V}^*\underline{V}\right) + \tilde{\nu}(1-\varphi_1^2)\right\} d^3x$$

$$\geqslant (3/16(8\pi)^2)\ \frac{m}{e}\ |\,Q\,| \tag{11}$$

with $\tilde{\nu} = 8\gamma^2/m^4 e^2$. Such an estimate guarantees the stability of the solution with minimum energy. The maximal compact group admitted by the Hamiltonian for which the invariant field n_a gives $Q \neq 0$ is /4/

$$G = \mathrm{diag}\ (\,O_s(2)\otimes O_I(2)\,)\ , \tag{12}$$

where $O_s(2)$ and $O_i(2)$ are respectively the rotation groups around the axes z and n_3. The G-invariant n-field in spherical coordinates (r, ξ, α) takes the form

$$\beta = \beta(r, \xi), \qquad \gamma = \pm\alpha - v(r, \xi) \tag{13}$$

(β, γ) being the polar angles of vector $n^a \in S^2$. Ansatz (13) leads to the following expressions:

$$\begin{aligned} &X = \cos(\beta/2)\exp(iv), \qquad Y = \sin(\beta/2)\exp(i\alpha) \\ &n_1 + in_2 = \sin\beta \exp(i\gamma), \qquad n_3 = \cos\beta \equiv w \\ &\underline{B} = \mathrm{rot}\,\underline{A} = -2\left[\underline{\nabla} w, \underline{\nabla}\gamma\right], \qquad \gamma = \alpha - v \\ &A_\mu = 2\left\{(1+w)\partial_\mu v + (1-w)\partial_\mu \alpha\right\} \end{aligned} \tag{14}$$

Putting (14) directly in static Hamiltonian (10) of the system (since permited by Coleman's principle /5/) one gets

$$\begin{aligned} H_{st} = (2m\pi/e)\iint r^2 \sin\xi\, dr\, d\xi \Big\{ (\underline{\nabla}\lambda)^2 + \cos^2\lambda(\underline{\nabla} v)^2 + \\ \sin^2\lambda (r\sin\xi)^{-2}\left[1 + (\underline{\nabla}\lambda)^2 + \cos^2\lambda(\underline{\nabla} v)^2\right] + \\ \cos^2\left[\underline{\nabla}\lambda, \underline{\nabla} v\right]^2 + 2\nu(\sin^2\lambda + \cos^2\lambda \sin^2 v)\Big\} \end{aligned} \tag{15}$$

where $\lambda = \beta/2$ and $\nu = \tilde{\nu}/32$. The Euler-Lagrange equations with respect to λ and v with the help of ansatz

$$\begin{aligned} \cos\beta &= 1 - 2\sin^2\theta(r)\sin^2\xi \\ v &= \tan^{-1}(\tan\theta(r)\cos\xi) + c \end{aligned} \tag{16}$$

reduce to a single equation relative to $\theta(r)$:

$$\theta''(r^2 + 2\sin^2\theta) + 2r\theta' + \sin 2\theta\left(\theta'^2 - (1 + \nu r^2 + \sin^2\theta/r^2)\right) = 0. \tag{17}$$

If we demand the following boundary condition

$$\theta(0) = n\pi, \qquad \theta(\infty) = 0 \tag{18}$$

then the topological charge (7) is equal to

$$Q = \mp\pi^{-1}\int_0^{\pi}\sin\xi\, d\xi \int_0^{n\pi}\sin^2\theta\, d\theta = \mp n. \tag{19}$$

Note that equation (17) also coincides with the Skyrme's equation for spherically symmetric ansatz /6/. Therefore, the result of our investigation is also applicable to the energy-spectrum of Skyrme's model. The uniqueness and regularity of the solution of equation (17) for boundary condition (18) were established in /7/. Hence we may take up the numerical investigation of the problem with full confidence. The energy of the system is given by

$$E_n = 4\pi(m/e)\int_0^{\infty}\mathcal{E}_n(\theta, \theta')\, dr = 4\pi(m/e)\, I_n[\theta], \tag{20}$$

where $\mathcal{E}_n(\theta, \theta') = (\theta' r)^2 + 2\sin^2\theta(1 + \theta'^2 + \nu r^2) + \sin^4\theta/r^2$

The solution of (17) has the following asymptotics:

$$\begin{aligned} &\lim_{r\to 0}\theta = n\pi + ar + br^3 + \ldots \\ &\lim_{r\to\infty}\theta = \exp(-cr)/\sqrt{r}, \qquad a, b, c = \mathrm{const}. \end{aligned} \tag{21}$$

Equation (17) with (18) was integrated by numerical methods for $\nu = 1$ and $n = 1, 2, \ldots, 5$ and the energy of the system was calculated for these cases. The results are shown in the table. Comparing coloumbs III and IV in the table one can observe immediately an interesting depe ndence of the energy-spectrum on topological charge:

$$E_n \approx E_1\, n(n+1)/2, \qquad \text{for} \quad Q = \pm n$$

which holds with a fair accuracy (error less than 1%).

3. Conclusion

We have investigated a nonlinear field model with nontrivial topological charge in three space-dimensions and numerically found the energy dependence on the topological charge of the system. This energy-spectrum coincides with that of a quantum rotator. The above fact is difficult to explain theoretically. We may only speculate that the existence of such a nice energy dependence on topological charge shows probably the existence of an exact solution of the system, the discovery of which would be a real achievement.

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Topological charge $\|Q\|$ = n	Field derivative for different solutions at r=0 $\theta'(0)$	Result of numerical experiment E_n/E_1	n(n+1)/2
I	II	III	IV
1	-2.59088	1	1
2	-4.77865	2.999	3
3	-6.84426	5.975	6
4	-8.85182	9.909	10
5	-10.82435	14.791	15

Table

Energy of the system E_n for different values of the topological charge. The energy value for $|Q| = 1$ is $E_1 = 4\pi(m/e)\, I_1$, $I_1 = 13.6896$.

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