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ABSTRACT

Duality transformation of the vacuum expectation value of the Wilson loop operator is performed in the radial gauge ($x_\mu A^a_\mu(x) = 0$). It is found to be equal, up to a multiplicative constant, to $< 0 \mid \mathrm{Tr}\ P \exp O(c) \mid 0 >$, where $O(c)$ is a line integral along the loop c (defining the Wilson loop operator) of a function of the dual field variables. In the weak coupling region self duality is recovered in the sense that the Lagrangian is local gauge invariant defined in terms of the dual gauge potentials but with g(the coupling constant) replaced by $1/g$, and $O(c)$ is simply the line integral of the dual gauge potentials. For large g, a strong coupling expansion is suggested (but the theory is not local gauge invariant).

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1. INTRODUCTION

Duality transformation is a very well known concept in lattice models [1]. There, one expresses the Hamiltonian in terms of variables defined on the dual lattice, in which case g (the coupling constant) goes into 1/g . Hence it is a useful device when the strong coupling region of the theory is explored, and in particular it is important in finding the critical points of the model. A self dual theory would be one where the Hamiltonian has the same functional dependence on the dual variables as on the original ones. Another concept of duality transformation is used in Abelian gauge models when the roles of electric and magnetic fields are interchanged, and for the free electromagnetic field the Lagrangian in Euclidean space-time is known to be invariant under this interchange. When electric charges are introduced the equations of motion are invariant under this transformation only if magnetic monopoles exist. These two independent concepts of duality transformations can be connected to each other if one realizes that the transformation to the dual variables on the lattice can be performed by Fourier transform in function space [2]. When this is done in the U(1) gauge model one finds that this exchanges the roles of electric and magnetic fields, and electric currents become magnetic ones for the dual variables, hence the theory is invariant if both electric and magnetic currents exist.

This same method can be used [3] to perform duality transformation of a non-Abelian gauge theory. Here, though, the gauge choice is very important because only when an inversion formula, which expresses the gauge potentials in terms of the field strengths, exists it is possible to write the Lagrangian in terms of the dual gauge potentials, and to decide whether the theory is self dual or not. Such a gauge condition is the radial gauge [4] ($x_\mu A^a_\mu(x) = 0$), where for non-singular gauge potentials an inversion formula exists which expresses them in terms of the field strengths. Using this gauge, when the duality transformation is performed, the Lagrangian (in 4 dimensional Euclidean

space-time) is found to be invariant in the weak coupling region [5] (small g). That is to say, when g is small we get a non-Abelian gauge theory of the dual potentials satisfying the same gauge condition but with g replaced by 1/g . In the strong coupling region it is possible to get a consistent strong coupling expansion (in terms of 1/g) for the dual variables. In this gauge, however, the important role played by monopoles and vortices cannot be singled out in the pure Yang-Mills theory because they are singular field configurations not attainable in the radial gauge. This can be remedied by considering a spontaneously broken gauge theory [6], where part of the singularities of monopoles and vortices are smeared out, hence they can be transformed to the radial gauge and the inversion formula is valid. And indeed when the duality transformation is employed on the spontaneously broken gauge theory it is found that the theory is self dual in the weak coupling region only in directions of symmetry restorations, or in regions where monopoles and vortices are concentrated. This seems to be consistent both with the result found in the pure non-Abelian theory [5], and with 't Hooft's [7] ideas that self duality can be shown to hold if one considers contributions of monopoles and vortices only to the generating functional. It is not yet clear whether the duality transformation used by 't Hooft is exactly the one considered above [2]-[6], even though for the Abelian theory they seem to give the same results. Hence investigating the properties of the Wilson loop operator under this transformation can give a hint as to the relation between the two transformations and maybe one can also find a way of expressing 't Hooft loop operator in terms of the dual as well as the original variables. A by-product will be a strong coupling expansion of the vacuum expectation value (VEV) of the Wilson loop operator. The importance of the Wilson loop operator in understanding confinement is very well known, hence any information about its behaviour in the strong coupling region should be welcomed.

In the following we will perform the duality transformation of the VEV the Wilson loop operator, and will find its behaviour in the weak coupling region. In the strong coupling region a systematic expansion in powers of $1/g$ is suggested. In particular we will show that in terms of the dual fields the VEV of the Wilson loop operator is equal (up to a multiplicative constant) to the VEV of $\mathrm{TrPexp}O(c)$, where $O(c)$ is an operator expressed in terms of the dual fields. In the weak coupling limit self duality holds in the sense that this operator becomes a line integral along the loop c (defining the Wilson loop) of the dual gauge potentials satisfying the radial gauge condition (with a local gauge invariant Lagrangian as a functional measure). This appears in Section 2, Section 3 contains the concluding remarks, and in an Appendix we suggest an alternative way to get the same results.

2. DUALITY TRANSFORMATION OF WILSON LOOP OPERATOR

Consider the generating functional of the Yang-Mills theory in 4 Euclidean space-time dimension.

$$Z(J)= \int[dA_\mu^a]\, \delta(x_\mu A_\mu^a)\, \exp\left(-\int d^4x\, \mathcal{L}(x)\right) \tag{1}$$

$$\mathcal{L}(x)= \frac{1}{4g^2} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + i A_\mu^a(x) J_\mu^a(x) \tag{2}$$

$$F_{\mu\nu}^a(x)= \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + C_{abc} A_\mu^b(x) A_\nu^c(x) \tag{3}$$

Here we are using the scaled field variables:

$$\begin{aligned} A_\mu^a(x) &= g\, A_\mu^{(0)a}(x) \\ F_{\mu\nu}^a(x) &= g\, F_{\mu\nu}^{(0)a}(x) \end{aligned} \tag{4}$$

$J_\mu^a(x)$ are the external sources used to define the n point functions via functional derivatives with respect to the external source, at zero external

source.

$$x_\mu A_\mu^a(x) = 0 \tag{5}$$

is the gauge condition. In this gauge the Faddeev-Popov ghosts are independent of the fields hence they were omitted.

Using the following relation

$$\exp\left(-\frac{1}{4g^2}\int d^4x\, F_{\mu\nu}^a F_{\mu\nu}^a\right) \propto \int[dW_{\mu\nu}^a]\, \exp\left[-\int d^4x \left(\frac{g^2}{4} W_{\mu\nu}^a W_{\mu\nu}^a + \frac{i}{2} F_{\mu\nu}^a \tilde{W}_{\mu\nu}^a\right)\right] \tag{7}$$

where
$$\tilde{W}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} W_{\rho\sigma}^a \tag{8}$$

and $W_{\mu\nu}^a$ are six independent antisymmetric tensor variables (multiplied by the number of group generators), it has been shown by Itabashi [5] that

$$\begin{aligned} Z(J) = &\int[dB_\mu^a]\, \delta(x_\mu B_\mu^a)\, \exp\left(-\frac{g^2}{4}\int d^4x\, G_{\mu\nu}^a G_{\mu\nu}^a\right)(\det T^{-1})^{1/2} \\ &\cdot \exp\left\{-i\int \frac{d^4x}{x_4}\left[\frac{1}{2} j_i^a (T^{-1}j)_i^a + j_i^a B_i^a\right]\right\} \\ &\int[d\omega_{\mu\nu}^a] \prod_{i=1}^{3} \delta(x_j \omega_{ji}^a)\, \exp\left[-\frac{g^2}{4}\int d^4x \left(\omega_{\mu\nu}^a \omega_{\mu\nu}^a + 2\omega_{\mu\nu}^a G_{\mu\nu}^a\right)\right] \\ &\cdot \exp\left\{-\frac{i}{2}\int \frac{d^4x}{x_4}\left[Y_i^a (T^{-1}Y)_i^a + 2B_i^a Y_i^a + 2Y_i^a (T^{-1}j)_i^a\right]\right\} \end{aligned} \tag{9}$$

where the equality holds up to a multiplicative constant. Here B_μ^a are the dual gauge potentials, and $G_{\mu\nu}^a$ are the dual gauge fields, namely

$$G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + C_{abc} B_\mu^b B_\nu^c \quad . \tag{10}$$

The $\omega_{\mu\nu}^a$ are antisymmetric tensor fields connected to $W_{\mu\nu}^a$ via the following relation

$$W_{\mu\nu}^a = G_{\mu\nu}^a + \omega_{\mu\nu}^a \quad , \tag{11}$$

j_i^a are linear in the external sources

$$j_i^a = x_4 J_i^a - x_i J_4^a \,, \tag{12}$$

T_{ij}^{ab} is a tensor with two sets of indices, space-time (i,j = 1,2,3) and internal (a,b = 1,...N, dimension of the algebra)

$$T_{ij}^{ab} = \varepsilon_{ijk}\, C_{abc}\, x_\rho\, G_{\rho k}^{c} \tag{13}$$

antisymmetric independently in ij and in a,b and

$$Y_i^a = \varepsilon_{ijk}\,(2 + x_\lambda \partial_\lambda)\,\omega_{jk}^a \tag{14}$$

Sum over repeated indices should be understood, $\mu,\nu = 1,2,3,4$ $i,j = 1,2,3$ and $a = 1,2,\ldots,N$.

The expression for $Z(J)$ has been proven with the aid of the relations (inversion formula)

$$x_\rho F_{\rho\mu}^a(x) = (1 + x_\rho \partial_\rho) A_\mu^a(x) \tag{15a}$$

$$A_\mu^a(x) = \int_0^1 d\alpha \;\; \alpha x_\rho F_{\rho\mu}^a(\alpha x) \tag{15b}$$

which are valid in the radial gauge [5].

Going back to the unscaled fields:

$$B_\mu^a = \frac{1}{g} B_\mu^{(0)a} \quad , \quad G_{\mu\nu}^a = \frac{1}{g} G_{\mu\nu}^{(0)a} \quad , \quad Y_i^a = \frac{1}{g} Y_i^{(0)a} \tag{16}$$

it is easily seen [5] that $Z(J)$ reduces in the small g limit to a functional integration over the pure Yang-Mills Lagrangian of the dual potentials satisfying the same gauge condition but with g replaced by $1/g$. Hence the celebrated self duality of the theory in the small g limit.

To get the vacuum expectation value of the Wilson loop operator, we note the following relation *

* This representation of the Wilson loop operator was pointed out to me by John Strathdee. I would like to thank him for that.

$$A(c) = \mathrm{Tr}\, P \exp\left(i \oint_c A_\mu^a(\xi)\, \tau^a\, d\xi_\mu\right) =$$
$$= \int [d\alpha^+][d\alpha]\; \alpha(1)\, \alpha^+(0) \exp\left\{-\alpha^+(1)\alpha(1) + i \int_0^1 dt\, \alpha^+(t)\left(-i\partial_t + \dot{\xi}_\mu(t) A_\mu^a(\xi)\tau^a\right)\alpha(t)\right\} \tag{17}$$

where $\alpha(t)$ is a complex multiplet of bose operators, and $0 < t < 1$ parametrizes the path $\xi(t)$. A proof of this representation for the Wilson loop operator can be found in Ref. [8], and the equality in (17) is up to a multiplicative constant. Therefore

$$\langle A(c)\rangle = \int [dA_\mu^a]\, \delta(x_\mu A_\mu^a)\, \mathrm{Tr}\, P \exp\left(i \oint_c A_\mu^a(\xi)\tau^a d\xi_\mu\right) \cdot \exp\left(-\frac{1}{4g^2}\int d^4x\, F_{\mu\nu}^a F_{\mu\nu}^a\right)$$
$$= \int [d\alpha][d\alpha^+][dA_\mu^a]\, \delta(x_\mu A_\mu^a)\, \alpha(1)\, \alpha^+(0) \exp\left(-\frac{1}{4g^2}\int d^4x\, F_{\mu\nu}^a F_{\mu\nu}^a\right)$$
$$\cdot \exp\left\{-\alpha^+(1)\alpha(1) + i\int_0^1 dt\, \alpha^+(t)\left(-i\partial_t + \dot{\xi}_\mu(t) A_\mu^a(\xi)\tau^a\right)\alpha(t)\right\} \tag{18}$$

Define

$$\mathcal{J}_\mu^a(x) = \int_0^1 \delta^4(x - \xi(t))\, \dot{\xi}_\mu(t)\, \alpha^+(t)\, \tau^a\, \alpha(t)\, dt \tag{19}$$

Then

$$\int_0^1 dt\; \dot{\xi}_\mu(t)\, A_\mu^a(\xi)\, \alpha^+(t)\, \tau^a\, \alpha(t) = \int d^4x\; A_\mu^a(x)\, \mathcal{J}_\mu^a(x) \tag{20}$$

So that

$$\langle A(c)\rangle = \int [d\alpha][d\alpha^+][dA_\mu^a]\; \delta(x_\mu A_\mu^a)\, \alpha(1)\, \alpha^+(0) \exp\left\{-\alpha^+(1)\alpha(1) + \int_0^1 dt\, \alpha^+(t)\partial_t \alpha(t)\right\}$$
$$\cdot \exp\left[-\int d^4x \left(\frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a - i A_\mu^a \mathcal{J}_\mu^a\right)\right] \tag{21}$$

By comparing $\langle A(c)\rangle$ in (21) with $Z(\mathcal{J})$ in (1) we find that the part of $\langle A(c)\rangle$ which contains the integrations over A_μ^a is similar to $Z(J)$ but with J_μ^a replaced by $-\mathcal{J}_\mu^a$. Hence we can write

$$\langle A(c)\rangle = \int [d\alpha][d\alpha^+]\; \alpha(1)\, \alpha^+(0) \exp\left[-\alpha^+(1)\alpha(1) + \int_0^1 dt\, \alpha^+(t)\partial_t \alpha(t)\right] Z(-\mathcal{J}) \tag{22}$$

Performing the duality transformation on $Z(-\mathfrak{z})$ we find that in (9) one has simply to insert $-\zeta_i^a$ instead of j_i^a where

$$\zeta_i^a = x_4 \mathfrak{z}_i^a - x_i \mathfrak{z}_4^a \quad . \tag{23}$$

We would like to show now that the expression for $< A(c) >$ in terms of the dual variables can be written as the VEV of TrPexpO(c), where O(c) is an operator defined in terms of the dual fields. To that end we point out that the term $\int \frac{d^4x}{x_4} \zeta_i^a (T^{-1}\zeta)_i^a$ in (9) may spoil this scheme because it is quartic in $\alpha(t)$, but fortunately it is zero. The other terms, which contain ζ_i^a linearly in the exponential, can be expressed as TrP expO(c), by use of (17). So we will prove now

$$\int \frac{d^4x}{x_4} \zeta_i^a (T^{-1}\zeta)_i^a = 0 \tag{24}$$

$$\begin{aligned} \int \frac{d^4x}{x_4} (x_4 \mathfrak{z}_i^a - x_i \mathfrak{z}_4^a) T^{-1}{}^{ab}_{ij} (x_4 \mathfrak{z}_j^b - x_j \mathfrak{z}_4^b) &= \\ = \int \frac{d^4x}{x_4} x_i \mathfrak{z}_4^a T^{-1}{}^{ab}_{ij} x_j \mathfrak{z}_4^b + \int d^4x\, x_4 \mathfrak{z}_i^a T^{-1}{}^{ab}_{ij} \mathfrak{z}_j^b & \\ - \int d^4x \left(\mathfrak{z}_i^a x_j \mathfrak{z}_4^b + \mathfrak{z}_4^a x_i \mathfrak{z}_j^b\right) T^{-1}{}^{ab}_{ij} & \end{aligned} \tag{25}$$

But according to (19)

$$\int d^4x\, x_4 \mathfrak{z}_i^a T^{-1}{}^{ab}_{ij} \mathfrak{z}_j^b = \int d^4x\, x_4 \int_0^1 dt_1 \int_0^1 dt_2\, \alpha^+(t_1) \dot{\xi}_i(t_1) \tau^a \alpha(t_1)$$

$$\cdot\, \delta^4(x-\xi(t_1))\, T^{-1}{}^{ab}_{ij}(x)\, \alpha^+(t_2) \tau^b \dot{\xi}_j(t_2) \alpha(t_2)\, \delta^4(x-\xi(t_2))$$

From the δ function factors we find that $\xi(t_1) = \xi(t_2) = x$, and $T^{-1}{}^{ab}_{ij}$ is antisymmetric in i,j because of the antisymmetry of T^{ab}_{ij} in (13), hence multiplying it by a symmetric tensor gives zero, namely:

$$\dot{\xi}_i(t_1) T^{-1}{}^{ab}_{ij}(x) \dot{\xi}_j(t_2) = \dot{x}_i\, T^{-1}{}^{ab}_{ij} \dot{x}_j = 0$$

For the same reason the first term of (25) vanishes.

The last term of (25) reads

$$\int d^4x \left(\mathfrak{z}_i^a x_j \mathfrak{z}_4^b + \mathfrak{z}_4^a x_i \mathfrak{z}_j^b\right) T^{-1}{}^{ab}_{ij} =$$

$$= \int d^4x \int_0^1 dt_1 \int_0^1 dt_2 \left[\dot{\xi}_i(t_1) x_j \dot{\xi}_4(t_2) + \dot{\xi}_4(t_1) x_i \dot{\xi}_j(t_2) \right] T^{-1}{}^{ab}_{ij}$$

$$\cdot\, \delta^4(x-\xi(t_1))\, \alpha^+(t_1) \tau^a \alpha(t_1)\, \delta^4(x-\xi(t_2))\, \alpha^+(t_2) \tau^b \alpha(t_2).$$

Once again because of the δ function factors $\xi(t_1) = \xi(t_2) = x$, so the term in the square bracket is symmetric in i,j whereas $T^{-1}{}^{ab}_{ij}$ is antisymmetric, hence the integral vanishes.

Going back to (9) we find that the term quartic in α does not contribute, and we have only terms quadratic in α, namely

$$\frac{1}{x_4} \zeta_i^a B_i^a = \mathfrak{z}_\mu^a B_\mu^a \qquad i = 1,2,3\ ;\ \mu = 1,2,3,4 \tag{26}$$

which results from the definition of ζ_i^a (23) and the validity of the gauge condition $x_\mu B_\mu^a(x) = 0$ and

$$\begin{aligned} \frac{1}{x_4} Y_i^a (T^{-1}\zeta)_i^a &= \frac{1}{x_4} Y_i^a T^{-1}{}^{ab}_{ij} (x_4 \mathfrak{z}_j^b - x_j \mathfrak{z}_4^b) \\ &= \mathfrak{z}_\mu^a R_\mu^a \end{aligned} \tag{27}$$

with

$$R_\mu^a = \begin{cases} (T^{-1}Y)_i^a & \text{for} \quad \mu = i = 1,2,3 \\ -\frac{1}{x_4} x_j (T^{-1}Y)_j^a & \text{for} \quad \mu = 4 \end{cases} \tag{28}$$

But

$$\int \left(B_\mu^a(x) + R_\mu^a(x)\right) \mathfrak{z}_\mu^a(x)\, d^4x =$$

$$= \int_0^1 dt\; \dot{\xi}_\mu(t) \left[B_\mu^a(\xi) + R_\mu^a(\xi) \right] \alpha^+(t) \tau^a \alpha(t)$$

which follows from the definition of $\mathfrak{z}_\mu^a(x)$ in (19), hence:

$$\langle A(c)\rangle = \int [d\alpha][d\alpha^*][dB^a_\mu][d\omega^a_{ij}]\, \delta(x_\rho B^a_\rho)\, \delta(x_j \omega^a_{ji})\, (\det T^{-1})^{1/2}$$
$$\cdot\, \alpha(1)\alpha^*(0)\, \exp\left\{-\alpha^*(1)\alpha(1) + i\int_0^1 dt\, \alpha^*(t)\left[-i\partial_t + \dot\xi_\mu(t)\left(B^a_\mu(\xi)+R^a_\mu(\xi)\right)\tau^a \alpha(t)\right]\right\}$$
$$\cdot\, \exp\left\{-\int d^4x \left[\frac{g^2}{4}\left(G^a_{\mu\nu}+\omega^a_{\mu\nu}\right)\left(G^a_{\mu\nu}+\omega^a_{\mu\nu}\right) + \frac{i}{\partial x_4} Y^a_i\left(T^{-1}Y+2B\right)^a_i\right]\right\} \quad (29)$$

$$\langle A(c)\rangle = \int [dB^a_\mu][d\omega^a_{\mu\nu}]\, \delta(x_\rho B^a_\rho)\, \delta(x_j\omega^a_{ji})\, (\det T^{-1})^{1/2}$$
$$\mathrm{Tr}\, P \exp\left[i\oint_c (B^a_\mu + R^a_\mu)\tau^a d\xi_\mu\right] \exp\left(-\int d^4x\, \mathcal{L}_D(x)\right). \quad (30)$$

where the dual Lagrangian $\mathcal{L}_D(x)$ is:

$$\mathcal{L}_D(x) = \frac{g^2}{4}\left(G^a_{\mu\nu}+\omega^a_{\mu\nu}\right)\left(G^a_{\mu\nu}+\omega^a_{\mu\nu}\right) + \frac{i}{\partial x_4} Y^a_i\left(T^{-1}Y+2B\right)^a_i \quad (31)$$

Hence we find that the operator

$$B(c) = \mathrm{Tr}\, P \exp\left[i\oint_c (B^a_\mu + R^a_\mu)\tau^a d\xi_\mu\right] \quad (32)$$

is dual to the Wilson loop operator $A(c)$, in the sense that

$$\langle A(c)\rangle = \langle B(c)\rangle\ . \quad (33)$$

But for $B(c)$ the functional measure is

$$[dB^a_\mu][d\omega^a_{\mu\nu}]\, \delta(x_\rho B^a_\rho)\, \delta(x_j\omega^a_{ji})\, (\det T^{-1})^{1/2} \exp\left(-\int d^4x\, \mathcal{L}_D(x)\right)$$

which is the functional measure for $Z(J)$. In an Appendix we show an alternative method to get (30).

Unlike the original theory the dual one is not local gauge invariant due to the ordinary derivatives acting on ω^a_{ij} defining Y^a_i (14), which appear in $\mathcal{L}_D(x)$. However in the weak coupling limit this local gauge invariance is recovered. Consider the unscaled fields in (16). In the weak coupling limit(small g)

$$Y^a_i (T^{-1}Y)^a_i \sim \theta(1/g)$$
$$T^{-1} \sim \theta(1/g) \quad (34)$$

Hence

$$(\det T^{-1})^{1/2} \exp\left[-\frac{i}{2}\int \frac{d^4x}{x_4} Y^a_i (T^{-1}Y)^a_i\right] \sim \prod_{i,a} \delta(Y^a_i) \quad (35)$$

where use of
$$\delta(x) = \frac{e^{i\pi/4}}{\sqrt{\pi}} \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\varepsilon}}\, e^{-ix^2/\varepsilon} \quad (36)$$
has been made. Using (34), (35) we get

$$\langle A(c)\rangle \underset{\text{small } g}{\approx} \int [dB^a_\mu]\, \delta(x_\mu B^a_\mu)\, \mathrm{Tr}\, P \exp\left(i\oint_c B^a_\mu \tau^a d\xi_\mu\right) \exp\left(-\frac{g^2}{4}\int d^4x\, G^a_{\mu\nu} G^a_{\mu\nu}\right) \quad (37)$$

This is again the self duality of the theory in the weak coupling limit. We see that not only the form of the Lagrangian is left invariant (it is local gauge invariant, but with g replaced by 1/g), but we also get

$$\langle A(c)\rangle \underset{\text{small } g}{\approx} \left\langle \mathrm{Tr}\, P \exp\left(i\oint_c B^a_\mu \tau^a d\xi_\mu\right)\right\rangle, \quad (38)$$

with the same functional measure, for the original and the dual fields.

In the strong coupling limit, on the other hand, a strong coupling expansion for $\langle A(c)\rangle$ can be written, that is:

$$\langle A(c)\rangle \underset{\text{large } g}{\approx} \int [dB^a_\mu]\, \delta(x_\mu B^a_\mu)\, (\det T^{-1})^{1/2} \exp\left(-\frac{g^2}{4}\int d^4x\, G^a_{\mu\nu} G^a_{\mu\nu}\right)$$
$$\cdot \int [d\omega^a_{\mu\nu}]\, \delta(x_j\omega^a_{ji})\, \mathrm{Tr}\, P \exp\left(i\oint_c R^a_\mu \tau^a d\xi_\mu\right)$$
$$\cdot\, \exp\left[-\frac{g^2}{4}\int d^4x \left(\omega^a_{\mu\nu}\omega^a_{\mu\nu} + 2G^a_{\mu\nu}\omega^a_{\mu\nu}\right)\right] \quad (39)$$

or

$$\langle A(c)\rangle \underset{\text{large } g}{\approx} \left\langle \mathrm{Tr}\, P \exp\left(i\oint_c R^a_\mu \tau^a d\xi_\mu\right)\right\rangle \quad (40)$$

Where terms of the order of $1/g$ and $1/g^2$ were neglected. Note that the functional measures for $A(c)$ and $\mathrm{Tr}P \exp\left(i\oint_c R^a_\mu \tau^a d\xi_\mu\right)$ are different. Of course higher order corrections can be systematically added, which will allow calculating $< A(c) >$ as a power series in $1/g$. The point to be made here is that the dual theory in the strong coupling limit is not local gauge invariant (which can easily be seen when the terms $o(1/g)$, $o(1/g^2)$ are added), and in particular $\mathrm{TrP}\exp(i\oint_c R^a_\mu \tau^a d\xi_\mu)$ is not invariant under local gauge transformations.

3. CONCLUDING REMARKS

Duality transformation of the VEV of the Wilson loop operator in the radial gauge $x_\mu A^a_\mu(x) = 0$ has been performed. It is found that this VEV is equal up to a multiplicative constant to the VEV of $\mathrm{TrP}\exp O(c)$, where $O(c)$ is a line integral along the loop c, of a function of the dual fields. As for the generating functional also in the duality transformed Wilson loop operator g is replaced by $1/g$. Generally the theory is not self dual but for small g, self duality is recovered in the sense that the dual Lagrangian is local gauge invariant expressed in terms of the dual gauge potentials but with g replaced by $1/g$, and the operator dual to the Wilson loop operator, is a Wilson loop operator defined in terms of the dual gauge potentials (in other words $O(c)$ reduces to a line integral of the dual gauge potentials). In the strong coupling limit (large g), these nice features are lost because the theory is no longer local gauge invariant (though it is invariant under global gauge transformations), and the operator dual to the Wilson loop operator, is a complicated function of the dual field variables. However, it is the strong coupling region which is the most important here, hence a systematic strong coupling expansion in terms of $1/g$ is suggested. The importance of the Wilson loop operator in understanding confinement and in the string model is very well known, therefore any information about its behaviour, in particular in the strong coupling region should be welcomed. But this is still incomplete because for the Wilson loop operator,

as for the generating functional, there are important singular field configurations (like monopoles and vortices) which are not included in the path integral because they are not attainable in the radial gauge, and the inversion formula is not valid. Part of them can be included when an interaction with a scalar field is added, because it smears the singularities [6], the others should be treated independently when a more complete information about the duality transformation of the Wilson loop operator is sought.

The inclusion of monopoles and vortices in the spontaneously broken gauge theory is important in understanding the weak coupling limit (38). For the spontaneously broken theory it was found [6] that self duality in the weak coupling region is gained in directions of algebra space of symmetry restoration, or in regions where monopoles and vortices are concentrated. This singles out the important role played by these field configurations in rendering the theory self dual. The same result is also found when the duality transformation of the VEV of the Wilson loop operator is performed in the spontaneously broken gauge theory, because (22) still holds, but now $Z(-\mathfrak{J})$ should be defined in terms of the standard spontaneously broken gauge Lagrangian. Expressing $Z(-\mathfrak{J})$ in terms of the dual fields [6], and using (24) it is possible to find the operator dual to $A(c)$. In the weak coupling limit it reduces to $\mathrm{TrPexp}(i\oint_c B^a_\mu(\xi)\tau^a d\xi_\mu)$ and (38) still holds. However the dual theory has an effective coupling constant $1/g$, which is strong for $g \to 0$, hence it is the non-perturbative effects (dual magnetic interactions) rather than the perturbative ones (dual electric interactions) which contribute to the right-hand side of (38). These dual magnetic effects behave like $e^{-(1/g)^{-2}} = e^{-g^2}$. On the other hand, in the left-hand side of (38) there are the contributions of the electric effects (perturbative) and the magnetic ones (non-perturbative). The latter ones can be neglected for $g \to 0$ because they behave like e^{-1/g^2}. It is true that their accumulative effect may be finite because we have to sum over all the contributions, but the smaller the g is, the more unlikely it is to sum them to order e^{-g^2} which is the order of the right-hand side of (38). Therefore we find that (38) means that the magnetic effects of the dual theory contribute to $\langle A(c) \rangle$ as much as the electric effects of the original theory. Of course it is a qualitative argument only and to check it carefully one has to perform an explicit calculation of the contributions of the electric and magnetic effects in the original as well as in the dual theory. For the strong coupling region we have no such understanding yet because of the complicated form of the dual Lagrangian.

Finally the method used here to define a duality transformation of a non-Abelian gauge theory is one of many. Its advantage is that for the Abelian case it gives both the duality transformation as defined in lattice models and the duality transformation between electric and magnetic fields and for the non-Abelian case it replaces g by $1/g$. Hence it may be possible to prove that it is equivalent to the duality transformation defined by 't Hooft (interchanging electric and magnetic vortices in the non-Abelian theory) so that the duality transformation of the Wilson loop operator could be a step towards that.

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APPENDIX

In this Appendix I would like to show an alternative way to get (30).

$$\langle A(C)\rangle = \int [dA_\mu^a]\, \Gamma(x_\mu A_\mu^a)\, \mathrm{Tr}\, P \exp\left(i\oint_c A_\mu^a \tau^a d\xi_\mu\right) \exp\left(-\frac{1}{4g^2}\int d^4x\, F_{\mu\nu}^a F_{\mu\nu}^a\right)$$

$$= \int [dA_\mu^a]\, \Gamma(x_\mu A_\mu^a)\, \mathrm{Tr}\, P \exp\left(\oint_c \frac{\delta}{\delta J_\mu^a(\xi)} \tau^a d\xi_\mu\right) \cdot$$

$$\cdot \exp\left[-\int d^4x \left(F_{\mu\nu}^a F_{\mu\nu}^a - i J_\mu^a A_\mu^a\right)\right]\Big|_{J_\mu^a = 0}$$

$$\langle A(C)\rangle = \mathrm{Tr}\, P \exp\left(\oint_c \frac{\delta}{\delta J_\mu^a(\xi)} \tau^a d\xi_\mu\right) Z(-J) \qquad \text{(A.1)}$$

Using (9) we get

$$\langle A(C)\rangle = \mathrm{Tr}\, P \exp\left(\oint_c \frac{\delta}{\delta J_\mu^a} \tau^a d\xi_\mu\right) \int [dB_\mu^a]\, \delta(x_\mu B_\mu^a)\, \delta(x_j \omega_{ji}^a)$$

$$\cdot (\det T^{-1})^{1/2} \exp\left(-\int d^4x\, \mathcal{L}_D(x)\right)$$

$$\cdot \exp\left[-i\int \frac{d^4x}{x_4}\left(\frac{1}{2} j_i^a (T^{-1} j)_i^a - j_i^a B_i^a - j_i^a (T^{-1} Y)_i^a\right)\right]\Big|_{J_\mu^a = 0} \qquad \text{(A.2)}$$

where $L_D(x)$ is defined in (31), and j_i^a in (12).

$$\mathrm{Tr}\, P \exp\left(\oint_c \frac{\delta}{\delta J_\mu^a} \tau^a d\xi_\mu\right) = \mathrm{Tr} \sum_{n=0}^{\infty} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\cdot \tau^{a_1} \tau^{a_2} \cdots \tau^{a_n} \cdot \dot{\xi}_{\mu_1}(t_1) \frac{\delta}{\delta J_{\mu_1}^{a_1}(\xi(t_1))} \dot{\xi}_{\mu_2}(t_2) \frac{\delta}{\delta J_{\mu_2}^{a_2}(\xi(t_2))} \cdots \dot{\xi}_{\mu_n}(t_n) \frac{\delta}{\delta J_{\mu_n}^{a_n}(\xi(t_n))} \qquad \text{(A.3)}$$

For the linear term in the external source in (A.2) we get

$$\mathrm{Tr}\, P \exp\left(\oint_c \frac{\delta}{\delta J_\mu^a(\xi)} \tau^a d\xi_\mu\right) \exp\left\{i\int \frac{d^4x}{x_4}\, j_i^a(x)\left[B(x) + T^{-1}(x) Y(x)\right]_i^a\right\} =$$

$$= \mathrm{Tr}\, P \exp\left\{i\int_0^1 \frac{\eta_i(t)}{\xi_4(t)} \left[B(\xi) + T^{-1}(\xi) Y(\xi)\right]_i^a \tau^a dt\right\}$$

$$\cdot \exp\left[i\int \frac{d^4x}{x_4}\, j_i^a \left(B + T^{-1} Y\right)_i^a\right] \qquad \text{(A.4)}$$

where

$$\eta_i(t) = \xi_4(t)\,\dot\xi_i(t) - \xi_i(t)\,\dot\xi_4(t) \tag{A.5}$$

To get (A.4) we used the expansion (A.3) and the definition (12) of j_i^a. Using the gauge condition $x_\mu B_\mu^a(x)=0$, and the definition (28) for R_μ^a we get

$$\mathrm{Tr}\,P\exp\left(\oint_c \frac{\delta}{\delta J_\mu^a}\tau^a d\xi_\mu\right)\exp\left[i\int\frac{d^4x}{x_4}\, j_i^a\left(B + T^{-1}y\right)_i^a\right] =$$

$$= \mathrm{Tr}\,P\exp\left(i\oint_c\left(B_\mu^a + R_\mu^a\right)\tau^a d\xi_\mu\right)\exp\left[i\int\frac{d^4x}{x_4}\, j_i^a\left(B + T^{-1}y\right)_i^a\right] \tag{A.6}$$

For the term in (A.2) quadratic in the external source we get

$$\dot\xi_{\mu_1}(t_1)\frac{\delta}{\delta J_{\mu_1}^{a_1}(\xi(t_1))}\cdot\ldots\cdot\dot\xi_{\mu_n}(t_n)\frac{\delta}{\delta J_{\mu_n}^{a_n}(\xi(t_n))}\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}\, j_i^a(x)\left(T^{-1}(x)j(x)\right)_i^a\right]$$

$$= \alpha^1\cdot\alpha^2\cdot\ldots\cdot\alpha^n\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}\, j_i^a\left(T^{-1}j\right)_i^a\right] \tag{A.7}$$

where

$$\alpha^\ell = -\frac{i}{\xi_4(t_\ell)}\eta_i(t_\ell)\left[T^{-1}(\xi(t_\ell))\,j\,(\xi(t_\ell))\right]_i^{a_\ell} \qquad i = 1,2,3 \quad \ell = 1,2,\ldots,n$$

and $\eta_i(t_\ell)$ is defined in (A.5).

To prove (A.7) we need

$$\dot\xi_\mu(t)\frac{\delta}{\delta J_\mu^a(\xi(t))}\alpha^\ell = 0 \tag{A.8}$$

so we prove (A.8) first.

$$\dot\xi_\mu(t)\frac{\delta}{\delta J_\mu^a(\xi(t))}\alpha^\ell =$$

$$= \frac{-i}{\xi_4(t_\ell)}\eta_i(t_\ell)\,T^{-1}{}_{ij}^{a_\ell a}(\xi(t_\ell))\,\eta_j(t)\,\delta^4\left(\xi(t_\ell) - \xi(t)\right) \tag{A.9}$$

where we used (12) to apply the functional derivative. From the δ function factor in (A.9) $\xi(t_\ell) = \xi(t)$, hence $\eta_i(t_\ell)\eta_j(t)$ is symmetric in ij. But $T^{-1}{}_{ij}^{a_\ell a}$ is antisymmetric in ij therefore we get (A.8).

(A.7) can be easily proven by induction.

$$\dot\xi_\mu(t)\frac{\delta}{\delta J_\mu^a(\xi(t))}\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}\, j_i^b(x)\left(T^{-1}(x)\,j(x)\right)_i^b\right] =$$

$$= -\frac{i}{\xi_4(t)}\, j_4^c(\xi(t))\,T^{-1}{}_{4j}^{ca}(\xi(t))\,\eta_j(t)\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}\, j_i^b\left(T^{-1}j\right)_i^b\right]$$

where we used

$$T^{-1}{}_{ij}^{ab} = -T^{-1}{}_{ij}^{ba} = T^{-1}{}_{ji}^{ba}$$

and the definitions (12) and (A.5). Therefore each functional derivative acting on the exponential gives a factor α^ℓ, and since (A.8) holds we get that (A.7) holds generally.

But

$$\alpha^\ell\Big|_{J_\mu^a = 0} = 0$$

hence for zero external source (A.7) vanishes. Applying now the expansion (A.3) on the term quadratic in the external source, for zero external source gives zero. So we are left only with the contribution (A.6) to (A.2). That is,

$$\langle A(c)\rangle = \int[dB_\mu^a][d\omega_{\rho\sigma}^a]\,\delta(x_\mu B_\mu^a)\,\delta(x_\rho\omega_{\rho i}^a)\,(\det T^{-1})^{1/2}$$

$$\cdot\,\mathrm{Tr}\,P\exp\left[i\oint_c\left(B_\mu^a + R_\mu^a\right)\tau^a d\xi_\mu\right]\exp\left(-\int d^4x\,\mathcal{L}_0(x)\right)$$

which is exactly (30).

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