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METRIC AND TOPOLOGY ON A NON-STANDARD REAL LINE AND NON-STANDARD SPACE-TIME*

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ABSTRACT

We study metric and topological properties of extended real line R^* and compare it with the non-standard model of real line $^*R$. We show that some properties, like triangular inequality, cannot be carried over R^* from R. This confirms F. Wattenberg's result for measure theory on Dedekind completion of $^*R$. Based on conclusion from these results we propose a non-standard model of space-time. This space-time is without undefined objects like singularities.

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1. INTRODUCTION

A very large part of mathematics is concerned with the study of "standard" mathematical systems such as natural numbers, the rationals, the real and complex numbers etc. One finds that within first order predicate calculus[1] (FPC) axiomatization of these systems cannot be categorical[2] and there exist models of our axiomatic theory which are not isomorphic to the system we wish to study. The study of such models, called non-standard models, is very important for physical, economical and sociobiological theories. One obvious reason is that the concepts of infinitely large and infinitely small numbers are not expressible in standard mathematics based on FPC. In physical theories one ends up very often with infinite and unbounded quantities. Two well-known examples are the quantum field theory with non-renormalizable interaction and the classical theory of gravitation. In both these cases the treatment of infinites is so far an unsolved problem in a rigorous sense, although there are various techniques to deal with them, for instance, regularization of fields and construction of boundary of space-time. Furthermore, in the generalization of dynamical systems and flows (on a real line for instance) one needs explicit knowledge of topologies on $^*R$ if they are to be of any use in physics. One of the purposes of this note is to construct some topologies on the non-standard extension of real line taking into account the physical aspects involved. Since the real line is the simplest topological space with rich enough structure, as well as the fact that the results of experiments are always real numbers, we would like to initiate our use of non-standard mathematics with real line.

In 1961 A. Robinson[1] showed that the non-standard model of the formal theory of analysis provides a system of hyper-real numbers and moreover provides rigorous foundations for intuitive use of infinitely large and infinitely small numbers. He showed the existence of proper extensions $^*R$ of the field of real numbers R. The $^*R$ and R have the same formal properties in FPC, however, they differ in higher orders. It was shown that $^*R$ is a totally ordered field as well as non-Archimedean and consequently all the operations on R have corresponding notions in $^*R$ as well. In an example constructed below we show that the notion of triangular inequality cannot be carried over to $^*R$ (of a particular type) if the[5] restriction of non-standard distance function $^*d$ to R must have usual distance properties. This is interesting to know if one wants to construct a non-standard model of either physical space or space-time as well as dynamical systems on it.

In Section 2 we give general properties of $^*R$ and monadic topology on it. In Section 3 we construct an example of $^*R$ with topologies. Some discussion on NS-space-time, changes of topologies and comparison between monadic topology and our example is given in Section 4. The purpose of section 2 is to collect known material for the sake of completeness and comparison with results of other sections.

2. PROPERTIES OF $^*R$ AND MONADIC TOPOLOGY

Let $^*N$ be a higher order non-standard model of arithmetic having the following properties(1):

a. All the meaningful notions of natural numbers N are also meaningful for $^*N$.

b. Every true and meaningful statement in N is also true for $^*N$, provided one interprets them in terms of their subsets called 'internal' for a given type. Thus the quantifier '$\forall x$' changes to '$\forall x$ internal' and '$\exists x$' goes to '$\exists x$ internal' and so on.

c. $^*N$ properly contains N; $\exists n\epsilon{}^*N$(8) such that $n > m \forall m\epsilon N$.

d. If S is an internal set of relations, then all elements of S are internal.

All individuals of $^*N$ are to be called natural numbers and those belonging only to N as finite while all others as 'infinite'. The finite natural numbers are the <u>standard</u> natural numbers. However, the set of finite as well as the set of infinite natural numbers are 'external sets' in $^*N$.

Consider now $^*R$ which contains both infinitely large and infinitely small numbers, called hyper-reals. An element $\ell\epsilon{}^*R$ is said to be:

1. Infinitely large if for every $n\epsilon N$, $|\ell| > n$;
2. Finite if for some $m_0\epsilon N$, $|\ell| \leqslant m_0$;
3. Infinitesimally (or infinitely) small if for every $n\epsilon N$, $n \neq 0$, $|\ell| \leqslant \frac{1}{n}$. Zero is the only standard infinitesimal.

For any finite real $a\epsilon{}^*R$, a uniquely determined standard real number which is infinitely close to a, is called the *standard part of* a; denoted as $r = st(a)$ or $\overset{\circ}{\ell}$. The set of real numbers which are infinitely close to a is called the <u>monad</u> of a, denoted as $\mu(a)$. Thus $\mu(a)$ is the equivalence class of a in $^*R$. Consider now a topological space X and $^*X$ be its non-standard enlargement. Let $^*A$ be the set of individuals of $^*X$

called points and $A \subset {}^*A$ where A is the set of individuals, i.e. points of X. The points of A are called standard points. The set of open sets in $^*X$, denoted as $^*\Omega$, is a standard set. The elements of $^*\Omega$ are internal sets in $^*X$. Let p be any standard point and let Ω_p be the set of all open sets which contain p (open neighbourhood of p). Then we define the monad of p by

$$\mu(p) = \bigcap_{U_\nu \in \Omega_p} {}^*U_\nu$$

Thus, $\mu(p)$ is the intersection of all standard sets in $^*X$ which are open neighbourhoods of p. For any standard point p, there exists an open set V in $^*X$ (i.e., V is internal and element of $^*\Omega$) such that V contains p and is contained in $\mu(p)$. Given two standard points p and q such that $q\epsilon\mu(p)$, then $\mu(q) \subset \mu(p)$. A set of points S in X is open if and only if for every point $p\epsilon S$, $\mu(p)\epsilon{}^*S$. A set of points $S \subset X$ is closed if for any standard p, which does not belong to S, the monad $\mu(p)$ has no point in common with $^*S$. A point p in X belongs to the closure $\bar{S}$ of a set S in X if $\mu(p)$ has a point in common with $^*S$. A point p belongs to the boundry of a set $S \subset X$ if $\mu(p)$ has points in common with both $^*S$ and $^*(A-S) = {}^*A - {}^*S$. The space X is a Hausdorff (or T_2) if for any two distinct points p and q in X, the monad $\mu(p)$ and $\mu(q)$ are disjoint.

Let S be any set in X. Then there exists an internal open set V in $^*X$ such that $^*S \subset V \subset \mu(S)$. In order that the topological space X be regular it is necessary and sufficient that for every set of points $S \subset X$ and for every point p in X, $\mu(S) \bigcap \mu(p) = \phi$. X is normal if $\mu(S_1) \bigcap \mu(S_2) = \phi$ for $S_1 \bigcap S_1 = \phi$ and $S_1 \subset X$, $S_2 \subset X$.

Let $\{p_n\}$ be an infinite sequence in the Hausdorff space X. In order that p be a limit of $\{p_n\}$ in X it is necessary and sufficient that, in $^*X$, $p_n\epsilon\mu(p)$ for all infinite n.

The metric properties can be studied by defining non-standard metric. For any point p in $^*X$, the monad $\mu(p)$ can be defined as the set of all points q such that $\rho(p,q)$ is infinitesimal. Distinct monads are disjoint. A point p (in $^*X$) is called finite, if there exists a standard q, such that $\rho(p,q)$ is finite. p is *near-standard* if p is in the monad of a standard point q. Let B be any (internal or external) set of points in $^*X$. Then the set of points ^oB in X is defined by

$$\mathcal{B} = \{ p \mid B \cap \mu(p) \neq \phi \}$$

If B is an internal set of $^*X$ then $^{\circ}B$ is closed. If B is a standard set then $^{\circ}(^*B) = \bar{B}$ where $\bar{B}$ is the closure of B in X. Let D be an internal open set in $^*X$. For every point $p \in D$, there exists an open ball B with radius $r > 0$ and centre p such that $B \subset D$. One defines a topology called Q-topology where internal open sets in $^*X$ constitute a base for topology. The set of all open balls in $^*X$ can be taken as the basis for Q-topology in $^*X$. The Q-topology turns $^*X$ into a Hausdorff space.

A monad is, therefore, the union of the open balls contained in it and is thus a Q-open set. A monad is also the intersection of all S-balls that contain it. Hence, it is the intersection of all S-open sets that contain it. A point p belongs to the S-interior of B iff $\mu(p) \subset B$. Furthermore, p belongs to the S-closure of B iff $\mu(p)$ has a point common with B. And p belongs to the S-boundry of B iff $\mu(p)$ has points in common with both B and complement of B.

3. AN EXAMPLE OF $(R + \partial R) = R^*$ (see ref. 7)

We construct here an example which is different from the usual one-point compactification of R (but consider only two infinite points $+\infty$ and $-\infty$). These two points are taken to be sort of "boundry" points of R. Let us consider (R, d) one-dimensional Euclidean space R with metric d and also the set $R_B = \{+\infty, -\infty\}$. R_B is an external set in $^*R$ and its elements are infinitesimals. In essence $R \subset R^* \subset {}^*R$. We then define

$$\bar{d} : R_B \times R_B \longrightarrow R_0^+ \quad \text{where} \quad R_0^+ = \{ x \mid x \in R,\ x \geqslant 0 \}$$

in the following manner:

$$\bar{d}(+\infty, +\infty) = \bar{d}(-\infty, -\infty) = 0$$
$$\bar{d}(\mp\infty, \pm\infty) \in R^+ \qquad \text{where} \quad R^+ = \{ x \mid x \in R,\ x > 0 \}$$

$\bar{d}$ is then a metric (in the conventional sense) on R_B and satisfies all the axioms of a metric (see, e.g. Dieudonné, ref 3). It is well known in conventional topology that, if (X,d) is a metric space (for any set X and metric d) and if we take the open set

$$P_x^{\gamma} = \{ y \mid d(x,y) < \gamma \}$$

then the collection $\{P_x^{\gamma}\}_{\substack{\gamma \in R^+ \\ x \in X}}$ forms a basis for the metric topology of X.

Consequently, one can see from the above definition that the topology on the metric space $(R_B, \bar{d})$ is a discrete topology. In fact, the open balls are

$$P_{\pm\infty}^{\gamma} = \begin{cases} \{\pm\infty\} & \text{if } \gamma \leqslant \bar{d}(+\infty, -\infty) \\ \{+\infty, -\infty\} = R_B & \text{if } \gamma > d(+\infty, -\infty) \end{cases} \tag{1}$$

Let us now take $R^* = R \cup R_B$. One would like to see now whether it is possible to define a distance function d^* on R^* such that

$$\begin{aligned} d^* \big|_{R \times R} &= d \text{ (i.e., restriction to } R \text{ coincides with Euclidean metric)} \\ d^* \big|_{R_B \times R_B} &= \bar{d} \text{ (i.e., restriction to } R_B \text{ coincides with } \bar{d}). \end{aligned} \tag{2}$$

The reason for this restriction is that one would like to recover back all the local properties when d^* is projected back to real line due to physical reasons. Now, we are confronted with the problem as to how we can define a function $R \to R^+$ which associates to every point of R a positive number which is to be called distance from that point to $+\infty$ (respectively, $-\infty$), i.e.

$$d^*(\cdot\ , +\infty) : R \to R^+$$
$$d^*(\cdot\ , -\infty) : R \to R^+$$

Here the distance is from a fixed point $+\infty$ (respectively, $-\infty$) to a variable point, such that

$$d^*(\cdot\ , \pm\infty) = d^*(\pm\infty, \cdot\ ) \qquad \text{(symmetry)}$$

For any $p \in {}^*X$, the monad $\mu(p) = \{q \mid \rho(p,q) = \text{infinitesimal}\}$. In this case

$$\mu(\pm\infty) = \left\{ q \mid d^*(\pm\infty, q) = \text{infinitesimal} \right\}$$

This is true in general but we choose $d^*(1, \pm\infty) : R \to R^+$. Because all the physical measurements are real we impose this restriction.

The usual properties of the distance function are taken to be valid for d^*, i.e.

1. $d^*(x,y) > 0$ if $x = y$ and 0 if $x = y$ $\forall x,y \in R^*$
2. $d^*(x,y) = d^*(y,x)$
3. $d^*(x,y) < d^*(x,z) + d^*(z,y)$ $\forall x,y,z \in R^*$

However, we shall see that condition 3 is not satisfied[5] for the cases investigated below. This is not only due to the fact that we want $+\infty$ and $-\infty$ to be adherent points or points of accumulation with the usual meaning, i.e. we want all the infinitesimally large (respectively, small) distances to be "near $+\infty$" (respectively, $-\infty$) to R in R^*, but also that $d^*(x,+\infty)$ be a non-increasing function of x, or that $d^*(x,-\infty)$ be a non-decreasing function of x.

Theorem.

If for every $x\ y \in R$

a. $\exists M_1$, and $x, y > M_1$ such that $x > y$ implies $d^*(x, +\infty) \leq d^*(y, +\infty)$

b. $\exists M_2$ and $x, y < M_2$ such that $x > y$ implies $d^*(x, -\infty) \geq d^*(y, -\infty)$

then property 3 leads to contradiction and d^* is not usual distance function.

Proof.

Let us assume that property 3 holds true. Taking $x, y \in R$ such that $x > M_1$ and $x > 0$, we take $x > |y|$. The triangular inequality for triple $(x, y, +\infty)$

$$d^*(x,y) \leq d^*(x,+\infty) + d^*(y,+\infty)$$

Consider now a point z of R:

$$z = x + d^*(x,+\infty) + d^*(y,+\infty)$$

which satisfies $z > x$ and hence $z > M_1$. Therefore

$$d^*(z,+\infty) \leq d^*(x,+\infty)$$

while for triple $(z, y, +\infty)$ the triangular inequality is

$$d^*(z,y) \leq d^*(z,+\infty) + d^*(y,+\infty)$$

From the condition $d^*\big|_{R \times R} = d$, we get

$$|z-y| \leq d^*(z,+\infty) + d^*(y,+\infty) \leq d^*(x,+\infty) + d^*(y,+\infty)$$

and from the definition of point z, one has

$$|x + d^*(x,+\infty) + d^*(y,+\infty) - y| \leq d^*(x,+\infty) + d^*(y,+\infty)$$

Therefore $|x + d^*(x,+\infty) + d^*(y,+\infty)| - |y| \leq d^*(x,+\infty) + d^*(y,+\infty)$

Here we use the relation $|a| - |b| \leq |a - b|$ which is certainly valid for points belong to R. Since x, $d^*(x,+\infty)$ and $d^*(y,+\infty)$ are positive one finds that $x - |y| \leq 0 \Rightarrow x \leq |y|$ which contradicts the initial choice of y. Similar proof goes for (b) as well. It does not come as a great surprise that the triangular inequality is not satisfied on R^*. This inequality is strictly a property of linear geometry and we do not know our R^* preserves all the properties of R or not. In general, on a non-standard real line ${}^*R$ it should be satisfied, however, because of peculiar restrictions due to physical reasons we have imposed, this is not the case.

This example thus shows that the construction of boundry of a manifold as a set of singular points could lead to contradictions[4] if it is utilized in physics. However, in the case of space-time singularities it will not lead to inconsistencies if one uses non-standard manifolds instead of constructing boundry by various techniques.

Comparing carefully the metricization procedure for extended real line R^* followed by Dieudonne[3], we note that in his case the condition $d^*\big|_{R \times R} = d$ is not satisfied. Given R, $I = (-1,+1)$ the open interval, $f : R \to I : f(x) = \frac{x}{1+|x|}$, then f is a bijection and its inverse is $g(y) = (f^{-1}(y)) = \frac{y}{1-|y|}$. Putting now $J = [-1,+1]$, one can define

$\bar{g} : J \to R^* = R \cup \{\infty\} \cup \{\infty\}$, such that $\bar{g}\big|_I = g$, $\bar{g}(1) = +\infty$, $\bar{g}(-1) = -\infty$ and $f = \bar{g}^{-1}$. As a conclusion $\bar{f}\big|_R = f$, $\bar{f}(+\infty) = 1$ and $\bar{f}(-\infty) = -1$. Since J is a metric space with respect to the Euclidean distance $|x - y|$, R^* is metrized by carrying the distance of J on R^* via mapping $\bar{f}$ by putting $d(x, y) = d_{\text{Euclidean}}(\bar{f}(x), \bar{f}(y)) = |\bar{f}(x) - \bar{f}(y)|\ \forall\, x,y \in R^*$. One realizes that this distance for two points in R is different from the Euclidean one. In fact, supposing $|x - y| = |\bar{f}(x) - \bar{f}(y)|$ one gets false equality.

Since it is not possible to give d^* all the usual properties of a metric, we shall now explore the possibility of defining a topology $\mathcal{T}$ on R^* in such a manner that topology induced on the subset $R \subset R^*$ by the topology $\mathcal{T}$ is Euclidean and that on subset R_B R^* a discrete one. In general topology on R_B should be different from that on R induced by $\mathcal{T}$. This means that R^* is a union of two topological spaces with non-equivalent induced topologies. To study this problem we can define open sets of the topology on $R \cup R_B$ in the following manner.

Given the open balls P_x^γ with a real point as centre and positive radius γ, we can call open the following sets:

$$P_x^\gamma\ (\forall x \in R, \forall \gamma \in R^+),\ \{+\infty\},\ \{-\infty\}$$

and also their arbitrary unions and finite intersections. It is obvious that $\emptyset$ and R are open sets. The open sets of the induced topology on R are then (by definition) the intersections of R with the open sets of $(R^*, \mathcal{T})$, i.e., the arbitrary unions and finite intersections of the balls P_x^γ, since $+\infty$, $-\infty$ R.

One can verify this for each case, for instance

$$[(\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cup \{+\infty\}] \cap R = [(\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R] \cup (\{+\infty\} \cap R)$$

$$= [(\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R] \cup \emptyset = (\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R = (\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R = \bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma$$

In a similar fashion the topology induced on R_B by $\mathcal{T}$ is the discrete topology, in fact none of the balls P_x^γ has elements in R_B. One should note that the $\mathcal{T}$ is a Hausdorff topology. Also R^* is not connected due to the fact that R is open (union of balls P_x^γ), R_B is open consisting of

$\{+\infty\} \cup \{-\infty\}$ and $R \cap R_B = \phi$.

Let us give a topology $\mathcal{T}$ on R^* in such a manner that the closure of R in R^* is R^* itself. To do this consider,

$$d^*(x, \pm\infty) = \frac{\pi}{2} \mp \operatorname{arctg} x \quad \forall\ x \in R$$
$$d^*(\pm\infty, \mp\infty) = \pi = \bar{d}(\pm\infty, \mp\infty)$$

and

$$d^*\big|_{R\times R} = d \ ;\ i.e.,\ d^*(x,y) = |x-y|\ \forall\, x, y \in R$$
$$d^*\big|_{R_B \times R_B} = \bar{d}\ ;\ i.e.,\ d^*(\pm\infty, \pm\infty) = 0$$

The metric d^* satisfies conditions 1 and 2 but not 3, in general, unless the triples of R^* are chosen in a particular manner. In fact if $x, y, z \in R$, then condition 3 holds true, and also if we take an element of R and the other two of R_B. One has for instance

a. $d^*(x, +\infty) \leq d^*(x, -\infty) + d^*(+\infty, -\infty)$ since $\frac{-\pi}{2} \leq \operatorname{arctg} x$ is true for every $x \in R$.

b. $d^*(x, -\infty) \leq d^*(x, +\infty) + d^*(-\infty, +\infty)$ since $\operatorname{arctg} x \leq \pi/2\ \forall x \in R$.

c. $d^*(+\infty, -\infty) \leq d^*(+\infty, x) + d^*(x, -\infty)$.

Let us define as open balls with centre $+\infty$ ($-\infty$ respectively) and radius $\gamma > 0$ the following sets:

$$P_{\pm\infty}^\gamma = \{x \in R^* \mid d^*(x, \pm\infty) < \gamma\}$$

Here, if $\gamma > \pi$ then $P_{+\infty}^\gamma = R \cup \{+\infty\} \cup \{-\infty\} = R^* = P_{-\infty}^\gamma$ and if $\gamma \leq \pi$ then

$$P_{+\infty}^\gamma = \{x \in R \mid \operatorname{arctg} x > \pi/2 - \gamma\} \cup \{+\infty\}$$

We shall denote by P_y^γ the usual balls of the Euclidean topology, i.e., give $y \in R$, $P_y^\gamma = \{x \in R \mid d^*(x,y) = |x-y| < \gamma\}$. Hence none of the P_y^γ contain points of R_B while every $P_{+\infty}^\gamma$ and every $P_{-\infty}^\gamma$ contains points of R. We observe that R is an open set, in fact $R = \bigcup_{x\in Z} P_x^2$. To see this consider $x \in Z$ (where Z = relative integers), $P_x^2 \subset R$ implies $\bigcup_{x \in Z} P_x^2 \subset R$. For all $x \in R$, there exists $n(x)$ such that $x \in P_{n(x)}^2$, where $n(x)$ is the largest integer less than or equal to x.

Observe that all the open balls, those with real numbers as centre and those with centre $+\infty$ or $-\infty$ have non-empty intersection with R. Also it is trivial to verify this for balls whose centres are reals, but we shall verify this for $P^{\gamma}_{+\infty}$ or $P^{\gamma}_{-\infty}$, i.e., we prove that $P^{\gamma}_{+\infty} \cap R \neq \phi$ for $\gamma > 0$. From the density of $\mathbb{Q}$ (the rationals p/q in R, p/q such that $\gamma > \frac{p}{q} > 0$ and also $\gamma > p/q > \frac{1}{p+q} > 0$) one has

$$P^{\gamma}_{+\infty} = \{x \in R^* \mid d^*(x, +\infty) < \gamma\} \supseteq \{x \in R \mid \tfrac{\pi}{2} - \operatorname{arctg} x < \gamma\} \cup \{+\infty\}$$

Let $x \epsilon R$ such that $\operatorname{arctg} x = \frac{\pi}{2} - \frac{1}{p+q}$ (such a point always exists) then

$$\frac{\pi}{2} - \operatorname{arctg} x = \frac{1}{p+q} < \gamma \quad \text{implies} \quad x \in P^{\gamma}_{+\infty}$$

The consequences of the above observations are that

1. R_B is not open but closed;
2. R_B is the boundry of R.

Proof.

1. We do not hope to find R_B from the balls P^{γ}_{x} because these have the empty intersection with R_B. Therefore, we examine $P^{\gamma}_{+\infty}$ and $P^{\gamma}_{-\infty}$ taking their intersections

$$P^{\gamma}_{+\infty} \cap P^{\gamma'}_{+\infty} = P^{\operatorname{Min}(\gamma,\gamma')}_{+\infty} \ni \text{Real points}$$

$$P^{\gamma}_{+\infty} \cap P^{\gamma'}_{-\infty} = \begin{cases} \text{proper or improper subsets of } R \ (\gamma = \gamma' = \pi \Rightarrow R) \\ R^* \end{cases}$$

The argument goes through for any intersection as far as it is finite. Hence all points of R_B belong to the boundry of R. Now consider the set R_B. In fact, in every ball having a point of R_B as centre there exist points which do not belong to R_B. Since the boundry of a set is closed, therefore R_B is closed.

2. In every neighbourhood of $+\infty$ (or $-\infty$) there are points of R and of R_B, therefore R_B is the boundry of R. It follows that the closure of

R in R^* is $R\ R_B = R^*$. One sees immediately that the topology induced by $\mathcal{T}$ on R_B is discrete: $P^{\gamma}_{+\infty} \cap R_B = \begin{cases}\{+\infty\} \\ R_B\end{cases}$ and $P^{\gamma}_{-\infty} \cap R_B = \begin{cases}\{-\infty\} \\ R_B\end{cases}$ and $P^{\gamma}_{x} \cap R_B = \phi$. The topology induced on R is Euclidean because $P^{\gamma}_{x} \cap R = P^{\gamma}_{x}$, are the 'Old' Euclidean balls and

$$P^{\gamma}_{+\infty} \cap R = \begin{cases} R^* \cap R = R & \text{Union of Euclidean balls if } \gamma > \pi \\ R & \text{if } \gamma = \pi \\ \{x \in R \mid x > tg(\pi/2 - \gamma)\} & \text{if } \gamma < \pi \end{cases}$$

Every set of the type $\{x \epsilon R \ x > M\}$ is a union of open balls

$$\{x \in R \mid x > M\} = P^{|n(x)+1-M|}_{n(x)+1} \cup \left(\bigcup_{k \in N} P^{1}_{n(x)+1+k} \right)$$

Proposition 1.

The topology $\mathcal{T}$ is Hausdorff.

Proof.

We want to prove that $\forall\ x \neq y \epsilon R^*$, $\exists\ P^{\gamma}_{x}$ and $P^{\gamma'}_{y}$ such that $P^{\gamma}_{x} \cap P^{\gamma'}_{y} = \phi$. There are three distinct cases

1. $x \neq y \epsilon R$
2. $x \neq y \epsilon R_B$
3. $x \epsilon R$, $y \epsilon R_B$

Case 1.

If $x = y \epsilon R$ then $|x - y| > 0$ and $P^{\gamma}_{x} = \{\ \epsilon R \mid |x - z| < \gamma\}$ and $P^{\gamma'}_{y} = \{z \epsilon R \mid |y - z| < r'\}$. Putting $r < \frac{1}{2}|x - y|$ and $r' < \frac{1}{2}|x - y|$ results in $P^{\gamma}_{x} \cap P^{\gamma'}_{y} = \phi$. In fact we know that $z \epsilon P^{\gamma}_{x} \cap P^{\gamma'}_{y}$ implies $|x - y| < |x - z| + |z - y| < \frac{1}{2}|x - y| + \frac{1}{2}|x - y| < |x - y|$ which is absurd.

Case 2.

Take $x = +\infty$ and $y = -\infty$ and $\gamma, \gamma' < \pi$ (to simplify the computations).

$\bar{g} : J \to R^* = R \cup \{\infty\} \cup \{-\infty\}$, such that $\bar{g}\big|_I = g$, $\bar{g}(1) = +\infty$, $\bar{g}(-1) = -\infty$ and $f = \bar{g}^{-1}$. As a conclusion $\bar{f}\big|_R = f$, $\bar{f}(+\infty) = 1$ and $\bar{f}(-\infty) = -1$. Since J is a metric space with respect to the Euclidean distance $|x - y|$, R^* is metrized by carrying the distance of J on R^* via mapping $\bar{f}$ by putting $d(x, y) = d_{\text{Euclidean}}(\bar{f}(x), \bar{f}(y)) = |\bar{f}(x) - \bar{f}(y)| \; \forall x, y \in R^*$. One realizes that this distance for two points in R is different from the Euclidean one. In fact, supposing $|x - y| = |\bar{f}(x) - \bar{f}(y)|$ one gets false equality.

Since it is not possible to give d^* all the usual properties of a metric, we shall now explore the possibility of defining a topology $\mathcal{T}$ on R^* in such a manner that topology induced on the subset $R \subset R^*$ by the topology $\mathcal{T}$ is Euclidean and that on subset R_B R^* a discrete one. <u>In general topology on R_B should be different from that on R induced by $\mathcal{T}$</u>. This means that R^* is a union of two topological spaces with non-equivalent induced topologies. To study this problem we can define open sets of the topology on $R \cup R_B$ in the following manner.

Given the open balls P_x^γ with a real point as centre and positive radius γ, we can call open the following sets:

$$P_x^\gamma (\forall x \in R, \forall \gamma \in R^+), \; \{+\infty\}, \; \{-\infty\}$$

and also their arbitrary unions and finite intersections. It is obvious that $\emptyset$ and R are open sets. The open sets of the induced topology on R are then (by definition) the intersections of R with the open sets of $(R^*, \mathcal{T})$, i.e., the arbitrary unions and finite intersections of the balls P_x^γ, since $+\infty$, $-\infty$ R.

One can verify this for each case, for instance

$$[(\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cup \{+\infty\}] \cap R = [(\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R] \cup (\{+\infty\} \cap R)$$

$$= [(\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R] \cup \emptyset = (\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R = (\bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma) \cap R = \bigcap_{\substack{x\in I\\ \gamma\in J}} P_x^\gamma$$

In a similar fashion the topology induced on R_B by $\mathcal{T}$ is the discrete topology, in fact none of the balls P_x^γ has elements in R_B. One should note that the $\mathcal{T}$ is a Hausdorff topology. Also R^* is <u>not connected</u> due to the fact that R is open (union of balls P_x^γ), R_B is open consisting of

$\{+\infty\} \cup \{-\infty\}$ and $R \cap R_B = \phi$.

Let us give a topology $\mathcal{T}$ on R^* in such a manner that the closure of R in R^* is R^* itself. To do this consider,

$$d^*(x, \pm\infty) = \frac{\pi}{2} \mp \operatorname{arctg} x \quad \forall\, x \in R$$

$$d^*(\pm\infty, \mp\infty) = \pi = \bar{d}(\pm\infty, \mp\infty)$$

and

$$d^*\big|_{R\times R} = d \;;\; i.e., \; d^*(x, y) = |x - y| \; \forall\, x, y \in R$$

$$d^*\big|_{R_B \times R_B} = \bar{d} \;;\; i.e., \; d^*(\pm\infty, \pm\infty) = 0$$

The metric d^* satisfies conditions 1 and 2 but not 3, in general, unless the triples of R^* are chosen in a particular manner. In fact if $x, y, z \in R$, then condition 3 holds true, and also if we take an element of R and the other two of R_B. One has for instance

a. $d^*(x, +\infty) \le d^*(x, -\infty) + d^*(+\infty, -\infty)$ since $-\frac{\pi}{2} \le \operatorname{arctg} x$ is true for every $x \in R$.

b. $d^*(x, -\infty) \le d^*(x, +\infty) + d^*(-\infty, +\infty)$ since $\operatorname{arctg} x \le \pi/2 \; \forall x \in R$.

c. $d^*(+\infty, -\infty) \le d^*(+\infty, x) + d^*(x, -\infty)$.

Let us define as open balls with centre $+\infty$ ($-\infty$ respectively) and radius $\gamma > 0$ the following sets:

$$P^\gamma_{\pm\infty} = \{x \in R^* \mid d^*(x, \pm\infty) < \gamma\}$$

Here, if $\gamma > \pi$ then $P^\gamma_{+\infty} = R \cup \{+\infty\} \cup \{-\infty\} = R^* = P^\gamma_{-\infty}$ and if $\gamma \le \pi$ then

$$P^\gamma_{+\infty} = \{x \in R \mid \operatorname{arctg} x > \pi/2 - \gamma\} \cup \{+\infty\}$$

We shall denote by P_y^γ the usual balls of the Euclidean topology, i.e., give $y \in R$, $P_y^\gamma = \{x \in R \mid d^*(x,y) = |x-y| < \gamma\}$. Hence none of the P_y^γ contain points of R_B while every $P^\gamma_{+\infty}$ and every $P^\gamma_{-\infty}$ contains points of R. We observe that R is an open set, in fact $R = \bigcup_{x\in Z} P_x^2$. To see this consider $x \in Z$ (where Z = relative integers), $P_x^2 \subset R$ implies $\bigcup_{x\in Z} P_x^2 \subset R$. For all $x \in R$, there exists $n(x)$ such that $x \in P^2_{n(x)}$, where $n(x)$ is the largest integer less than or equal to x.

Observe that all the open balls, those with real numbers as centre and those with centre $+\infty$ or $-\infty$ have non-empty intersection with R. Also it is trivial to verify this for balls whose centres are reals, but we shall verify this for $P^{\gamma}_{+\infty}$ or $P^{\gamma}_{-\infty}$, i.e., we prove that $P^{\gamma}_{+\infty}\cap R \neq \phi$ for $\gamma > 0$. From the density of $\mathbb{Q}$ (the rationals p/q in R, p/q such that $\gamma > \frac{p}{q} > 0$ and also $\gamma > p/q > \frac{1}{p+q} > 0$) one has

$$P^{\gamma}_{+\infty} = \{x \in R^* \mid d^*(x,+\infty) < \gamma\} \supseteq \{x \in R \mid \tfrac{\pi}{2} - \operatorname{arctg} x < \gamma\} \cup \{+\infty\}$$

Let $x \in R$ such that $\operatorname{arctg} x = \frac{\pi}{2} - \frac{1}{p+q}$ (such a point always exists) then

$$\frac{\pi}{2} - \operatorname{arctg} x = \frac{1}{p+q} < \gamma \quad \text{implies} \quad x \in P^{\gamma}_{+\infty}$$

The consequences of the above observations are that

1. R_B is not open but closed;
2. R_B is the boundry of R.

Proof.

1. We do not hope to find R_B from the balls P^{γ}_{x} because these have the empty intersection with R_B. Therefore, we examine $P^{\gamma}_{+\infty}$ and $P^{\gamma}_{-\infty}$ taking their intersections

$$P^{\gamma}_{+\infty} \cap P^{\gamma'}_{+\infty} = P^{\operatorname{Min}(\gamma,\gamma')}_{+\infty} \ni \text{real points}$$

$$P^{\gamma}_{+\infty} \cap P^{\gamma'}_{-\infty} = \begin{cases} \text{proper or improper subsets of } R \ (\gamma=\gamma'=\pi \Rightarrow R) \\ R^* \end{cases}$$

The argument goes through for any intersection as far as it is finite. Hence all points of R_B belong to the boundry of R. Now consider the set R_B. In fact, in every ball having a point of R_B as centre there exist points which do not belong to R_B. Since the boundry of a set is closed, therefore R_B is closed.

2. In every neighbourhood of $+\infty$ (or $-\infty$) there are points of R and of R_B, therefore R_B is the boundry of R. It follows that the closure of

R in R^* is $R \cup R_B = R^*$. One sees immediately that the topology induced by $\mathcal{T}$ on R_B is discrete: $P^{\gamma}_{+\infty}\cap R_B = \begin{cases}\{+\infty\}\\ R_B\end{cases}$ and $P^{\gamma}_{-\infty}\cap R_B = \begin{cases}\{-\infty\}\\ R_B\end{cases}$ and $P^{\gamma}_{x}\cap R_B = \phi$. The topology induced on R is Euclidean because $P^{\gamma}_{x}\cap R = P^{\gamma}_{x}$, are the 'old' Euclidean balls and

$$P^{\gamma}_{+\infty}\cap R = \begin{cases} R^*\cap R = R & \text{Union of Euclidean balls if } \gamma > \pi \\ R & \text{if } \gamma = \pi \\ \{x \in R \mid x > tg(\pi/2 - \gamma)\} & \text{if } \gamma < \pi \end{cases}$$

Every set of the type $\{x \in R \mid x > M\}$ is a union of open balls

$$\{x \in R \mid x > M\} = P^{|n(x)+1-M|}_{n(x)+1} \cup \Big(\bigcup_{k\in N} P^{1}_{n(x)+1+k}\Big)$$

Proposition 1.

The topology $\mathcal{T}$ is Hausdorff.

Proof.

We want to prove that $\forall\, x \neq y \in R^*$, $\exists\, P^{\gamma}_{x}$ and $P^{\gamma'}_{y}$ such that $P^{\gamma}_{x}\cap P^{\gamma'}_{y} = \phi$. There are three distinct cases

1. $x \neq y \in R$
2. $x \neq y \in R_B$
3. $x \in R$, $y \in R_B$

Case 1.

If $x \neq y \in R$ then $|x-y| > 0$ and $P^{\gamma}_{x} = \{z \in R \mid |x-z| < \gamma\}$ and $P^{\gamma'}_{y} = \{z \in R \mid |y-z| < \gamma'\}$. Putting $\gamma < \frac{1}{2}|x-y|$ and $\gamma' < \frac{1}{2}|x-y|$ results in $P^{\gamma}_{x}\cap P^{\gamma'}_{y} = \phi$. In fact we know that $z \in P^{\gamma}_{x}\cap P^{\gamma'}_{y}$ implies $|x-y| < |x-z| + |z-y| < \frac{1}{2}|x-y| + \frac{1}{2}|x-y| < |x-y|$ which is absurd.

Case 2.

Take $x = +\infty$ and $y = -\infty$ and $\gamma,\gamma' < \pi$ (to simplify the computations).

Then

$$P^{\gamma}_{+\infty} = \{x \in R \mid \operatorname{arctg} x > \tfrac{\pi}{2} - \gamma\} \cup \{+\infty\}$$

$$P^{\gamma'}_{-\infty} = \{x \in R \mid \operatorname{arctg} x < \gamma' - \pi/2\} \cup \{-\infty\}$$

and the intersection

$$P^{\gamma}_{+\infty} \cap P^{\gamma'}_{-\infty} = \{x \in R \mid \operatorname{arctg} x > \tfrac{\pi}{2} - \gamma \text{ and } x < \gamma' - \pi/2\}$$

It is enough to put $\gamma = \gamma' \leqslant \pi/2$ to obtain $P^{\gamma}_{+\infty} \cap P^{\gamma}_{-\infty} = \phi$. In fact one obtains the system $\begin{cases} \operatorname{arctg} x > 0 \\ \operatorname{arctg} x < 0 \end{cases}$ which does not admit solutions.

Case 3.

Let $x \in R$, $y = +\infty$ and choose $\gamma' < \pi$, then

$$P^{\gamma}_{x} = \{z \in R \mid |x - z| < \gamma\}, \quad P^{\gamma'}_{+\infty} = \{z \in R \mid \operatorname{arctg} z < \tfrac{\pi}{2} - \gamma'\} \cup \{+\infty\}$$

and $P^{\gamma}_{x} \cap P^{\gamma'}_{+\infty} = \{z \in R \ |x - z| < \gamma$ and $\operatorname{arctg} z > \pi/2 - \gamma'\} = \phi$ when $\gamma' < \operatorname{arctg}(x + \gamma)$.

It is easy to see that

$$|x - z| < \gamma \text{ is equivalent to } \begin{cases} \operatorname{arctg} z > \operatorname{arctg}(x - \gamma) \\ \operatorname{arctg} z < \operatorname{arctg}(x + \gamma) \end{cases}$$

Therefore the solution of the system of inequalities is

$$\begin{cases} \operatorname{arctg} z > \operatorname{arctg}(x - \gamma) \\ \operatorname{arctg} z < \operatorname{arctg}(x + \gamma) \\ \operatorname{arctg} z > \pi/2 - \gamma' > \operatorname{arctg}(x + \gamma) \end{cases}$$

belongs to $P^{\gamma}_{x} \cap P^{\gamma'}_{+\infty}$. This system does not have any solution since $\operatorname{arctg} z > \operatorname{arctg}(x + \gamma)$ and $\operatorname{arctg} z < \operatorname{arctg}(x + \gamma)$ contradict each other.
A similar proof goes through for $-\infty$.

We would now like to give a topology on R^*, a topology $\mathcal{T}^*$ which conserves all the properties given above except the closure of R_B. Let us define d^* without triangular inequality as a condition and $P^{\gamma}_{\pm\infty}$ as

$$P^{\gamma}_{\pm\infty} = \{x \in R^* \mid d^*(x, \pm\infty) < \gamma\}$$

but with a new definition of the open balls with real numbers as centre: precisely $\forall\, y \in R$, $\gamma \in R^+$

$$P^{\gamma}_{y} = \{x \in R^* \mid d^*(x, y) < \gamma\} = \{x \in R \mid |x - y| < \gamma\} \cup \{x \in R_B \mid d^*(x, y) < \gamma\}$$

$$= \{x \in R \mid |x - y| < \gamma\} \cup \begin{cases} \{+\infty\} & \text{if } \operatorname{arctg} y > \pi/2 - \gamma \\ \{-\infty\} & \text{if } \operatorname{arctg} y < \gamma - \pi/2 \end{cases}$$

This tells us that the balls with real numbers as centre can be "richer" in structure than those of Euclidean balls with the same centre and radius.

Let us now define the open sets of the new topology as the arbitrary unions and finite intersections of the balls P^{γ}_{y}, $P^{\gamma}_{+\infty}$ and $P^{\gamma}_{-\infty}$ $\forall\, \gamma \in R^+$. One gets immediately that R^* is open and also ϕ.

Proposition 2.

The set R_B is open in the new topology: $R_B = P^{\pi}_{0} \cap P^{\pi}_{3\pi}$

Proof.

$$P^{\pi}_{0} = \{x \in R \mid |x| < \pi\} \cup \{+\infty\} \cup \{-\infty\}$$

Use

$d^*(0, +\infty) < \pi$ *implies* $\frac{\pi}{2} - \operatorname{arctg} 0 < \pi$ *and* $d^*(0, -\infty) < \pi$

implies $\frac{\pi}{2} + \operatorname{arctg} 0 < \pi$.

$$P^{\pi}_{3\pi} = \{x \in R \mid |x - 3\pi| < \pi\} \cup \{+\infty\} \cup \{-\infty\}$$

with metric

$d^*(3\pi, +\infty) < \pi$ *implies* $-\operatorname{arctg} 3\pi < 0 < \pi/2$

$d^*(3\pi, -\infty) < \pi$ *implies* $\operatorname{arctg} 3\pi < \pi/2$.

Proposition 3.

R is open in the new topology $\mathcal{T}^*$.

Proof.

$\forall$ $x \in R$, define $\gamma(x) = \mathrm{Min}\ \{\pi/2 - \mathrm{arctg}\ x, \pi/2 + \mathrm{arctg} x\}$ and $\gamma(x) > 0$. We shall then verify that

$$P_x^{\gamma} = \{y \in R \mid |x-y| < \gamma(x)\}$$

Since $d^*(x, +\infty) = \frac{\pi}{2} - \mathrm{arctg}\ x \nless \gamma(x)$ and $d^*(x, -\infty) - \frac{\pi}{2} + \mathrm{arctg}\ x \nless \gamma(x)$ therefore $R = \bigcup_{x \in R} P_x^{\gamma}(x)$. One concludes that $R^* = R \ R_B$ with R open and $R \ R_B = \phi$. This means that R^* is not connected. The topology induced by $\mathcal{T}^*$ on R is Euclidean, in fact.

$$R \cap P_x^{\gamma} = \begin{cases} P_x^{\gamma} & \text{if } \pm\infty \notin P_x^{\gamma} \\ P_x^{\gamma} - \{\text{points of } R_B\} & \text{if } +\infty \text{ or } -\infty \in P_x^{\gamma} \end{cases}$$

Analogously, the induced topology on R_B is a discrete one.

Theorem 4.

The topological space $(R^*, \mathcal{T}^*)$ is Hausdorff.

Proof.

As above we have three cases:

1. $x \nmid y \in R$
2. $x \nmid y \in R_B$
3. $x \in R$, $y \in R_B$

Cases (2) and (3) are derived from the above discussion and hence we take case (1). It follows from case (3) that there exists an (open) ball with centre x which does not contain $+\infty$ (it is enough that $\gamma < \pi/2 - \mathrm{arctg}\ x$) and a ball with centre x that does not contain $-\infty$. The intersection of these balls is another ball with centre x and is $P_x^{\gamma} = \{z \in R \ |z - x| < \gamma\}$. Given $P_y^{\gamma'}$, the intersection of the two is empty when $\gamma + \gamma' < d^*(x,y)$.

4. THE NON-STANDARD SPACE-TIME AND TOPOLOGY CHANGE

Why non-standard space-time? There are two main reasons behind this choice. Firstly, that one would like to have a space-time free of singularities. This can be done by either identifying singularities and then removing them to the boundry of space-time manifold, or start with a complete space-time where the meaning of singularities is well defined. The second reason which is of equal importance that with the boundry construction one ends up with undesirable behaviour, e.g. non-Hausdorff behaviour of the manifold in some cases[4]. This follows also from the example we have constructed for extended real line where triangular inequality cannot be carried over to extended manifold. A similar demonstration is given by F. Wattenberg[5].

Consider now the well-known Schwarzschildspace-time metric whose radial metric component

$$g_{rr} = \left(1 - \frac{2m}{r}\right)^{-1} \in {}^*R \quad \text{given } r \approx 2m$$

where the symbol $\approx$ means 'infinitesimally close'. Similarly the time taken for light to get out from radius r (of a collapsing star) is

$$t = \int_{r_1}^{r_2} \frac{dr}{1 - \frac{2m}{r}} = (r_2 - r_1) + 2Gm \ln\left(\frac{r_2 - 2Gm}{r_1 - 2Gm}\right)$$

as $r_1 \approx r_S$ where $r_S = r_{\text{Schwarzschild}}$, then

$$t = (r_2 - r_S) + 2m\left[\ln(r_2 - 2Gm) + \ln A \in {}^*R\right]$$

In physical context the question is then whether infinite time t is measurable. The answer is in the affirmative if the measurements are in non-standard metric, i.e. global measurement and not local. Here global means on the whole manifold including singular regions. However, the results of 'local' measurements are always to be real.

As we have seen in the example constructed above, going from real to hyper real or vice-versa there is a change of topology. Thus the functions defined in real region with respect to Euclidean topology for instance become singular in hyper real region and vice-versa. However, these functions which are singular with respect to topology $\mathcal{T}_1$, become regular with respect to $\mathcal{T}_2$.

The hyper real part of Schwarzschild space-time (i.e., singular region) is

$$\mu(r_s) = \{q \mid d^*(r_s, q) = \text{infinitesimal}\}$$

Any regular function for $r > 2m$ becomes regular in the region $\mu(r = 2m)$ in new topology.

It is obvious from the discussion above that the only solution to the singularity problem is to consider non-standard space-time. To give an idea we briefly suggest how to construct a two-dimensional space-time in non-standard analysis.

Consider $^*R^3 = {}^*R \times {}^*R \times {}^*R$ and using the implicit function theorem[6] we can define a manifold $^*M^2$, a two-dimensional non-standard manifold. Also we can give it a Lorentz structure, that is to endow it with a non-standard Lorentz metric $^*g$. Then $(^*M, {}^*g)$ is a two-dimensional space-time manifold. Here there are two possibilities to define metric $^*g$.

1. Define $^*g : {}^*M \times {}^*M \to {}^*R$, $^*g(x,y) \geqslant 0$ or $^*g(x,y) < 0$ and $^*g(x,y) > m$, $\forall m \epsilon R$ and $^*g(x,y) < n, \forall n \epsilon R$. Here $x, y \epsilon {}^*M$. This means the distance could be time-like, light-like, space-like, time-like infinitesimal and space-like infinitesimal. Since zero is the only standard infinitesimal, light-like real and infinitesimal coincides. The light cone has standard and non-standard parts. The non-standard part is inside the singular region with respect to the natural topology of regular part of space-time.

2. One can define $^*g$ as $^*g : {}^*M \times {}^*M \to R$, however, this does not give anything interesting because it is essentially restriction of case (1) when measurements are made locally.

All the known solutions of Einstein's equation with singularity are of the non-standard type. As a concluding remark we think that use of non-standard mathematics is perhaps an appropriate solution for physical theories and a possible cure for the singularity problem.

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FOOTNOTES AND REFERENCES

1. A Robinson, Non-Standard Analysis, North-Holland (1970).
2. A theory T which is complete and consistent is called categorical. This means all of its models are isomorphic and no non-standard model exists. See, e.g. J. Monk, Mathematical logic, Springer-Verlag 1976.
3. J. Dieudonne, Elements d'Analyse (G. Villars, Paris).
4. Waterloo conference on singularities, in GRG 10, no. 12 (1979). The original g-boundry proposals as well as that of bundle completions of the Schwarzschild and closed Friedmann solution exhibit problems. In fact, R.A. Johnson, GRG 10, 967-969 (1979) shows that: for each $p \epsilon M$, there is a point q in the bundle boundry $\bar{M}$ of M such that p is in every neighbourhood of q. Here (M, g) denote either Schwarzschild or closed Friedmann space-time with corresponding Lorentz metric g. This is a typical non-Hausdorff behaviour.
5. A similar result is proved by F. Wattenberg for measures on Dedekind completion $^\#R$ of non-standard real line $^*R$ (see Pacific Jour. Math 90, 223 (1980)). There are some other properties also which cannot be carried over to $^\#R$. Although our $R^* = R \cup \{+\infty\} \cup \{-\infty\}$ is not the same as $^\#R$ but the result suggests that not all of the properties of R can be carried over to this sort of extension.
6. W.A.J. Luxumburg and K.D. Stroyan, Introduction to the Theory of Infinitesimals, Academic Press (1976).
7. We use R^* to distinguish it from the non-standard $^*R$. Notice $*$ on the right above R. All the notions are standard in this section.
8. $n \epsilon {}^*N$ means 'n is an individual of $^*N$'. When using standard mathematics 'ϵ' means "element of".

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