AT8300032 | RN 1067

UWThPh-1982-4

INTERTWINERS AND THEIR APPLICATIONS

M. Breitenecker and H.-R. Grümm
Institut für Theoretische Physik
Universität Wien

Abstract

The concept of "intertwiner" - less familiar to physicists - appears to be quite useful for the applications of group theory. We present as examples the change of ground field as well as a basis-free version of the Wigner-Eckart theorem.

Zusammenfassung

Der Begriff des "intertwiners" - in der Physik weniger geläufig - eignet sich vorzüglich für die Anwendungen der Gruppentheorie. Als Beispiel werden der Wechsel des Grundkörpers und eine basisfreie Version des Wigner-Eckart Theorems gegeben.

1. Introduction

The concept of intertwining operators (shortly intertwiners) is well known to mathematicians. Despite of the fact that intertwiners serve as excellent tool in the analysis of group representations and the formulation of prominent objects (e.g. Clebsch-Gordan operators) and theorems (e.g. Wigner-Eckart theorem) they are not so well known among theoretical physicists. It is the purpose of this paper to shed some light onto this subject.

Let $\Pi(G,K)$ be the category of linear representations (in the sequel we omit "linear") of a group G on vector spaces over the (skew) field K (K will always be **R**, **C**, or **H**, the skew field of quaternions). If K is a skew field (e.g. **H**) we have to distinguish between left and right vector spaces, depending on the multiplication with the scalars from the left or from the right. In this case we choose to deal with left-linear representations on left vector spaces over K (= **H**).

Then we have the

Definition: The morphisms in $\Pi(G,K)$ are called intertwiners.

Remark: Let us recall that a representation T of a group G is a homomorphism of G into the automorphisms of some representation space V over the skew field K:

$$T: G \to \mathrm{aut}_K V , \qquad g \to T(g): V \to V .$$

Let T_1, T_2 be two such representations on the spaces V_1, V_2 resp. Then for each intertwiner $S \in \mathrm{Mor}(T_1,T_2)$ the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{T_1(g)} & V_1 \\ S \downarrow & & \downarrow S \\ V_2 & \xrightarrow[T_2(g)]{} & V_2 \end{array} \qquad (1.1)$$

commutes for all $g \in G$.

We express some well known concepts in this language (see e.g. [1]):

1.1 If S is an isomorphism then T_1 and T_2 are equivalent representations.

1.2 If $S, S' \in Mor(T_1,T_2)$ and $a, a' \in$ center of K (i.e. R, if $K = H$), then $aS + a'S' \in Mor(T_1,T_2)$: the set $Mor(T_1,T_2)$ of all intertwiners for the representations T_1 and T_2 is a linear space over the center of K (hence, for $K = H$, the H-left linear transformations from $V_1 \to V_2$ form only a <u>real</u> vector space!); we shall denote it by $C(T_1,T_2)$, its dimension $c(T_1,T_2) = \dim C(T_1,T_2)$ is called the intertwining number.

1.3 If there is no (non zero) intertwiner for T_1 and T_2, i.e. $c(T_1,T_2) = c(T_2,T_1) = 0$, then T_1 and T_2 are said to be disjoint.

1.4 Lemmata of Schur:

a) If T_1 and T_2 are irreducible representations (shortly: irreps)[1], then either T_1 and T_2 are disjoint or each intertwiner is invertible and hence T_1 and T_2 are equivalent.

b) If T is an irrep, then $C(T) := C(T,T)$ is a skew field over the center of K.

1.5 Let T_1 and T_2 act on V_1 and V_2 resp.[2]; then $T_1 \times T_2$ acts on $V_1 \underset{K}{\times} V_2$. Hence the formation of the tensor product within $\Pi(G,K)$ is defined.

1.6 For the interwining number we have

$$\begin{aligned} &a)\ c(T_1+T_2,T_3) = c(T_1,T_2) + c(T_1,T_3) \\ &b)\ c(T_1,T_2+T_3) = c(T_1,T_2) + c(T_1,T_3) \end{aligned} \qquad (1.2)$$

c) If $\dim T_2 < \infty$ then $c(T_1 \times T_2,T_3) = c(T_1,T_2^t \times T_3)$ where the conjugate representation T^t acting on the dual space V^* is defined by

$$T^t(g) := (T(g^{-1}))^t .$$

d) If $\dim T_1 < \infty$ and $\dim T_2 < \infty$ then

$$c(T_1,T_2) = c(T_2,T_1) . \qquad (1.3)$$

---

1) "irreducible" means algebraically irreducible, i.e. the representation space V does not contain a non-trivial invariant subspace.

2) Here we assume V_1 and V_2 to be bi-vector spaces over K with $(kv)k' = k(vk')$, $v \in V_1$ or V_2 and $k,k' \in K$. $V_1 \underset{K}{\times} V_2$ is then again a bi-vector space over K.

2. Change of Scalars

2.1 Let K be a subfield of the (skew) field L (i.e. K = R, L = C or H or K = C, L = H); we may consider the categories $\Pi(G,K)$ and $\Pi(G,L)$ and the functors "change of scalars" between them (see diagram): the functor "extension",

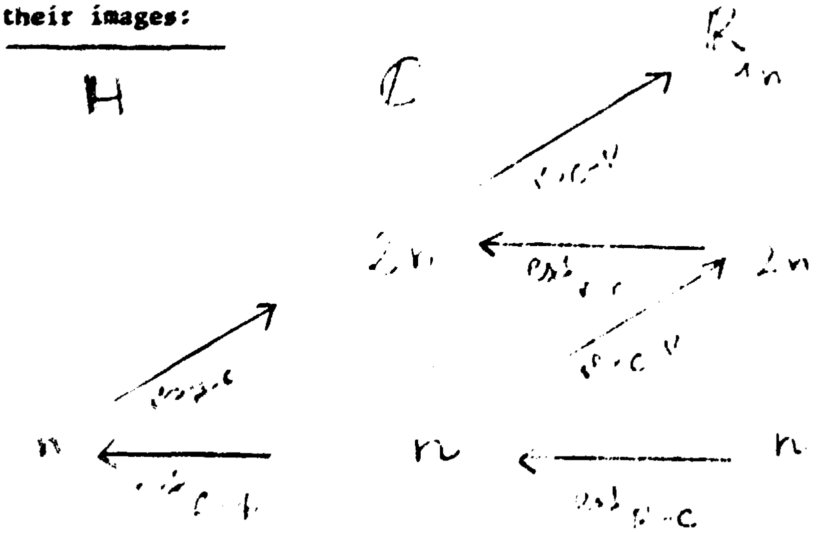
$$\text{ext}_{K-L}: \Pi(G,K) \to \Pi(G,L)$$

assigns to the representation space V over K the space $_LV := L \underset{K}{\times} V$ (L considered as a vector space over K) and to the representation T on V the representation $T_L := 1_L \times T$ on $_LV$; the functor "restriction",

$$\text{res}_{L-K}: \Pi(G,L) \to \Pi(G,K)$$

assigns to the representation space V over L the space V_K over K which has the same additive group as V but restricts the scalars from L to the subfield K. Hence these functors turn L-linear representations into K-linear representations and vice versa[3].

For K = R, L = C both functors lead to additional structures on their images:



(2.1)

Diagram for the functors F_{ext} and F_{res}. n denotes the dimension of the vector spaces with respect to R, C or H.

3) For K = R and L = C these functors are better known as complexification and realification resp.

On $ext_{R-C}(V) = C \underset{R}{\times} V \cong {}_CV$ we have an operator $*: {}_CV \to {}_CV$, $*(z \times v) :=$ $:= \bar{z} \times v$ which is obviously antilinear with $*^2 = 1$. The complex dimension, $\dim_C({}_CV)$ of ${}_CV$ is equal to the real dimension, $\dim_R(V)$ of V. The functor res_{C-R}, however, doubles the dimension: for any $v \neq 0$ the elements v and iv are R-linearly independent. Hence $\dim_R(res_{C-R}(V)) = 2\dim_C(V)$ and we have an R-linear operator I with $I^2 = -1$ on $res_{C-R}(V)$; it corresponds to the multiplication with i on V and it may be represented as

$$I = \begin{pmatrix} 0 & 1_V \\ -1_V & 0 \end{pmatrix} .$$

In contrast, restriction from V over H to $V_C = res_{H-C}(V)$ yields naturally on V_C an antilinear operator J with $J^2 = -1$: $J: V_C \to V_C$, $J(v) := jv$ (considered as an element of the underlying additive group) with $J(zv) = jzv = \bar{z}jv = \bar{z}J(v)$, $z \in C$ and $j \in H$ with $j^2 = -1$. Finally on $ext_{C-H}(V) = H \underset{C}{\times} V \cong {}_HV$ we have an operator $\tilde{*}: {}_HV \to {}_HV$, $\tilde{*}(q \times v) =$ $= \tilde{q} \times v$ where $\tilde{q} := q_1 - jq_2$ if $q = q_1 + jq_2$, $q_1, q_2 \in C$ and j is a quaternion with $j^2 = -1$ and $ij + ji = 0$. We remark that $\tilde{*}^2 = 1$ but that it is not canonically defined. It depends on our choice of embedding C in H as the (maximal commutative) subfield generated by i.

$*$, I, $\tilde{*}$ and J allow us to reconstruct the original vector spaces:

$$V \simeq \{w \in {}_CV \mid *(w) = w\} ;$$

complex multiplication on V_R can be defined using I: $z \cdot w := \mathrm{Re}\, z \cdot w +$ $+ \mathrm{Im}\, z \cdot Iw$, $w \in V_R$; $V \simeq \{w \in {}_HV \mid \tilde{*}(w) = w\}$ and finally J can be used to define a quaternion structure on V_C by $q \cdot w := q_1 w + q_2 Jw$.

For further use we notice that $res_{C-R}(ext_{R-C}(V)) \simeq V \oplus V$ while $ext_{R-C}(res_{C-R}(V)) \simeq V \oplus \bar{V}$. $\bar{V}$, in this notation, means the vectorspace with the same underlying additive group as V but with the scalar multiplication $(z,v) \to \bar{z}v$. Considering $\bar{z}v$ as an element of V, we may define an operator C: $\bar{V} \to V$ by $C(v) = v$; C is obviously antilinear. It allows us to define the representation $\bar{T}$ on $\bar{V}$, called the representation complex conjugate to T on V by $\bar{T} := C^{-1} \circ T \circ C$; it is C-linear of course. A similar procedure assigns to each left H-vector space/ representation T a right H-vector space V^o/representation T^o, using the antiautomorphism of H: $q \to \bar{q}$.

2.2 Under the assumption, that the dimension of the irrep T is not larger than countable we have the

Theorem: If K = C, then $C(T)$ = C but if K = R, then $C(T)$ = R or C or H. Therefore the intertwining number c(T) of an irrep T is equal to 1 for K = C and to 1, 2 or 4 for K = R. (For a proof see [1], p. 119.) If T is an irrep over H then $C(T)$ is again a skew field over R and again we have the possibilities $C(T)$ = R, C or H resp.

2.3 A real irrep is called a representation of type R, C or H resp., if $C(T)$ = R, C or H. An irreducible quaternion representation is called a representation of type R, C or H if $C(T)$ = R, C or H resp.

2.4 The classification of complex irreps is a bit more involved (compare [2]). If T is not equivalent to $\bar{T}$ we call it of type III. If $T = \bar{T}$ then, by composing an intertwiner $S \in C(T,\bar{T})$ with the canonical antilinear C: $\bar{V} \to V$, we obtain an antilinear operator K on V, commuting with T ($K \notin C(T)$ because intertwiners are linear). Since $K^2 \in C(T)$, $K^2 = c1$; c is real because $KK^2 = K^2K$; changing K to $\sqrt[+]{|c|}K$ we can normalize $K^2 = \pm 1$.

Case a): $K^2 = +1$; then, according to 2.1 , K (= *) shows T to be of of form $ext_{R-C}(T')$, T' some real representation; T is said to be of type I.

Case b): $K^2 = -1$; then K (= J) shows T to be of the form $res_{H-C}(T'')$, T" some H-representation; T is said to be of type II.

Examples: type I: spin-1 representation of SU(2)
type II: spin-1/2 representation of SU(2)
type III: self representation of SU(3).

2.5 The actions of the various res- and ext-functors on the previously defined types are summarized in the following diagram: (For reducible representations: ext and res commute with direct sums!)

$$
\begin{array}{lll}
\mathrm{ext}_{R-C}(R) = I & \mathrm{ext}_{R-C}(C) = III \oplus \overline{III}\ ^{i)} & \mathrm{ext}_{R-H}(H) = 2II \\
\mathrm{ext}_{R-H}(R) = H & \mathrm{ext}_{R-H}(C) = 2\,C & \mathrm{ext}_{R-H}(H) = 4\,R \\
\mathrm{res}_{C-R}(I) = 2\,R & \mathrm{res}_{C-R}(III) = C & \mathrm{res}_{C-R}(II) = H \\
\mathrm{ext}_{C-H}(I) = H & \mathrm{ext}_{C-H}(III) = C & \mathrm{ext}_{C-H}(II) = 2\,R \\
\mathrm{res}_{H-C}(H) = 2I & \mathrm{res}_{H-C}(C) = III \oplus \overline{III} & \mathrm{res}_{H-C}(R) = II \\
\mathrm{res}_{H-R}(H) = 4\,R & \mathrm{res}_{H-R}(C) = 2\,C\ ^{ii)} & \mathrm{res}_{H-R}(R) = H
\end{array}
\tag{2.2}
$$

Notation: E.g. i) means that the extension to C of an R-irrep of type C splits into the direct sum of a type III irrep with its complex conjugate. ii) means that the restriction to R of an H-irrep of type C splits into two copies of an R-irrep of type C.

The proof of the statements contained in this table is based on points 2.2 to 2.4 and the general fact that $C(\mathrm{ext}_{K-L}T) = L \underset{K}{\times} C(T)$ together with $C \underset{R}{\times} C = C \oplus C$, $C \underset{R}{\times} H = \mathrm{Mat}_2(C)$ and $H \underset{R}{\times} H = \mathrm{Mat}_4(R)$.

Remark: We observe that ext and res act only within a single column of the table. This suggests the definition of intrinsic type: an irrep over K (= R/C/H) is of intrinsic type K, if it stays irreducible after res or ext. An irrep over L (= R/C/H) is of intrinsic type K ($\neq$ L) if it is the restriction or extension of an irrep over K of intrinsic type K. The intrinsic type of an irrep gives its "natural" ground field.

Examples: An irrep of type III is of intrinsic type C. The spin-1/2 representation of SU(2) is of intrinsic type H. The adjoint representation of any compact semisimple Lie group is of intrinsic type R. Any nontrivial irrep of an abelian group is of intrinsic type C.

Application: We clarify and extend a statement in Hamermesh [2] (chap. 4): Even if only one-dimensional irreducible representations occur, one can have double degeneracy if the Hamiltonian is invariant under time reversal. In general, if a unitary representation induced by a group action on R^n contains a type III irrep T^σ, then a real invariant Hamiltonian H will have twice the degeneracy suggested by dim T^σ. We see this as follows:

The group action on R^n induces an orthogonal representation O on the real Hilbert space $L^2_R(R^n)$ as well; in the absence of a magnetic field H is a self-adjoint operator on $L^2_R(R^n)$ as well. If O contains an irrep of type C, H restricted to the corresponding subspace is a scalar (although R is not algebraically closed, H is self-adjoint). The corresponding analysis over C yields only half the degeneracy since the orthogonal irrep splits (into III + $\widehat{\mathrm{III}}$) under extension to C.

2.6 From now on we restrict ourselves to representations of compact groups (how far our results can be extended to arbitrary representations will be described in the appendix). Therefore we can assume all representations to be unitary. (By this we mean "orthogonal" if K = R or H resp.)

Thus all representations T may be decomposed as $T = \bigoplus_\sigma a_\sigma T^\sigma$ with T^σ irreps. It is well known that for K = C this decomposition can be obtained by inspecting the character $\chi^T(g) = \mathrm{tr}\, T(g) = \sum_\sigma a_\sigma \chi^\sigma(g)$; thus by the orthogonality relations

$$\int dg\, \bar{\chi}^\sigma(g)\, \chi^{\sigma'}(g) = \delta_{\sigma\sigma'}\,, \qquad a_\sigma = \int dg\, \bar{\chi}^\sigma(g)\, \chi^T(g)\,. \tag{2.3}$$

We give the analoga for the cases K = R and H resp.

a) K = R: we start by remarking that $\mathrm{tr}\,(1_C \times A) = \mathrm{tr}\, A$, thus $\chi^{\mathrm{ext}_{R\text{-}C}T} = \chi^T$; our table yields that $\int dg\, \chi^\sigma(g)\, \chi^{\sigma'}(g) = \delta_{\sigma\sigma'}\, c_\sigma$ ($c_\sigma := c(T^\sigma) = 1, 2,$ or 4).

b) K = H: we define tr A ($A \in \mathrm{End}(V)$, V a left H-vector space) as $\mathrm{tr}_{H/R} \sum_i A_{ii}$ with $\mathrm{tr}_{H/R}(q_0 + iq_1 + jq_2 + kq_3) = 2q_0$ (i.e. the usual trace in an extension skew field; see Bourbaki: Algebra, Book V) [4]. It is easy to see that $\chi^T = \chi^{\mathrm{res}_{H\text{-}C}T}$, since in a matrix representation of H over C, i, j and k have zero trace. As before, $\int dg\, \chi^\sigma(g)\, \chi^{\sigma'}(g) = \delta_{\sigma\sigma'}\, c_\sigma$. Therefore in all cases $a_\sigma = \frac{1}{c_\sigma} \int dg\, \bar{\chi}^\sigma(g)\, \chi^T(g)$.

2.7 The functors res and ext respect unitary equivalence, f.i. $\mathrm{ext}_{R\text{-}C}(T_1) \simeq \mathrm{ext}_{R\text{-}C}(T_2) \Leftrightarrow T_1 \simeq T_2$ with the exception in two cases: let T_1 be a C-irrep of type III, then $\mathrm{res}_{C\text{-}R}T_1 \simeq \mathrm{res}_{C\text{-}R}T_2 \Leftrightarrow \mathrm{ext}_{C\text{-}H}T_1 \simeq \mathrm{ext}_{C\text{-}H}T_2 \Leftrightarrow T_1 \simeq T_2$ <u>or $T_1 \simeq \bar{T}_2$</u>. This special case comes about because the outer automorphism $\lambda \to \bar{\lambda} \in C$ becomes trivial under $\mathrm{res}_{C\text{-}R}$ resp. becomes inner under $\mathrm{ext}_{C\text{-}H}$.

2.8 The exterior tensor product of the representations $T_1 \times T_2 \in \Pi(G_1 \times G_2, K)$ if $T_i \in \Pi(G_i, K)$, i = 1,2 with $(T_1 \times T_2)(g_1, g_2) = T_1(g_1) \otimes T_2(g_2)$ behaves well with respect to intrinsic type, i.e. its intrinsic type depends only on the intrinsic types of the factors.

We give explicit formulas for K = R, C and H separately:

---

4) Since End(V) is only an R-vector space tr is real valued; further, without $\mathrm{tr}_{H/R}$, we would not have tr BA = tr AB.

K = C: Only in this case $T_1 \times T_2$ is necessarily an irrep if T_1 and T_2 are irreps. (Remember: $\chi^{T_1 \times T_2}(g_1,g_2) = \chi^{T_1}(g_1)\,\chi^{T_2}(g_2)$.)
We have in a suggestive notation

$$\begin{aligned} &\mathrm{III} \times \mathrm{I} = \mathrm{III} \times \mathrm{II} = \mathrm{III} \times \mathrm{III} = \mathrm{III} \\ &\mathrm{I} \times \mathrm{I} = \mathrm{II} \times \mathrm{II} = \mathrm{I} \\ &\mathrm{I} \times \mathrm{II} = \mathrm{II}\ . \end{aligned} \tag{2.4}$$

$\chi^{T_1 \times T_2}$ is real iff both χ^{T_1} and χ^{T_2} are real which shows the first line; for the rest we use the fact that the tensor product of two antilinear operators is again antilinear and argue as in 2.4.
K = R: In this case we have

$$C(T_1 \times T_2) \simeq C(T_1) \underset{R}{\otimes} C(T_2)$$

and obtain

$$\begin{aligned} &R \times R = R\ , \quad R \times C = C\ , \quad R \times H = H \\ &C \times C = C \oplus C'\ , \quad C \times H = 2C\ , \quad H \times H = 4R\ ; \end{aligned} \tag{2,5}$$

$C \oplus C'$: direct sum of two inequivalent irreps of $G_1 \times G_2$ of type C.
K = H: Some care has to be taken forming the tensor product of H-representations: we form only products like $T_1^o \times T_2$, since we need both a right and a left H-vector space for the tensor product (an R-vector space). Using the formula $C(T_1^o \times T_2, T_3^o \times T_4) \simeq C(T_1,T_3) \underset{R}{\otimes} C(T_2,T_4)$, we obtain for the exterior tensor multiplication of types over H the same table as for K = R.
Note: Unfortunately the restriction of a representation to a subgroup does not respect intrinsic type; thus neither does the usual tensor product representations (= restriction of $T_1 \times T_2$ to the diagonal in $G_1 \times G_2$). F.i. the self representations of SU(3) and SU(4) are both of intrinsic type C, however in the first case restriction to SU(2) yields intrinsic types R and H, in the second case restriction to SU(3) yields the types R and C.

3. Clebsch-Gordan Operators

In the following we restrict ourselves to representations over R or C. We assume the reader to be familiar with elementary theory of the reduction of tensor products (e.g. Edmonds: Angular Momentum in Quantum Mechanics). Our aim is this chapter is to translate these results into the basis-free language of intertwiners.

For each unitary equivalence class σ ("unitary" in the sense of 2.6) of irreps of G over K (together forming the dual $\hat{G}_K$) we choose a standard representative T^σ acting on V^σ. Under the formation of tensor products they decompose as follows:

$$T^\mu \otimes T^\nu \simeq \bigoplus_\sigma c_{\mu\nu,\sigma}\, T^\sigma \qquad (\mu,\nu,\sigma \in \hat{G}_K) \tag{3.1}$$

From chap. 1 we have $c_{\mu\nu,\sigma} = c(T^\mu \otimes T^\nu, T^\sigma)/c_\sigma$ $\left(= \frac{1}{c_\sigma}\int dg\ \bar{\chi}^\mu(g)\ \bar{\chi}^\nu(g)\ \chi^\sigma(g)\right)$. We call a Clebsch-Gordan operator $C^\sigma_{\mu\nu}$ any coisometry in $C(T^\mu \otimes T^\nu, c_{\mu\nu,\sigma}T^\sigma)$ (thus a projection $V^\mu \otimes V^\mu \to c_{\mu\nu,\sigma}V^\sigma$, mapping the σ-isotypic subspace[5] of $V^\mu \otimes V^\nu$ unitarily onto $c_{\mu\nu,\sigma}V^\sigma$) (see diagram). $C^\sigma_{\mu\nu}\cdot C^{\sigma\dagger}_{\mu\nu} = 1_{c_{\mu\nu,\sigma}V^\sigma}$.

$$\begin{array}{ccc}
V^\mu \otimes V^\nu & \xrightarrow{\ T^\mu \otimes T^\nu\ } & V^\mu \otimes V^\nu \\
\Big\downarrow C^\sigma_{\mu\nu} & & \Big\downarrow C^\sigma_{\mu\nu} \\
c_{\mu\nu,\sigma}V^\sigma & \xrightarrow[\ c_{\mu\nu,\sigma}T^\sigma\ ]{} & c_{\mu\nu,\sigma}V^\sigma \\
\Big\downarrow S & & \Big\downarrow S \\
c_{\mu\nu,\sigma}V^\sigma & \xrightarrow[\ c_{\mu\nu,\sigma}T^\sigma\ ]{} & c_{\mu\nu,\sigma}V^\sigma
\end{array} \tag{3.2}$$

$$S \in C(c_{\mu\nu,\sigma}T^\sigma) \simeq \mathrm{Mat}_{c_{\mu\nu,\sigma}}(C(T^\sigma)) \ .$$

---

5) Heuristically, this subspace consists of "all vectors in $V^\mu \times V^\nu$, which transform under T^σ".

To investigate the ambiguity in this definition we observe that $\mathrm{Mat}_{c_{\mu\nu,\sigma}}(C(T^\sigma))$ acts on $C(T^\mu \otimes T^\nu; c_{\mu\nu,\sigma} T^\sigma)$: in the diagram $C^\sigma_{\mu\nu} \cdot S \cong C^\sigma_{\mu\nu}$. Thus $U(c_{\mu\nu,\sigma}; C(T^\sigma))$[6] acts on the coisometric intertwiners (= Clebsch-Gordan operators). It is easy to see that this action is free (i.e. transitive and without fixed point); "a Clebsch-Gordan operator is defined up to a unitary matrix (for $C(T^\sigma) = \mathbb{C}$ and $c_{\mu\nu,\sigma} = 1$ up to a phase factor)".

Decomposing (non-uniquely) the isotypic subspace into irreducible subspaces we arrive at a decomposition

$$C^\sigma_{\mu\nu} = \bigoplus_{t=1}^{c_{\mu\nu,\sigma}} C^{\sigma;t}_{\mu\nu} \, . \tag{3.3}$$

Each of the $C^{\sigma;t}_{\mu\nu}$ is an intertwining coisometry from $V^\mu \otimes V^\nu$ onto V^σ. If $c_{\mu\nu,\sigma} = 1$, $C^\sigma_{\mu\nu}$ can be identified with a unitary element of $C(T^\sigma)$.

From now on we assume that for each triple (μ,ν,σ) a fixed Clebsch-Gordan operator as well as a fixed decomposition (3.3) has been chosen.

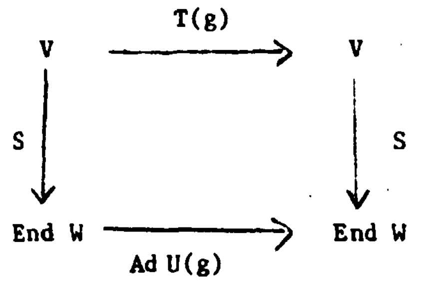
4. Tensor Operators and the Wigner-Eckart Theorem

We translate the concept of a tensor operator into the language of intertwiners. For this we remark that for any finite-dimensional representation U on W there is defined a representation Ad U on End W via

$$\mathrm{Ad}\, U(g)(A) := U(g)\, A\, U^{-1}(g) \qquad (A \in \mathrm{End}\, W) \, . \tag{4.1}$$

(For unitary U's actually $\mathrm{Ad}\, U = \bar{U} \otimes U$ over $\mathbb{C}$ and $\mathrm{Ad}\, U = U \otimes U$ over $\mathbb{R}$). A <u>tensor operator</u> S of type T on W (T acting on V) is an element of $C(T, \mathrm{Ad}\, U)$. S is called irreducible if T is

6) $U(n,\mathbb{R})$ denotes the orthogonal group, $U(n,\mathbb{H})$ the symplectic orthogonal group (i.e.: $\{A \in \mathrm{Mat}_n(\mathbb{H}) \mid \bar{A}\cdot A^T = 1\}$).



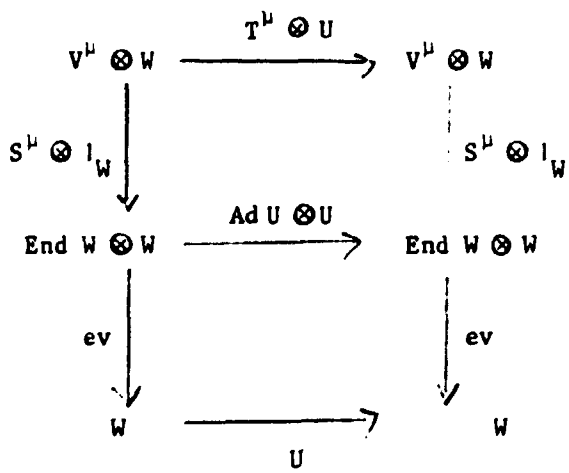
(4.2)

Introducing a basis $\{v_i\}$ in V and defining $S_i := S(v_i)$ we recover the usual definition of a tensor operator:

$$U(g)\ S_i\ U^{-1}(g) = \sum_j T_{ji}(g)\ S_j \qquad (4.3)$$

($T_{ji}(g)$ the matrix form of T in the basis chosen).

The Wigner-Eckart theorem corresponds now to the expansion of $S^\mu \in C(T^\mu, \mathrm{Ad}\,U)$ - an irreducible tensor operator - in terms of Clebsch-Gordan operators. We need some preliminaries. Consider the following diagram:



where ev denotes the evaluation map $\mathrm{ev}(A \otimes w) := Aw$: an intertwiner between $\mathrm{Ad}\,U \otimes U$ and U. If $\hat{S}^\mu := \mathrm{ev} \circ (S^\mu \otimes 1_W)$, $\hat{S}^\mu \in C(T^\mu \otimes U, U)$. Let $U \simeq \bigoplus_\sigma U^\sigma \simeq \bigoplus_\sigma u_\sigma T^\sigma$ be the (unique) decomposition of U into multiples of irreps. We have

$$C(T^\mu \otimes U,U) = \bigoplus_\sigma \bigoplus_\tau C(T^\mu \otimes U^\sigma,U^\tau) = \bigoplus_\sigma \bigoplus_\tau \mathrm{Mat}_{u_\sigma,u_\tau}(K) \times C(T^\mu \otimes T^\sigma,T^\tau);$$

let $\hat{S}^\mu_{\sigma,\tau}$ be a component of $\hat{S}^\mu$ in this direct sum decomposition. Since, by counting dimensions, the $C^{\tau;\ell}_{\mu\sigma}$ form a basis of $C(T^\mu \otimes T^\sigma,T^\tau)$ over $C(T^\tau)$ we can write

$$\hat{S}^\mu_{\sigma,\tau} = \sum_{j=1}^{u_\sigma} \sum_{k=1}^{u_\tau} \sum_{\ell=1}^{c_{\mu\sigma,\tau}} \langle\sigma;j \,\|\hat{S}^\mu\|\, \tau;k\rangle_\ell \, C^{\tau;\ell}_{\mu\sigma} \, E_{jk} ,$$

where $E_{jk} \in \mathrm{Mat}_{u_\sigma,u_\tau}(K)$ denotes the usual matrix unit and $\langle\sigma;j \,\|\hat{S}^\mu\|\, \tau;k\rangle_\ell \in C(T^\tau)$ is the "ℓ-th reduced matrix element". This is the abstract Wigner-Eckart theorem. To convert this into the usual form, introduce a basis $\{v_i\}$ in V as in (4.3), as well as bases $\{\phi_r\}$ in the j-th irreducible subspace of U^τ and $\{\psi_s\}$ in the k-th irreducible subspace of U^σ. Then

$$\langle\psi_s|S_i|\phi_r\rangle = \langle\psi_s|\hat{S}(v_i \times \phi_r)\rangle =$$

$$= \sum_{\ell=1}^{c_{\mu\sigma,\tau}} \langle\sigma;j \,\|\hat{S}^\mu\|\, \tau;k\rangle_\ell \, \langle s|C^{\tau;\ell}_{\mu\sigma}|i,r\rangle$$

and we identify the usual Clebsch-Gordan coefficients as matrix elements of $C^{\tau;\ell}_{\mu\sigma}$: $V^\mu \otimes V^\sigma \to V^\tau$ in the basis chosen.

Appendix

The statements in this paper can be generalized - mutatis mutandis - to the following situations:

A) G is a semi-simple non-compact group. Since finite-dimensional representations of G are again direct sums of irreps - determined by their characters - we can again proceed as in chap. 3 and 4. Without this complete reducibility, a reducible representation T could very well fulfill $c(T) = 1$. Since the representations are necessarily non-unitary, we only require a Clebsch-Gordan operator to be an intertwining surjection $T^{\mu} \otimes T^{\nu} \rightarrow c_{\mu\nu,\sigma} T^{\sigma}$; it is determined up to an element of $GL(c_{\mu\nu,\sigma}; C(T^{\sigma}))$. The expansion of an irreducible tensor operator as a linear combination of Clebsch-Gordan operators remains valid.

B) U of chap. 4 is a "unitary" representation on a (real or complex) ∞-dimensional Hilbert space. Of course U decomposes into an orthogonal direct sum of irreps. If, as usual, the representation comes from an action of G on a homogeneous space, the multiplicities are finite. The evaluation map $L(H) \times H \rightarrow H$ is continuous if restricted to bounded sets; replacing tensor products by Hilbert tensor products one arrives at the statements of chap. 4.

References

[1] A.A. Kirillov, Elements of the Theory of Representations, Grundlehren Vol. 220, Springer, Berlin, Heidelberg, New York, 1976

[2] M. Hamermesh, Group Theory, Addison-Wesley, 1962