A NUMERICAL SOLUTION TO THE RADIAL EQUATION
OF THE TIDAL WAVE PROPAGATION

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1. INTRODUCTION

In the development of the tidal theory (Siebert, 1961 and Chapman and Lindzen, 1970), the assumption is made such that the dependence of the tides on height, co-latitude and longitude are separable.

The latitude-dependences of the tidal fields are expressed in terms of the Hough functions $\mathcal{H}(\theta)$, which are solutions of Laplace's tidal equation. It is an eigenvalue-eigenfunction problem and has been the subject of extensive study during the last decade (Kato, 1966; Lindzen, 1966; Longuett-Higgins, 1967 and Makarious, 1968).

The vertical structure of the tidal wave propagation, on the other hand, is expressed in terms of the wave function $y(x)$, which is the solution of the Radial Wave equation. For a given mode of oscillation, the variable coefficient, in that equation, depends solely on the vertical temperature distribution. Previous investigators (Fekete, 1937; Jacchia and Kopal, 1952; Butler and Small, 1963 and Lindzen, 1967) adopted, however, simplified vertical temperature structure in order to render the mathematical treatment more tractable.

Consequently classical tidal theory has limitations, arising out of the assumptions on which it is based, which need to be recognized when making close comparison with the observational data (Groves, 1977). With a realistic temperature structure, however, no closed form solution to the vertical wave equation exists and the solution has to be approached numerically. This is the goal of the present study.

Analytical expressions for the general solution, in conformity with the observed tidal oscillations in the atmosphere, have been obtained. The numerical solutions for the oscillatory and trapped modes of tidal wave propagation have also been presented and discussed.
The basic equations of the theory of atmospheric tides are given in Siebert (1961) and the derivations of the basic equations need not, therefore, be repeated. The wave function \( y_n(x) \) is expressed in terms of the velocity divergence \( J_n(x) \) and the rate of heating \( J_n(x) \) per unit mass as

\[
J_n(x) = \left[ \frac{\lambda_n - \frac{k_j}{2\pi}}{\frac{1}{\eta} H(x)} \right] \exp(x/z)
\]

where \( x \), the reduced height, is defined by

\[
x = \int_0^x \frac{d\xi}{H(\xi)}
\]

\( H(\xi) = \frac{RT(\xi)}{g} \) and \( k = \frac{2}{\gamma} = (\gamma - 1)/\gamma \).

The wave function \( y_n(x) \) is a solution to the Radial Wave Equation, written in the form:

\[
\frac{d^2 y_n}{dx^2} + \mu_n^2 y_n = Q_n(x)
\]

where

\[
\mu_n^2(x) = \frac{1}{\gamma} \left[ 1 - \frac{h_n}{h_n} \left( x/H(x) + \frac{dH}{dx} \right) \right]
\]

and

\[
Q_n(x) = \frac{\lambda_n}{\eta} \frac{J_n(x)}{\exp(x/z)}
\]

\( h_n \) is the equivalent depth of the mode, it is the eigenvalue obtained through the solution of Laplace's tidal equation, which depends only on the period of the mode.

In order to evaluate the reduced height (Eq. 2) realistic temperature data are utilized. These data are presented by Cole et al. (1965) for \( 0 < Z < 30 \) km and by Groves (1971) for \( 30 < Z < 120 \) km - at 5 km intervals. The scale height \( H(x) \) has been obtained, by using quadratic interpolating formula, at \( x = 0(0.1)17 \). By least-squares approximation \( H(x) \) has been found to be represented fairly accurately by a fourth degree polynomial in the region \( 0 < x < 15.6 \). At \( x > 15.6 \), a constant temperature gradient is assumed, i.e.

\[
\frac{dH}{dx} = \frac{1}{H} \frac{dH}{dx} = 0.25
\]

Consequently,

\[
\mu_n^2(x) = \alpha_n \frac{1}{\eta} \frac{H}{h_n} (x) - \frac{1}{\eta}
\]

on using (4), where \( \alpha_n = \frac{H(15.6)}{h_n} (k + 0.25)/\exp(3.9) \). Values of the approximated scale height \( H(x) \) as obtained from the assumed temperature model are given in Table 1.

As a second-order differential equation, two boundary conditions for \( y(x) \) must be satisfied. At high levels we shall require, first, that \( y \) be bounded as \( x \to \infty \) and secondly, that the flow of energy be in the upward direction. Following Wilkes (1949) and according to Jacchia and Kopal (1952), this requirement should be enforced not at infinity, but at a finite altitude \( Z^* \), beyond which the highly rarefied atmosphere can be expected to float more or less passively on the less dense strata. \( Z^* \) is taken, in the present study,

\* The subscript \( n \) will be dropped thereafter for simplicity in writing and we deal with each mode independently.
to correspond to 110 Km and to conform with the boundness of \( y \) it is taken as:

\[
y(x') \to 0 \quad a \quad x' > 11.2
\]

(8)

As a lower boundary condition, it is generally assumed that at the earth's surface, the vertical component of the velocity must vanish (Siebert, 1961); this implies that:

\[
\left( \frac{d}{dx} + \frac{y}{h} \right) \gamma \bigg|_{x=0} = 0
\]

(9)

The conventional tidal fields may all be conviently expressed in terms of the \( \gamma \)'s and \( \beta \)'s:

\[
u(x, \theta, t) = \frac{4 \beta_x \psi y (x \theta)}{4 \pi x \gamma \left( \frac{1}{2} \gamma \right)} \left( \frac{dy}{dx} - \frac{y}{2} \right) \left( \frac{2 \beta_x}{y} \frac{1}{2} \frac{2 \beta_x}{y} \right) e^{-i(t + \omega)}
\]

(10)

\[
u(x, \theta, t) = \frac{4 \beta_x \psi y (x \theta)}{4 \pi x \gamma \left( \frac{1}{2} \gamma \right)} \left( \frac{dy}{dx} - \frac{y}{2} \right) \left( \frac{2 \beta_x}{y} \frac{1}{2} \frac{2 \beta_x}{y} \right) e^{-i(t + \omega)}
\]

(11)

\[
u(x, \theta, t) = \frac{R(x) y \psi y (x \theta)}{4 \pi x \gamma \left( \frac{1}{2} \gamma \right)} \left( \frac{dy}{dx} - \frac{y}{2} \right) \right) e^{-i(t + \omega)}
\]

(12)

3. VERTICAL DEPENDENCE OF THE TIDAL FIELDS

The best tidal observations are those for surface pressure, since they are obtained for a long period of time. The amplitudes and phases of the solar diurnal and semi-diurnal oscillations have been calculated by Haurwitz (1965) and Haurwitz (1956) respectively and are presented in terms of the associated Legendre functions as:

\[
\sum_{l=1}^{n} \left[ P_{l}^{s} (\theta) + \beta_{l} \right] \left[ P_{l}^{s} (\theta) + \beta_{l} \right] \sin \left( lt + \omega_{l} \right)
\]

(13)

For comparison with the theory, a transformation of associated Legendre polynomials to Legendre functions is required

\[
P_{l}^{s} (\theta) = \sum_{n=0}^{l} \gamma_{n} \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right)
\]

(14)

\( \lambda^{-1} \) of the solar day defines the period of oscillation; \( \lambda = 1 \) for diurnal and \( \lambda = 2 \) for semi-diurnal. Substituting (14) into (13) and comparing with (12) we get:

\[
y(0) = \frac{- \sigma A_{l} \left[ y_{l}^{s} + B_{l} \gamma_{l}^{s} \right]}{4 \pi x \gamma \left( \frac{1}{2} \gamma \right)} \left( \frac{dy}{dx} - \frac{y}{2} \right) e^{-i(t + \omega)}
\]

(15)

using the lower boundary condition (Eq. 9). The coefficients \( \gamma_{n} \) have been previously obtained through the solution of Laplace's tidal equation (Makarious, 1968). Therefore the two initial values \( y(0) \) and \( y'(0) \) have been evaluated at \( x = 0 \).

The vertical dependence terms in the horizontal wind fields (Eqs. 10, 11) may be expressed as:

\[
\left( \frac{dy}{dx} - \frac{y}{2} \right) \psi y (x \theta) = \zeta(x)
\]

(16)

This term had been previously evaluated (Makarious, 1979) in analysing 6 years of horizontal wind data and the results were obtained at \( z = 2.5 \) (2.5) 60 Km. By Lagrangian quadratic interpolation, and using the lower boundary condition \( \zeta(x) \) values of \( \psi y(x) \) at \( x = 0 \) (0.1) 8.4 are evaluated.

The solution \( y(x) \) is expressed as a complex variable function, whose real and imaginary parts are evaluated independently.
The initial-value problem for the inhomogeneous first order D.E. is to determine \( y(x) \) that satisfies (16) with the initial value \( y(0) \) as given in (15). Multiplying both sides by the integrating factor \( \exp(-x) \) and integrate both sides of the resulting equation we get:

\[
y(x) = y(0) \exp(x) + \int_0^x y'(s) \exp(x-s) \, ds
\]

The integral on the right-hand side has been evaluated using Simpson's rule of integration with the spacings \( 4x = 0.1 \).

By inspection, it was found that \( z(x) \) can be approximated fairly accurately by Complex Fourier Series expansion in the form:

\[
z(x) = \sum_{n=0}^{N} C_n e^{i\omega_n x}
\]

The coefficients \( C_n \) are obtained for the two regions \( 0 < x < 4.8 \) and \( 4.8 < x < 9.6 \) using Complex Fast Fourier Transform Technique. The two regions are chosen this way as they are characterized by the different temperature behaviours and consequently different wavelengths for each mode.

The periodic values of \( z(x) \), expressed by Eq. (18), are therefore included in the integral of (17) and solution \( y(x) \) is also obtained. The two obtained solutions are found to be comparable, within differences \( \pm 0.5\% \) on the average, reflecting the validity of the approximation (18) and the numerical stability of the solution. The initial-value problem is considered to be well-set in the domain \( 0 < x < 9.6 \) and thus the solution \( y(x) \) obtained in (17) is the only solution for each given \( y(0) \) at \( x = 0 \).

The inhomogeneous term \( z(x) \) (Eq. 16) is the forcing function and corresponds to the applied tidal force. Differentiating both sides of (16) we get:

\[
\frac{d^2y}{dx^2} + \mu^2(x)y = 0
\]

In comparing (19) with the homogeneous D.E. (3), the source function \( Q(x) \) is expressed analytically, in terms of the solution \( y(x) \), as:

\[
Q(x) = \left\{ \int_0^x y(s) \, ds \right\} \exp(x)
\]

\[ \mu^2(x) = \frac{1}{14} \left[ \frac{1}{4} + \mu^2(x) \right] y(x) \]

1. THE REDUCED HOMOGENEOUS EQUATION

If \( Q(x) = 0 \), the inhomogeneous second-order differential equation (3) reduces to the homogeneous form:

\[
\frac{d^2y}{dx^2} + \mu^2(x)y = 0
\]

An examination of this reduced equation (21) shows that there is a simple analogy with the propagation of plane waves in a medium of varying refractive index (Wilkes, 1949). This state of affairs is described by the Helmholtz equation, which, in general, is a typical one for oscillating systems.

In order to obtain exact solutions to the Helmholtz equation, the variable coefficient \( \mu^2(x) \) can be replaced by a collection of continuous functions which look like acceptable approximation to \( \mu^2(x) \). This type of functions will allow us to investigate effects of the continuous stratification by means of simple solutions. Insight into the properties of such a media may be obtained by the use of a linear law for \( \mu^2(x) \) in the Helmholtz equation.
In a previous investigation*, the region below \( x = 15.6 \) has been unequally divided into six subregions, in each of which

\[
\mu' \psi(x) = \mu x + \eta
\]  \hspace{1cm} (22)

The present study extends the previous one by dividing the region \( x < 15.6 \) into equally spaced subregions. The results for \( \mu \) are given in Table I for the different modes of diurnal and semidiurnal oscillations. It can be inferred that the atmosphere is divided into 4 distinct regions with regard to the characteristics of its refractive index, namely (i) \( 0 < x < 4.8 \), (ii) \( 4.8 < x < 9.6 \), (iii) \( 9.6 < x < 12.0 \), and (iv) \( 12.0 < x < 16.8 \). For the region \( 15.6 < x < 16.8 \), the reduced homogeneous equation assumes the Bessel form equation, and for other layers it assumes Stokes' form.

The homogeneous solution has then been obtained in terms of Bessel functions in the form:

\[
\psi(x) = C_1 \psi_1(x) + C_2 \psi_2(x)
\]  \hspace{1cm} (23)

The initial-value problem for the second-order homogeneous linear differential equation (22) is to determine the general solution \( \psi(x) \) given by (23) that satisfies the initial conditions \( \psi(x_0) \) and \( \psi'(x_0) \). Since the two solutions \( \psi_1 \) and \( \psi_2 \) are linearly independent, then we get the Wronskian:

\[
W(x_0) = \psi_1(x_0) \psi_2'(x_0) - \psi_2(x_0) \psi_1'(x_0) 
\]  \hspace{1cm} (24)

Therefore we get for the two constants \( C_1 \) and \( C_2 \) (Eq. 23) the following expressions:

\[
C_1 = \frac{\psi_1'(x_0) \psi_2(x_0) - \psi_2'(x_0) \psi_1(x_0)}{W(x_0)}
\]  \hspace{1cm} (25)

and

\[
C_2 = \frac{\psi_2'(x_0) \psi_1(x_0) - \psi_1'(x_0) \psi_2(x_0)}{W(x_0)}
\]  \hspace{1cm} (26)

In order to get \( \psi(x_0) \) and \( \psi'(x_0) \), we call for the condition at \( x_0 = 9.6 \), \( J \) is assumed to vanish at that height, and thereafter, based on observations (Butler and Small, 1963 and Linzen, 1967). Therefore at \( x_0 = 9.6 \), the general solution to the reduced equation (21) is equivalent to that obtained by (20) in setting \( Q = 0 \). Thus we get:

\[
\psi(x_0) = \frac{1}{(\mu^2 + 1/4)} \int \frac{\psi_2(x)}{(x - \mu)} e^{i(x - \mu)/l} \, dx
\]  \hspace{1cm} (26)

and

\[
\psi'(x_0) = \int \frac{\psi_2(x)}{(x - \mu)} e^{i(x - \mu)/l} \, dx
\]  \hspace{1cm} (27)

Substituting in Eq. (25), we can easily obtain the constants \( C_1 \) and \( C_2 \) and hence the homogeneous solution can then be evaluated numerically. This completes the method of numerical solution to the differential Equation (Eq.3).

5. RESULTS AND DISCUSSION

In this paper we presented a numerical method of solving linear inhomogeneous second-order differential equation with a variable coefficient.
as applied to the vertical tidal-wave propagation. Two solutions are obtained, a particular solution that is derived from the observed tidal motion and a homogeneous solution using the characteristics of the atmospheric structure.

As an illustration to the vertical behaviour of the solutions, the results for two modes of the diurnal oscillation \( (1, -1) \) and \( (1, 1) \) and for the main semidiurnal mode \( (2, 2) \) are presented in Table 2.

It can be inferred from Table 2, that the particular solution grows exponentially with \( x \) and has greater values for the trapped mode \( (1, -1) \) than the oscillating one, reflecting the effectiveness of the former mode to the excitation. Although the \( (1,1) \) mode is a propagating one, the constancy in phase is due to its large wavelength which is about 43 km in the troposphere and 25 km in the stratosphere. For the main semidiurnal mode \( (2, 2) \), \( \nu_2^2(x) \) is almost zero through most of the stratosphere; i.e. extremely long vertical wavelength (\( \sim 150 \) km). Thus not only does this mode receive the bulk of semidiurnal oscillation, but it must also respond to the excitation with particular efficiency.

With respect to the homogeneous solution, the function \( y(x) \) has been found to be non-oscillatory and the amplitude is exponentially increasing for \( 0 < x < 1.0 \) and decreasing above; \( \nu_0 \) the latter region the energy is trapped; a feature associated with the negative mode. For the oscillating mode the function \( y(x) \) is in general oscillating and thus it is very sensitive to the variation in temperature.

To conclude, with such a method of solution, a general solution to the radial equation can be expressed analytically. It will be interesting to show how this solution can be used to evaluate the thermal drive for the atmospheric tides. This will be the subject of the forthcoming work.

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TABLE 2. THE NUMERICAL SOLUTION TO THE WAVE EQUATION (17)

<table>
<thead>
<tr>
<th></th>
<th>Diurnal Oscillation</th>
<th>Semidiurnal Oscillation</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(2,2)</td>
</tr>
<tr>
<td></td>
<td>y**</td>
<td>y*</td>
<td>y**</td>
</tr>
<tr>
<td>X</td>
<td></td>
<td></td>
<td>y*</td>
</tr>
<tr>
<td>0</td>
<td>0.016</td>
<td>0.016</td>
<td>0.016</td>
</tr>
<tr>
<td>1.2</td>
<td>0.159</td>
<td>0.178</td>
<td>0.178</td>
</tr>
<tr>
<td>2.4</td>
<td>0.066</td>
<td>0.577</td>
<td>0.352</td>
</tr>
<tr>
<td>3.6</td>
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<td>3.229</td>
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<td>1.969</td>
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<td>3.581</td>
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<td>8.4</td>
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<td>10.795</td>
<td>6.519</td>
</tr>
<tr>
<td>9.6</td>
<td>19.662</td>
<td>19.662</td>
<td>11.679</td>
</tr>
</tbody>
</table>

* Values obtained from the integration, using the approximation given by Eq. (18).

** Values obtained from the integration of \( z(x) \) as obtained from the observations.

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TABLE 1. THE MODEL ATMOSPHERE

<table>
<thead>
<tr>
<th>1.a The Reduced Height</th>
<th>1.b The Refractive Index Characteristic</th>
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</thead>
<tbody>
<tr>
<td>x</td>
<td>Z(x) (m)</td>
</tr>
<tr>
<td>0.0</td>
<td>3.0</td>
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<td>1.2</td>
<td>9.0027</td>
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<td>16.9021</td>
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</tr>
<tr>
<td>15.6</td>
<td>100.0805</td>
</tr>
</tbody>
</table>

* P as obtained using Eq. (22)

** The subscript \( (s, n) \) stands for the type of oscillation \( s = 1 \) for migrating solar wave and \( n \) is the mode number, (negative values stand for modes with negative equivalent depths).
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