IC/82/109
INTERNAL REPORT
(Limited distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ALGEBRA OF PSEUDO-DIFFERENTIAL OPERATORS OVER C^*-ALGEBRA *

Noor Mohammad **
International Centre for Theoretical Physics, Trieste, Italy.

MIRAMARE - TRIESTE
August 1982

* To be submitted for publication.

** Permanent address: Ph.D Scholar Moscow State University, USSR.

I. INTRODUCTION

Pseudo-differential operators have been extensively studied recently. Kumano-go [4,5] studied algebra of pseudo-differential operators complex valued through the notion of Hormander class [2] $S^m_{\rho,\delta}(\Omega)$, $0 \leq \delta \leq \rho \leq 1$, $\Omega = \mathbb{R}^n$. Mishchenko and Fomenko [2] defined pseudo-differential operators over C^*-algebras in a classical way.

In the present paper we shall study algebras of pseudo-differential operators over C^*-algebras for the special case when in Hormander class $S^m_{\rho,\delta}(\Omega)$ $\Omega = \mathbb{R}^n$; $\rho = 1$, $\delta = 0$, m any real number, and the C^*-algebra is infinite dimensional non-commutative. The results obtained in this paper, have been already obtained in the case when pseudo-differential operators are complex valued by Hormander [1,2], Kohn and Nirenberg [3], Kumano-go [4,5], Mishchenko and Fomenko [6] and Shubin [8].

In the present paper the space B i.e. the set of A-valued C^*-functions in $\mathbb{R}^n$ (or $\mathbb{R}^n \times \mathbb{R}^n$) whose derivatives are all bounded, plays an important role. Here A denotes C^*-algebra.

In Sec.I we define the operator class $S^m_{\phi,0}$ and through it, the class $L^m_{1,0}$ of pseudo-differential operators.

In Sec.II we state the basic asymptotic expansion theorems concerning adjoint and product of operators of class $S^m_{1,0}$.

Sec.III is devoted to the proofs of L_2-continuity theorem and to the main theorem which states that algebra of all pseudo-differential operators over C^*-algebras, is itself C^*- algebra.

II. DEFINITIONS AND NOTATIONS

Let $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and let A be infinite dimensional, non-commutative C^*-algebra and let $\zeta = \zeta(\mathbb{R}^n, A)$ denote the space of all A-valued C^∞-functions which together with all their derivatives decrease faster than any power of $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ as $|x| \to \infty$ i.e. $\varphi(x) \in S$ if for any pair of multi-indices α, β, $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\beta| \, \| \partial_x^\alpha \varphi(x) \| < \infty \; ; \quad \alpha_i, \beta_j \geq 0 \; ; \; i,j = 1,\dots,n \tag{2.1}$$

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots\dots \partial x_n^{\alpha_n}} \, , \qquad \partial x_j = \frac{\partial}{\partial x_j}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \, ; \quad x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n} \, .$$

For $\varphi \in \mathcal{S}$ we defined the Fourier transform of φ by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix.\xi} \varphi(x) \, dx$$
$$x.\xi = x_1\xi_1 + \quad + x_n \xi_n . \tag{2.2}$$

The inverse Fourier transform is defined as

$$\varphi(x) = \int e^{ix.\xi} \hat{\varphi}(\xi) \, đ\xi \tag{2.3}$$

where $đ\xi = \frac{1}{(2\pi)^n} d\xi \; , \quad \varphi \in S(\mathbb{R}^n, A)$.

We now take End(a,n)-space of all $n \times n$ matrices with elements in A which is obviously C^*-algebra and allow the functions $a(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ to take values in End(a,n).

Let $A^n = A \underbrace{\oplus \dots \oplus}_{\text{n-copies}} A$ denote the direct sum of n-copies of A.

<u>Definition 2.1</u>

We say that a C^∞-function $a(x,\xi)$ belonging to $C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \text{End}(A,n))$ belongs to the class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ when for any multi-indices α, β $\exists$ a constant $C_{\alpha,\beta}$ s.t.

$$\| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|} . \tag{2.4}$$

Here we used the Friedrichs notation $\langle \xi \rangle = (1 + \sum_{i=1}^n \xi_i^2)^{1/2}$ and m is any real number. (See [4,5]).

Definition 2.2

For any $a(x,y,\xi) \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathrm{End}(A,n))$ we define an operator $\mathcal{A}$ Called pseudo-differential operator by

$$\mathcal{A}u(x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n} e^{i(x-y)\xi}\, a(x,y,\xi)\, u(y)\, dy\, d\xi \tag{2.5}$$

where $u \in S(\mathbb{R}^n, A^n)$ and $x,y,\xi \in \mathbb{R}^n$.

We set $L^m_{1,0} = L^m$-class of all pseudo-differential operators of type (2.5).

Remark Let $\mathcal{A} = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ be a differential operator of order m with coefficients $a_\alpha(x) \in B$-space of all A-valued C^∞-functions, defined in $\mathbb{R}^n$ whose derivatives are all bounded.

Then $\mathcal{A} \in L^m$ i.e. $\mathcal{A}$ is a pseudo-differential operator of order m.

Let us show that integral (2.5) converges. Here convergence is taken in the sense that firstly integration with regard to y and then with regard to ξ (after partial differentiation).

Firstly let $a(x,y,\xi) = a(x,\xi)$.

$$\mathcal{A}u(x) = \iint e^{i(x-y)\xi}\, a(x,\xi)\, dy\, d\xi \tag{2.6}$$

Since

$$\left(1-\sum_{i=1}^n \left(\frac{\partial}{\partial y_i}\right)^2\right)^N e^{i(x-y)\xi} = \left(1+\sum_{i=1}^n \xi_i^2\right)^N e^{i(x-y)\xi}$$

where N is sufficiently large whole number. Substituting for $e^{i(x-y)\xi}$ in (2.6) and integrating by parts, we obtain

$$\mathcal{A}u(x) = \iint e^{i(x-y)\xi}\, a(x,\xi)\, u(y)\, dy\, d\xi$$

$$= \iint \langle D\rangle^{2N} e^{i(x-y)\xi} \frac{a(x,\xi)}{\langle\xi\rangle^{2N}}\, u(y)\, dy\, d\xi$$

where

$$\langle D\rangle^{2N} = \left(1-\sum_{i=1}^n \left(\frac{\partial}{\partial y_i}\right)^2\right)^N$$

$$\langle \xi\rangle^{2N} = \left(1+\sum_{i=1}^n \xi_i^2\right)^N$$

$$\mathcal{A}u(x) = \iint e^{i(x-y)\xi} \frac{a(x,\xi)}{\langle\xi\rangle^{2N}} \langle D\rangle^{2N} u(y)\, dy\, d\xi$$

$$= \int e^{ix\cdot\xi} \frac{a(x,\xi)}{\langle\xi\rangle^{2N}} \langle\xi\rangle^{2N}\, \hat{u}(\xi)\, d\xi$$

$$= \int e^{ix\cdot\xi}\, a(x,\xi)\, \hat{\omega}(\xi)\, d\xi .$$

So we have shown the existence of integral (2.6) and similarly it can be shown for $a(x,y,\xi)$.

Theorem 2.1

The operators $\mathcal{A}: S(\mathbb{R}^n, A^n) \to S(\mathbb{R}^n, A^n)$ defined by (2.6), defines a continuous linear mapping and can be continuously extended of space $B = \left\{u(x) \in C^\infty(\mathbb{R}^n, A^n) : \sup_x \| \partial_x^\alpha u(x) \| < C_\alpha\right\}$ into itself.

Proof

Let $\mathcal{A}u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi$

where $u \in S(\mathbb{R}^n, A^n)$ and

$$a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{End}(A,n)).$$

By easy calculation we have

$$x^\alpha \partial_x^\beta \mathcal{A}u(x) = \int_{\mathbb{R}^n} x^\alpha \partial_x^\beta \left[e^{ix\cdot\xi} a(x,\xi)\, \hat{u}(\xi)\right] d\xi$$

$$= \int x^\alpha \sum_{\beta'\le\beta} C_{\beta,\beta'}\, \partial_x^{\beta'} e^{ix\cdot\xi}\, \partial_x^{\beta-\beta'} a(x,\xi)\, \hat{u}(\xi)\, d\xi$$

$$= \sum_{\beta'\le\beta} C_{\beta,\beta'} \int_{\mathbb{R}^n} (\partial_\xi^\alpha e^{ix\cdot\xi})\, \xi^{\beta'} \partial_x^{\beta-\beta'} a(x,\xi)\, \hat{u}(\xi)\, d\xi .$$

Integrating by parts we obtain

$$= \sum_{\beta'\le\beta} C_{\beta,\beta'} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \partial_\xi^\alpha \left\{ \xi^{\beta'} \partial_x^{\beta-\beta'} a(x,\xi)\, \hat u(\xi)\right\} d\xi$$

$$= \sum_{\beta'\le\beta} C_{\beta,\beta'} \sum_{\alpha'\le\alpha} C_{\alpha,\alpha'} \sum_{\alpha''\le\alpha-\alpha'} C_{\alpha-\alpha',\alpha''} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \partial_\xi^{\alpha-\alpha'-\alpha''} \xi^{\beta'} \times$$

$$\times\, \partial_\xi^{\alpha''} \partial_x^{\beta-\beta'} a(x,\xi)\, \partial_\xi^{\alpha'} \hat u(\xi)\, d\xi .$$

Therefore

$$|x^\alpha| . \| \partial_x^\beta \mathcal{A} u(x) \| \le \sum_{\beta'\le\beta} C_{\beta,\beta'} \sum_{\alpha'\le\alpha} C_{\alpha,\alpha'} \sum_{\alpha''\le\alpha-\alpha'} C_{\alpha-\alpha',\alpha''}\, C_{\alpha'',\beta-\beta'} \times$$

$$\times \int_{\mathbb{R}^n} \langle \xi \rangle^{m-|\alpha''|}\, \xi^{\beta'-\alpha+\alpha'+\alpha''} \| \partial_\xi^{\alpha'} \hat u(\xi) \|\, d\xi$$

Since
$$\| \partial_\xi^{\alpha''} \partial_x^{\beta-\beta'} a(x,\xi) \| \le C_{\alpha'',\beta-\beta'} \langle \xi \rangle^{m-|\alpha''|} .$$

From here follows the continuity of operator $\mathcal{A}$.

Let us show that $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is also continuous. Let $\mathcal{A}$ be given as

$$\mathcal{A} u(x) = \iint e^{i(x-y)\xi}\, a(x,\xi)\, u(y)\, dy\, d\xi$$

where $u \in \mathcal{B}$ and $a(x,\xi) \in S^m_{1,0}$. It is easy to show the existence of this integral. Let us prove that $\mathcal{A}$ maps $\mathcal{B}$ to itself continuously. Consider:

$$\partial_x^\alpha \mathcal{A} u(x) = \partial_x^\alpha \left[\iint e^{i(x-y)\xi} a(x,\xi)\, u(y)\, dy\, d\xi \right]$$

$$= \partial_x^\alpha \left[\iint \Big(1-\sum_{k=1}^n \partial_{\xi_k}^2\Big)^{M_1} \Big(1-\sum_{k=1}^n \partial_{y_k}^2\Big)^{M_2} e^{i(x-y)\xi} a(x,\xi)\, u(y)\, dy\, d\xi \right]$$

$$= \iint \partial_x^\alpha \Big[e^{i(x-y)\xi} (1+|x-y|^2)^{-M_1} (1+|\xi|^2)^{-M_2} \times$$

$$\times \left\{ (1-\Delta_\xi)^{M_1} a(x,\xi) \right\} . \left\{ (1-\Delta)^{M_2} u(y) \right\} \Big]\, dy\, d\xi$$

where
$$(1-\Delta_\xi)^{M_1} = \Big(1 - \sum_{k=1}^n \Big(\frac{\partial}{\partial \xi_k}\Big)^2 \Big)^{M_1}$$
and
$$(1-\Delta)^{M_2} = \Big(1 - \sum_{k=1}^n \Big(\frac{\partial}{\partial y_k}\Big)^2 \Big)^{M_2}$$

M_1 and M_2 sufficiently large whole numbers $M_1, M_2 > 0$.

Putting $x - y = z$, we obtain

$$= \iint \partial_x^\alpha \left[e^{iz.\xi} (1+|z|^2)^{-M_1} (1+|\xi|^2)^{-M_2} (1-\Delta_\xi)^{M_1} a(x,\xi) (1-\Delta_x)^{M_2} u(x+z)\, dz\, d\xi \right]$$

$$= \sum_{\alpha'+\alpha''=\alpha} C_{\alpha',\alpha''} \iint e^{i\xi.z} (1+|z|^2)^{-M_1} (1+|\xi|^2)^{-M_2} \times$$

$$\times \left[(1-\Delta_\xi)^{M_1} \partial_x^{\alpha'} a(x,\xi) \right] \left[(1-\Delta_x)^{M_2} \partial_x^{\alpha''} u(x+z) \right] dz\, d\xi$$

$$\Rightarrow \| \partial_x^\alpha \mathcal{A} u(x) \| \le \sum_{\alpha'+\alpha''=\alpha} C_{\alpha',\alpha''} \iint \frac{C_1 C_2 \langle \xi \rangle^m\, dz\, d\xi}{(1+|z|^2)^{M_1} (1+|\xi|^2)^{M_2}}$$

Since
$$\| (1-\Delta_x)^{M_2} \partial_x^{\alpha''} u(x+z) \| \le C_2$$
and
$$\| (1-\Delta_\xi)^{M_1} \partial_x^{\alpha'} a(x,\xi) \| \le C_1 \langle \xi \rangle^m .$$

This completes proof.

From here onwards we shall use abbreviation P.D.O. for Pseudo-differential operator.

III. THE SYMBOLIC CALCULUS OF PSEUDO-DIFFERENTIAL OPERATOR

Definition 3.1

Let $\mathcal{A}$ be P.D.O.. By symbol of P.D.O. $\mathcal{A}$ we mean the function $\sigma_{\mathcal{A}}(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ defined by the formula:

$$\sigma_{\mathcal{A}}(x,\xi) f = e^{-ix.\xi} \mathcal{A}(e^{ix.\xi} f) \tag{3.1}$$

where f is a constant which belongs to A^n. Obviously $f = \text{const} \Rightarrow f \in B$ so $e^{ix\xi} f \in B$. With the help of Fourier inverse transform we can write

$$\mathcal{A}u(x) = \iint e^{i(x-y)\xi} \sigma_{\mathcal{A}}(x,\xi)\, u(y)\, dy\, d\xi \tag{3.2}$$

Definition 3.2

Let $\mathcal{A}$ be P.D.O.. Since by theorem (2.1) $\mathcal{A}$ defines a continuous linear map of $\mathcal{S}(\mathbb{R}^n, A^n)$ into itself and of B into itself. The same is true of its transpose ${}^t\mathcal{A}$ which can be defined by

$$\langle \mathcal{A}u, v \rangle = \langle u, {}^t\mathcal{A}v \rangle \tag{3.3}$$

where $\langle u(x), v(x) \rangle = \int_{\mathbb{R}^n} \sum_{k=1}^{n} u_k^*(x)\, v_k(x)\, dx \in A.$

Therefore if $\mathcal{A} \in L^m_{1,0}$ is defined by

$$\mathcal{A}u(x) = \iint_{\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)\, u(y)\, dy\, d\xi$$

$$u \in S(\mathbb{R}^n, A^n) \text{ and } a(x,y,\xi) \in S^m_{1,0}$$

then from $\langle \mathcal{A}u, v \rangle = \langle u, {}^t\mathcal{A}v \rangle$

we have
$$\langle \mathcal{A}u, v \rangle = \iint e^{-i(x-y)\xi} u^*(y)\, a^{*t}(x,y,\xi)\, v(x)\, dy\, d\xi\, dx$$

$$= \int u^*(y) \left\{ \iint e^{-i(x-y)\xi} a^{*t}(x,y,\xi)\, v(x)\, dx\, d\xi \right\} dy$$

$$= \langle u, {}^t\mathcal{A}v \rangle$$

where

$${}^t\mathcal{A}v(y) = \iint e^{i(y-x)\xi} a^{*t}(x,y,\xi)\, v(x)\, dx\, d\xi \tag{3.4}$$

or $${}^t\mathcal{A}v(x) = \iint e^{i(x-y)\xi} a^{*t}(y,x,\xi)\, v(y)\, dy\, d\xi .$$

Here $a^{*t}(x,y,\xi)$ means the transpose of the matrix function $a(x,y,\xi)$. From (3.4), it is obvious that ${}^t\mathcal{A} \in L^m_{1,0}$.

Now we state some theorems, their proofs are omitted as they are quite easy and can be proved on the same lines in Kumano-go [5]. For the definition of asymptotic expansion see Kohn and Nirenberg [3].

Theorem 3.3

Let $\mathcal{A}$ be P.D.O., $\sigma(x,\xi)$ be its symbol and $\sigma'(x,\xi)$ be the symbol of the operator ${}^t\mathcal{A}$. Then

$$\sigma'(x,\xi) \sim \sum_{\alpha} \partial_\xi^\alpha D_x^\alpha \sigma^{*t}(x,\xi)/\alpha! \tag{3.5}$$

where " $\sim$ " means asymptotic expansion.

Definition 3.4

The dual symbol $\tilde{\sigma}(x,\xi)$ of operator $\mathcal{A}$ is defined in the following way: $\tilde{\sigma}(x,\xi) = \sigma'^{*t}(x,\xi)$ where $\sigma'(x,\xi)$ denotes the symbol of operator ${}^t\mathcal{A}$. We note that

$${}^t({}^t\mathcal{A}) = \mathcal{A}.$$

It follows from here that operator $\mathcal{A}$ can be written in term of dual symbol $\tilde{\sigma}(x,\xi)$ by the formula

$$\mathcal{A}u(x) = \iint e^{i(x-y)\xi} \tilde{\sigma}(y,\xi)\, u(y)\, dy\, d\xi . \tag{3.6}$$

Theorem 3.5

The dual symbol $\tilde{\sigma}(x,\xi)$ of operator $\mathcal{A}$ having symbol $\sigma(x,\xi)$ is given by

$$\tilde{\sigma}(x,\xi) \sim \sum_{\alpha} \partial_\xi^\alpha D_x^\alpha \sigma(x,\xi)/\alpha! . \tag{3.7}$$

Proof:

The proof immediately follows from

$$\tilde{\sigma}(x,\xi) = \sigma'^{*t}(x,\xi)$$

and

$$\sigma'(x,\xi) \sim \sum_{\alpha} \partial_{\xi}^{\alpha} D_x^{\alpha} \sigma^{*t}(x,\xi)/\alpha!$$

Theorem 3.6

Let $\mathcal{A}_1$ and $\mathcal{A}_2 \in L^m_{1,0}$ and have symbols $a(x,\xi)$, $b(x,\xi)$. Then the symbol of $\mathcal{A}_1\mathcal{A}_2$ is (x,ξ):

$$C(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a(x,\xi)\, D_x^{\alpha} b(x,\xi) \quad . \tag{3.8}$$

This theorem is a general form of the Leibniz formula.

IV. L_2-CONTINUITY THEOREM

In this section we firstly define a Hilbert A-module as given by Paschke [7].

Definition 4.1

Let A be a C^*- algebra. A pre-Hilbert A-module is a right A-module X equipped with a conjugate bilinear map $\langle .\,,\,. \rangle : X \times X \to A$ satisfying the following conditions:

(i) $\langle u,u\rangle \geq 0 \quad \forall u \in X$

(ii) $\langle u,u\rangle = 0$ only if $u = 0$

(iii) $\langle u,v\rangle = \langle v,u\rangle^* \quad \forall u,v \in X$

(iv) $\langle u\lambda,v\rangle = \lambda^*\langle u,v\rangle \; \forall u,v \in X,\ \lambda \in A.$

The map $\langle .\,,\,.\rangle$ will be called an A-valued inner product on X.

For a pre-Hilbert A-module X, define norm $||\cdot||_X$ on X by $||u||_X = ||\langle u,u\rangle||^{\frac{1}{2}}$.

The norm $||\cdot||_X$ satisfies

(v) $\|u\lambda\|_X \leq \|u\|_X \cdot \|\lambda\| \qquad u \in X,\ \lambda \in A$

(vi) $\|\langle u,v\rangle\| \leq \|u\|_X \cdot \|v\|_X \qquad \forall u, v \in X$.

A pre-Hilbert A-module X which is complete with respect to $||\cdot||_X$ will be called a Hilbert A-module.

Theorem 4.2

Let $L_2(\mathbb{R}^n,A^n) = \left\{ f : \int_{\mathbb{R}^n} f^*(x)\, f(x)\, dx \text{ converges} \right\}$.

Then

1) $L_2(\mathbb{R}^n,A^n)$ is a Hilbert A-module.

2) Pseudo-differential operator $\mathcal{A}: \mathcal{S}(\mathbb{R}^n,A^n) \to \mathcal{S}(\mathbb{R}^n,A^n)$ of order zero is bounded operator in the norm of $L_2(\mathbb{R}^n,A^n)$ i.e. $\mathcal{A}$ can be continuously extended of $L_2(\mathbb{R}^n,A^n)$ into itself.

3) Pseudo-differentail operator $\mathcal{A}$ of order zero admits conjugate operator defined by

$$\langle {}^t\mathcal{A}u, v\rangle = \langle u, \mathcal{A}v\rangle \;, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n, A^n) \;.$$

Proof

It is easy to show that $L_2(\mathbb{R}^n,A^n)$ is a pre-Hilbert A-module. We will prove at the end of the proof of this theorem that $L_2(\mathbb{R}^n,A^n)$ is complete with respect to L_2-norm.

2) Let $\mathcal{A}$ be pseudo-differential operator of order zero defined by

$$\mathcal{A}u(x) = \int e^{ix\cdot\xi} \sigma(x,\xi)\, \hat{u}(\xi)\, d\xi$$

Also we shall suppose that $\sigma(x,\xi)$ is a finite function with respect to x.

Then

$$\widehat{\mathcal{A}u}(\xi) = \int \hat{\sigma}(\xi-\eta, \eta)\, \hat{u}(\eta)\, d\eta$$

We put $||\mathcal{A}|| = \operatorname{Supp} ||\mathcal{A}u||$ where $||u|| \leq 1$.

$$\|\mathcal{A}u\|^2 = \|\langle \mathcal{A}u, \mathcal{A}u\rangle\| \,.$$

We define norm of $u(x) \in \mathcal{S}(\mathbb{R}^n, A^n)$ by :

$$\|u\|_s^2 = \left\| \int (1+|\xi|^2)^s (u(\xi), u(\xi))\, d\xi \right\|$$

where $\quad (u(\xi), u(\xi)) = \sum_{i=1}^{n} u_i^*(\xi)\, u_i(\xi), \quad s \in \mathbb{R}$.

We denote by $H_s(\mathbb{R}^n, A^n)$-Hilbert space obtained as the completion of space $\mathcal{S}(\mathbb{R}^n, A^n)$ in the norm $||\cdot||_s$ and call it Sobolev space. It is quite easy to see that $\| u \|_{L^2} = \| u \|_0$. Consider now

$$\| \hat{A}u(\xi), \hat{A}u(\xi)\rangle \|_s = \| \int (\hat{A}u(\xi), \hat{g}(\xi))(1+|\xi|^2)^s \, d\xi \|$$

where $\hat{A}u(\xi) = \hat{g}(\xi)$

$$= \iint (\sigma(\xi-\eta,\eta)\hat{u}(\eta), \hat{g}(\xi))(1+|\xi|^2)^s \, d\xi \, d\eta \| .$$

We have to prove the inequality :

$$\| \langle \hat{A}u(\xi), \hat{g}(\xi)\rangle \|_s \leq C \, \|u\|_s \cdot \|g\|_s \tag{4.1}$$

without loss of generality, it is sufficient to prove inequality (4.1) for the case when $\hat{\sigma}(\xi-\eta,\eta)$ is positive self-conjugate operatorsfor ξ, η. Then put $\hat{\sigma}(\xi-\eta,\eta) = [b(\xi,\eta)]^2$ and $b(\xi,\eta)^* = b(\xi,\eta)$. Let us prove that such a supposition is justified.

Lemma 4.1

Let A be C^*-algebra, non-commutative infinite. Suppose $f : X \to A$ is a continuous mapping where $X \in \mathbb{R}^n$. Then

$$f(x) = \sum_i \lambda_i f_i^2(x) \quad \text{and} \quad f_i^*(x) = f_i(x)$$

$$\lambda_i \text{ - constant }, \qquad X \in \mathbb{R}^n .$$

Proof of lemma 4.1

Let X be compact. Then $f(x) \in A$ can be written as

$$f(x) = f_1(x) + i f_2(x) \qquad f_1(x) = \frac{f(x) + f^*(x)}{2}, \quad f_2(x) = \frac{f(x) - f^*(x)}{2i}$$

f_1, f_2 are hermitians.

Since X is compact $\Rightarrow \| f_1(x) \| < C < \infty$. It is sufficient to prove the statement for $f_1(x)$. Now $f_1(x) = [C + f_1(x)] - C$, C-scalar. As $f_1(x) = f_1^*(x)$, then $\mathrm{spect}(C + f_1(x)) \subset [0, \infty]$. So $C + f_1(x) = g_1^2(x)$ and $C = h_1^2(x)$. Hence $f(x) = \sum_i \lambda_i f_i^2(x)$.

Let $X = \mathbb{R}^n$ and suppose $F(x) = \frac{f(x)}{1+\|f(x)\|} \Rightarrow \|F(x)\| \leq 1$ and

$F(x) = \Sigma_k \lambda_k F_k^2(x)$ where λ_k-constant and $F_k^*(x) = F_k(x)$.

Now $f(x) = 1 + \|f(x)\| \cdot \Sigma_k \lambda_k F_k^2(x)$ also $1 + \|f(x)\| = \rho^2(x)$ as it is real valued function then $f(x) = \Sigma \lambda_k f_k^2(x)$, $f_k(x) = \rho(x) F_k(x)$ which proves lemma.

Now come back to the proof of Theorem 4.1 we have

$$\| \langle \hat{A}u(\xi), \hat{g}(\xi)\rangle \|_s = C \| \iint (b^2(\xi,\eta)\hat{u}(\eta), \hat{g}(\xi))(1+|\xi|^2)^s \, d\xi \, d\eta \|$$

$$\leq C \| \iint (b(\xi,\eta)\hat{u}(\eta), b(\xi,\eta)\hat{u}(\eta))(1+|\xi|^2)^s \, d\xi \, d\eta \|^{1/2} \times$$

$$\times \| \iint (b(\xi,\eta)\hat{g}(\xi), b(\xi,\eta)\hat{g}(\xi))(1+|\xi|^2)^s \, d\xi \, d\eta \|^{1/2}$$

$$\leq C \| \iint (\hat{\sigma}(\xi-\eta,\eta)\hat{u}(\eta), \hat{u}(\eta))(1+|\xi|^2)^s \, d\xi \, d\eta \|^{1/2} \times$$

$$\times \| \iint (\hat{\sigma}(\xi-\eta,\eta)\hat{g}(\xi), \hat{g}(\xi))(1+|\xi|^2)^s \, d\xi \, d\eta \|^{1/2} . \tag{4.2}$$

Let us show that

$$\| \iint (\hat{\sigma}(\xi-\eta,\eta)\hat{u}(\eta), \hat{u}(\eta))(1+|\xi|^2)^s \, d\xi \, d\eta \|$$

$$\leq C \| \iint (\hat{u}(\eta), \hat{u}(\eta))(1+|\eta|^2)^s \, d\eta \| \tag{4.3}$$

We have $\| \iint (\hat{\sigma}(\xi-\eta,\eta)\hat{u}(\eta), \hat{u}(\eta))(1+|\xi|^2)^s \, d\xi \, d\eta \|$

$$= \| \iint (\hat{\sigma}(\xi-\eta,\eta)\frac{(1+|\xi|^2)^s}{(1+|\eta|^2)^s}\hat{u}(\eta), \hat{u}(\eta))(1+|\eta|^2)^s \, d\xi \, d\eta \|$$

$$= \| \int d\eta \left(\left\{ \int (\hat{\sigma}(\xi-\eta,\eta)\frac{(1+|\xi|^2)^s}{(1+|\eta|^2)^s} \, d\xi \right\} \hat{u}(\eta), \hat{u}(\eta) \right)(1+|\eta|^2)^s \| . \tag{4.4}$$

Since $\hat{\sigma}(\xi-\eta,\eta) \in \mathcal{S}(\mathbb{R}^n, A^n) \Rightarrow \|\sigma(\xi-\eta,\eta)\| \leq \frac{C_1}{(1+|\xi-\eta|^2)^N}$ and from

Peeters' inequality $\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^s \le 2^{|s|}(1+|\xi-\eta|^2)^{|s|}$

where N is sufficiently large whole number. It follows that if

$(\xi,\eta) = \partial(\xi,\eta)\left\{\frac{1+|\xi|^2}{1+|\eta|^2}\right\}^s$ then $\|(\xi,\eta)\| \le C(1+|\xi-\eta|^2)^{-N}$ for sufficiently large whole number N.

Then norm of integral $\int K(\xi,\eta)d\xi = B(\eta)$ is bounded by a constant which does not depend upon η. For the estimate of right side of (4.4) we shall use the representation of all elements $K(\xi,\eta)$, $\hat u(\eta)$ as operators on some Hilbert space H. Then putting $\bar{\bar u}(\eta) = \hat u(\eta)(1+|\xi|^2)^{s/2}$ we obtain

$$\left\|\int (B(\eta)\bar{\bar u}(\eta), \bar{\bar u}(\eta))\,d\eta = \sup_{\|x\|=1}\left(\left\{\int (B(\eta)\bar{\bar u}(\eta), \bar{\bar u}(\eta)\,d\eta\right\}x, x\right)\right. .$$

We can consider that all elements $u(\eta)$, $B(\eta)$ are matrices with operator elements. Then it is sufficient to show the estimate for norm of integral of operator

$$\int \bar{\bar u}(\eta)B(\eta)\bar{\bar u}^*(\eta)\,d\eta$$

Then

$$\left\|\int \bar{\bar u}(\eta)B(\eta)\bar{\bar u}^*(\eta)\,d\eta\right\| = \sup_{\|x\|=1}\left(\left\{\int \bar{\bar u}(\eta)B(\eta)\bar{\bar u}^*(\eta)\,d\eta\right\}x, x\right)$$

$$= \sup_{\|x\|=1}\int (B(\eta)\bar{\bar u}^*(\eta)x, \bar{\bar u}^*(\eta)x)\,d\eta \le \sup_{\|x\|=1}\int \|B(\eta)\|(\bar{\bar u}^*(\eta)x, \bar{\bar u}^*(\eta)x)\,d\eta$$

$$\le C\sup_{\|x\|=1}\int(\bar{\bar u}^*(\eta)x, \bar{\bar u}^*(\eta)x)\,d\eta =$$

$$= C\sup_{\|x\|=1}\left(\left\{\int \bar{\bar u}(\eta)\bar{\bar u}^*(\eta)\,d\eta\right\}x, x\right) \le$$

$$\le C\left\|\int \bar{\bar u}(\eta)\bar{\bar u}^*(\eta)\,d\eta\right\| .$$

Thus substituting the last in (4.4) we obtain (4.3).

Analogously it can be shown that

$$\left\|\iint (\hat\sigma(\xi-\eta),\eta)\hat g(\xi), \hat g(\xi))(1+|\xi|^2)^s\,d\xi\,d\eta\right\|$$
$$\le C\left\|\int(\hat g(\xi), \hat g(\xi))(1+|\xi|^2)\,d\xi\right\| . \qquad (4.5)$$

Substituting now (4.3) and (4.5) in (4.2) we get((4.1). So $\|\mathcal{A}u\|_s \le C\|u\|_s$

3) In Sec.III definition 3.2 we have shown that the conjugate operator ${}^t\mathcal{A}$ defined by

$$\langle {}^t\mathcal{A}u, v\rangle = \langle u, \mathcal{A}v\rangle, \quad \forall u,v\in S(\mathbb{R}^n, A^n)$$

is pseudo-differential operator. Now we want to show the completeness of L_2 with respect to L_2-norm.

We have shown above that $H_0(\mathbb{R}^n, A^n) \cong L_2(\mathbb{R}^n, A^n)$. In([6] lemma 4.1)it was proved that $H_0(\mathbb{R}^n, A^n)$ and $\ell_2(A)$ are isomorphic where $\ell_2(A)$ is the space of sequences $x = (x_1, x_2, \dots, x_n, \dots)$ $x_k \in A$, $1 < k < \infty$ satisfying the condition:
the series $\sum_{k=1}^{\infty} x_k^* x_k$ converges. We define norm in $\ell_2(A)$ by the formula:

$$\|x\|^2 = \left\|\sum_{k=1}^{\infty} x_k^* x_k\right\|$$

Hence the proof of Theorem (4.1) is complete.

Theorem 4.2

Algebra of all P.D.O. of order zero with coefficients in C^*-algebra is itself C^*-algebra.

Proof

Let us denote by $\mathcal{L}$, algebra of all P.D.O. of order zero. Following Theorem (4.1) we have only to show that $\|{}^t\mathcal{A}\mathcal{A}\| = \|\mathcal{A}\|^2$. We put
$\|\mathcal{A}\| = \sup_{\|u\|\le 1}\|\mathcal{A}u\|$ $\quad u \in S(\mathbb{R}^n, A^n)$

where $\|\cdot\|$ means L_2-norm. Then

$$\|\mathcal{A}\| = \sup_{\|u\|\le 1, \|v\|\le 1}\|\langle\mathcal{A}u, v\rangle\|$$

$\forall u,v \in S(\mathbb{R}^n, A^n)$

Certainly

$$\|\mathcal{A}\|^2 = \sup_{\|u\|\leq 1} \|\langle \mathcal{A}u, \mathcal{A}u\rangle\|$$

$$\leq \sup_{\|u\|\leq 1} \Big(\sup_{\|v\| = \|\mathcal{A}u\|} \|\langle \mathcal{A}u, v\rangle\| \Big) =$$

$$= \sup_{\|u\|\leq 1} \Big(\|\mathcal{A}u\| \sup_{\|v\|\leq 1} \|\langle \mathcal{A}u, v\rangle\| \Big) \leq$$

$$\leq \|\mathcal{A}\| \sup_{\|u\|\leq 1,\ \|v\|\leq 1} \|\langle \mathcal{A}u, v\rangle\|$$

So $\quad \|\mathcal{A}^*\| = \|\mathcal{A}\|$

Next

$$\|{}^t\!\mathcal{A}\,\mathcal{A}\| = \sup_{\|u\|\leq 1,\ \|v\|\leq 1} \|\langle {}^t\!\mathcal{A}\,\mathcal{A}u, v\rangle\|$$

$$= \sup_{\|u\|\leq 1,\ \|v\|\leq 1} \|\langle \mathcal{A}u, \mathcal{A}v\rangle\| \geq$$

$$\geq \sup_{\|u\|\leq 1} \|\langle \mathcal{A}u, v\rangle\| = \|\mathcal{A}\|^2$$

But on the other hand we have

$$\|{}^t\!\mathcal{A}\,\mathcal{A}\| = \sup_{\|u\|\leq 1} \|{}^t\!\mathcal{A}\,\mathcal{A}u\| \leq \|{}^t\!\mathcal{A}\| \sup_{\|u\|\leq 1} \|\mathcal{A}u\|$$

$$= \|{}^t\!\mathcal{A}\|\cdot\|\mathcal{A}\| = \|\mathcal{A}\|^2$$

Thus

$$\|{}^t\!\mathcal{A}\,\mathcal{A}\| = \|\mathcal{A}\|^2$$

which proves theorem.

ACKNOWLEDGMENTS

The author acknowledges with great pleasure his gratitude to Professor A.S. Mishchenko for his guidance, helful advice and encouragement. The author also wishes to express his hearty thanks to Dr. M.A. Shubin and Dr. Ju.P. Solorev for the invaluable assistance he received in the form of suggestions.

The author is also grateful to Professor M.S. Narasimhan who read the final manuscript and of his help in making it a preprint.

The author wishes to express his thanks to Professor Abdus Salam and to the International Centre for Theoretical Physics, Trieste for the hospitality extended to him.

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