IC/82/116

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

MASS SPLITTING INDUCED BY GRAVITATION

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1982 MIRAMARE-TRIESTE

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International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

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ABSTRACT

The exact combination of internal and geometrical symmetries and the associated mass splitting problem is discussed. A 10-parameter geometrical symmetry is defined in a curved space-time in such a way that it is a combination of de Sitter groups. In the flat limit it reproduces the Poincare-group and its Lie algebra has a nilpotent action on the combined symmetry only in that limit. An explicit mass splitting expression is derived and an estimation of the order of magnitude for spin zero mesons is made.

MIRAMARE - TRIESTE
August 1982

* To be submitted for publication.

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I. INTRODUCTION

The mass spectra of elementary particles is currently understood in terms of symmetry breaking scales. On the other hand, it has been known for some time that an exact combination of the Poincaré-group with an internal symmetry group leads to a mass spectra for particle multiplets which is either continuous or it consists of a single point. Thus, in a sense the symmetry breaking procedure provides a phenomenological bypass to the theoretical symmetry combination problem. A brief analysis of the various no go theorems shows that the basic difficulty lies in the fact that the Lie algebra of the Poincaré group acts nilpotently on the Lie algebra of the combined symmetry [1-5]. There are basically three ways to avoid this nilpotent action:

(a) The combined symmetry is not a Lie group. This alternative finds its most remarkable realization in supersymmetric theories. In fact the possibility of combining internal and space-time symmetries was one of the basic motivations for supersymmetry [6]. However, again, to avoid fermions and bosons having the same mass, supersymmetry in the end has to be broken.

(b) The combined symmetry is an infinite Lie group. This case may find its main difficulty in the classification of the unitary representations of an infinite Lie group. It is worth mentioning that one of the most general and recent versions of the no go theorems also applies to infinite Lie groups [4].

Finally in case (c) the space-time symmetry is not a Poincaré-like group. This alternative has been the subject of various conjectures specially in connection with the conformal and the de Sitter groups as space-time symmetries. Since the notions of mass and spin are deeply rooted in the structure of the Poincaré-group, any modification on that group should be made in association with a limiting process which restaures the Poincaré-group and its invariant operators. This in essence is the content of a modification suggested by Segal in 1967 [7]. More specifically, if E is the new space-time symmetry then the Poincaré-group P is a contraction of E (in the sense of Lie algebra [8]). Furthermore the nilpotent action of E on the combined symmetry occurs only after the contraction. The purpose of this note is to implement Segal's proposal within the context of a space-time of general relativity, regarded as a locally embedded manifold. The embedding is used as a device to generate symmetries. Physics remains a property of the four dimensional space-time.

Since a curved space-time admits in general no isometries or only small groups of isometries, Sec.II looks for a group E which satisfies Segal's conditions and also generates the possible space-time isometries. In Sec.III the mass splitting problem is dealt with. The problem is set up in an exact combination scheme where the internal group is at least in part provided by a group of pseudo rotations orthogonal to the space-time. Estimations of the order of magnitude of the mass splitting are made with Schwarzschild space-time (at the surface of the Earth) and it is compared with the results in a constant curvature universe.

II. COMBINED SYMMETRY IN CURVED SPACE-TIME

Group contraction is a parameter preserving operation [8]. Therefore the new space-time symmetry should contain 10 parameters. This fact suggests that the de Sitter cosmological symmetries SO(4,1) and SO(3,2) are good candidates for possible replacements of P. They have been used as such in some modern approaches to gauge theories and elementary particles, and they satisfy Segal's limiting conditions [9),10)]. Explicit mass splitting calculations show that it turns out to be proportional to $1/R^2$ where R is the curvature radius of the universe [11)]. This suggests that the gravitational field plays a significant role for the notion of mass and mass splitting. Indeed, if a cosmological gravitational field can induce a small (but not zero) mass splitting then a local gravitational field could induce a more pronounced result [12)]. Based on the requirement of a ten-parameter group and on the fact that only constant curvature space-times may provide such group of isometries, many authors have suggested the use of a de Sitter bundle on the space-times of general relativity [13)-15)]. More recently, de Sitter bundles were constructed by using the notion of contact of manifolds as applied to space-times [16)]. In this case the de Sitter symmetry arises naturally from tangent spaces of next order to Minkowski space (that is, 2nd order tangent spaces with constant curvature). More important, however, is that there is a small neighbourhood of space-time which shares the same geometry of the constant curvature tangent space. Thus the symmetries of the latter can be taken as a local symmetry of the former. Moreover, instead of a single tangent constant curvature manifold there is a varied number from one to four distinct tangent spaces, corresponding to the number of independent radii of curvature of the given space-time. Since there is no reason to choose one from the others these four tangent spaces, and their groups of isometries, have to be considered together. In the following it will be shown how to combine these local space-time de Sitter groups and also that this combination meets Segal's conditions.

Let S be any space-time of general relativity assumed to be locally, minimally and isometrically embedded in a p-dimensional pseudo Euclidean space M(m, n) with metric signature m + (-n). Since M(m, n) is flat, Cartesian coordinates Z^μ ($\mu=1...p$) may be used throughout. On the other hand, the use of an S based Gaussian coordinate system x^α ($\alpha=1...p$) may be more convenient for dealing with objects which are to be space-time defined. This Gaussian system can be constructed with the four space-time coordinates x^i, together with p-4 coordinates x^A, measured along p-4 unit direction $\eta_A(x^i)$ orthogonal to S and to themselves, with respect to the metric of M(m, n). Thus $\{x^\alpha\} = \{x^i, x^A\}$. The relationship between these two coordinate systems is given by

$$Z^\mu(x^i, x^A) = X^\mu(x^i) + x^A \eta^\mu_A(x^i), \tag{1}$$

where $X^\mu(x^i)$ are the Cartesian coordinates of a point in S and η^μ_A are the Cartesian components of η_A.

The index convention will be as follows: Unless explicitly stated otherwise all Greek indices run from 1 to p. Small case Latin indices run from 1 to 4 while capital Latin indices run from 5 to p. Summation is applied to all cases of repeated upper and lower indices.

From (1), the definition of S in the Gaussian system is given by $x^A = 0$. Let $\Omega(x^\alpha)$ be a geometrical object in M(m, n) Its space-time projection is defined by restriction to S:

$$\Omega(x^\alpha)\big|_S = \Omega(x^\alpha)\big|_{x^A=0}.$$

Denoting by $\eta_{\mu\nu}$ the Cartesian components of the metric tensor of M(m, n), its Gaussian components are given by

$$g_{\alpha\beta} = Z^\mu_{,\alpha} Z^\nu_{,\beta} \eta_{\mu\nu}. \tag{2}$$

In particular, separating the components and restricting to S,

$$g_{ij}\big|_S = X^\mu_{,i} X^\nu_{,j} \eta_{\mu\nu} = g_{ij}(S),$$

$$g_{iA}\big|_S = 0,$$

$$g_{AB}\big|_S = \epsilon^A \delta_{AB}. \tag{3}$$

Here $g_{ij}(S)$ denotes the metric tensor of S in the x^i coordinates. The last equation follows from the orthogonality if the η_A fields

$$\eta^\mu{}_A \eta^\nu{}_B \eta_{\mu\nu} = \varepsilon^A \delta_{AB} ,$$

where ε^A are the signature numbers which are determined by the Gauss-Codazzi equations [17),18)] and they are unique when the embedding is minimal [19)].

The Christoffel symbols $\Gamma_{\alpha\beta\gamma}$ of the metric affine connection ∇M of $M(m,n)$ do not vanish in the Gaussian frame and they are calculated in the usual way

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}) . \tag{4}$$

Then it follows that $\Gamma_{ijk}\big|_S = \Gamma_{ijk}(S)$ (the Christoffel symbols of $g_{ij}(S)$). Gauss-Codazzi equations are given by the various components of $R^\alpha{}_{\beta\gamma\delta}\big|_S = 0$, which results from the flatness of $M(m,n)$.

Consider now an infinitesimal displacement dx^i in S and calculate the variation on $Z^\mu(x^i,x^A)$

$$\Delta Z^\mu = (X^\mu{}_{,i} + x^A \eta^\mu{}_{A,i})\, dx^i .$$

If the direction of dx^i is such that it corresponds to a fixed Z^μ in the direction η_A, then it is called a principal direction[17)]. Thus $\Delta Z^\mu = 0$ gives after contraction with $X^\nu{}_{,j}\eta_{\mu\nu}$

$$(g_{ij}(S) + x^A b_{ijA})\, dx^i = 0 ,$$

where $b_{ijA} = \eta^\mu{}_{A,i} X^\nu{}_{,j} \eta_{\mu\nu} = \Gamma_{ijA}|_S$.
The above equation has a non trivial solution when

$$\det(g_{ij}(S) + x^A b_{ijA}) = 0 . \tag{5}$$

The solutions of this equation gives the radius of curvature $\rho^A_{(i)}$ corresponding to the normal η_A and principal direction on dx^i. These quantities vary from point to point of S and in the flat limit at least one of them goes into infinity. For each direction η_A define

$$\rho^A = (g^{ij} \rho^A_{(i)} \rho^B_{(j)})^{1/2} \tag{6}$$

and for each pricipal direction dx^i in S define

$$\rho_{(i)} = (g_{AB}\, \rho^A_{(i)} \rho^B_{(i)})^{1/2} \tag{7}$$

From (5), it follows that there are at most four independent radius $\rho_{(i)}$, corresponding to each of the four possible principal directions dx^i [18)]. Now as it was previously mentioned, with each curvature radius $\rho_{(i)}$ there is one associated constant curvature tangent space, satisfying a second order contact condition [17)]. This means that each of these constant curvature spaces share the same geometry of S at a neighbourhood of the embedding point. Thus at that point there are at most four tangent spaces to S with constant curvature. As four dimensional manifolds embedded in $M(m,n)$, each of these constant curvature tangent spaces requires only a 5-dimensional minimal embedding subspace of $M(m,n)$. Then the groups of isometries of these spaces are either $SO(4,1)$, $SO(3,2)$ which are subgroups of $SO(m,n)$, the group of pseudo rotations of $M(m,n)$ which leaves the isometric embedding invariant. Because of the common neighbourhood with S it is possible to consider the above groups or local space-time symmetries. However, they cannot be taken separately because the action of $SO(m,n)$ at the end mixes all their generators. Therefore, the four de Sitter type groups with their corresponding signature and curvature radii have to be considered together.

In order to construct the combination of these groups and also to apply Segal's limiting conditions consider the Lie algebra of $SO(m,n)$. In the Cartesian frame it reads

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\mu\rho} L_{\nu\sigma} + \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\rho} L_{\mu\sigma} ,$$

or transforming to the Gaussian frame,

$$L_{\alpha\beta} = Z^\mu{}_{,\alpha} Z^\nu{}_{,\beta} L_{\mu\nu} . \tag{8}$$

Then

$$[L_{\alpha\beta}, L_{\gamma\delta}] = g_{\alpha\gamma} L_{\beta\delta} + g_{\beta\delta} L_{\alpha\gamma} - g_{\alpha\delta} L_{\beta\gamma} - g_{\beta\gamma} L_{\alpha\delta} . \tag{9}$$

The projected Lie group $SO(m,n)\big|_S$ is generated by

$$\ell_{ij} = L_{ij}\big|_S , \quad \ell_{iA} = L_{iA}\big|_S , \quad \ell_{AB} = L_{AB}\big|_S , \tag{10}$$

with a Lie algebra given by the projection of (9).

Now modify this projected Lie algebra with the introduction of the operators

$$\Pi_i = \sum \frac{1}{\rho^A} \ell_{iA} \quad .$$

Then the modified Lie algebra $G|_S$ is given by (using (3), (9), (10):

$$\begin{aligned}
[\ell_{ij}, \ell_{k\ell}] &= g_{ik}(s)\ell_{j\ell} + g_{j\ell}(s)\ell_{ik} - g_{i\ell}(s)\ell_{jk} - g_{jk}(s)\ell_{i\ell} , \\
[\ell_{ij}, \Pi_k] &= g_{ik}(s)\Pi_j - g_{jk}(s)\Pi_i , \\
[\Pi_i, \Pi_j] &= \sum g^{AB}\frac{1}{\rho^A\rho^B}\ell_{ij} , \\
[\Pi_i, \ell_{BC}] &= \sum \frac{1}{\rho^A} g_{AC}\ell_{iB} - \sum \frac{1}{\rho^A} g_{AB}\ell_{iC} , \\
[\ell_{AB}, \ell_{CD}] &= g_{AC}\ell_{BD} + g_{BD}\ell_{AC} - g_{AD}\ell_{BC} - g_{BC}\ell_{AD} .
\end{aligned} \tag{11}$$

As it can be seen the operators ℓ_{ij} and π_i define a 10-parameter non-invariant subgroup $E|_S$ of $G|_S$ and the operators ℓ_{AB} define another $(p-4)(p-5)/2$ - parameter non-invariant subgroup $N|_S$ of $G|_S$. In short:

$$[E|_S, E|_S] = E|_S \ , \ [E|_S, N|_S] \in G|_S \ , \ [N|_S, N|_S] = N|_S$$

The infinitesimal transformation in $M(m,n)$ generated by $L_{\alpha\beta}$ are

$$x'^{\gamma} = x^{\gamma} + \xi^{\gamma},$$

where

$$\xi^{\gamma} = \Theta^{\alpha\beta} L_{\alpha\beta}(x^{\gamma})$$

and where $\Theta^{\alpha\beta}$ are the infinitesimal parameters of $SO(m,n)$. Because this is an isometry group of $M(m,n)$, then

$$\xi^{(\alpha,\beta)}(\nabla M) = 0 \quad . \qquad \text{(fixed origin)}$$

Therefore the infinitesimal transformation of the modified projected group $G|_S$ is $x'^{\gamma} = x^{\gamma} + \xi^{\gamma}|_S$, with

$$\xi^{\gamma}|_S = \Theta^{\alpha\beta}\ell_{\alpha\beta}(x^{\gamma})$$

and

$$\xi^{(\alpha,\beta)}(\nabla M)\Big|_S = 0 \quad . \qquad \text{(fixed origin)} \tag{12}$$

In particular, for the subgroup $E|_S$ the descriptors are

$$\xi^{\gamma}|_S = \Theta^{ij}\ell_{ij}(x^{\gamma}) + \Theta^i \Pi_i(x^{\gamma}) \ . \tag{13}$$

It is interesting to investigate the action of $E|_S$ on S. For that matter consider the particular transformations which send space-time points to space-time points. These transformations are characterized by the conditions

$$\xi^A\big|_S = 0 \tag{14}$$

(so that $x^A = 0$ implies in $x'^A = 0$). With this condition, (12) reads

$$\xi^{(i,j)}(\nabla M)\Big|_S = \left(g^{k(j}\xi^{i)}_{,k} + g^{k(i}\Gamma^{j)}_{\alpha k}\xi^{\alpha}\right)\Big|_S = \phi^{(i;j)}(s) = 0 ,$$

where $\phi^i = \xi^i|_S$ is therefore a Killing vector field in S. From (13) it follows that the first part of ϕ^i describes infinitesimal pseudo rotations around the embedding point in S. The second part, however, does not describe exact space-time translations as it would seem. This is due to the fact that $[\pi_i, \pi_j] \neq 0$. However, if the neighbourhood of S is taken sufficiently small so that it has a diameter d such that

$$d << \sum g^{AB}\frac{1}{\rho^A}\frac{1}{\rho^B} \ ,$$

then $[\pi_i, \pi_j] \approx 0$. In such neighbourhoods π_i describes approximate translations which become exact only in the flat limit. Therefore $E|_S$ is a ten parameter group which is capable of describing the local isometries of S. Notice that the condition (14) imposes constraints in the parameters θ^{ij}, θ^i so that in general the resulting group of isometries of S will have less than ten parameters. For example, if S is the Schwarzschild space-time only four paramters remain independent.

On the other hand the subgroup $N|_S$ is simply the Lorentz group acting on the subspace of $M(m,n)$ orthogonal to S. This follows from

$$\xi^C = \Theta^{AB}\ell_{AB}(x^C)$$

and from (12):

$$\xi^{(A;B)}\Big|_S = \left(g^{\alpha(A}\xi^{B)}{}_{,\alpha} + g^{\alpha(A}\Gamma^{B)}_{\alpha\beta}\xi^{\beta}\right)\Big|_S = g^{C(A}\xi^{B)}{}_{,C} + g^{C(A}\Gamma^{B)}_{CD}\xi^{D}\Big|_S = 0 .$$

However from (4), $\Gamma_{CD}{}^{B}|_S = 0$ so that

$$\xi^{(A,B)}\Big|_S = 0 . \tag{15}$$

Finally consider the group contraction of $G|_S$ with respect to the $E|_S$ subgroup with contracting factors $\frac{1}{\rho^A}$. The contraction proceeds as $\rho^A \to \infty$ so that it occurs in the flat limit of S. Under this limit the Lie algebra (11) becomes

$$
\begin{aligned}
[\mathring{l}_{ij}, \mathring{l}_{kl}] &= \mathring{g}_{ik}\mathring{l}_{jl} + \mathring{g}_{jl}\mathring{l}_{ik} - \mathring{g}_{il}\mathring{l}_{jk} - \mathring{g}_{jl}\mathring{l}_{ik} ,\\
[\mathring{l}_{ij}, \mathring{\pi}_k] &= \mathring{g}_{ik}\mathring{\pi}_j - \mathring{g}_{jk}\mathring{\pi}_i ,\\
[\mathring{\pi}_i, \mathring{\pi}_j] &= 0 ,\\
[\mathring{\pi}_i, \mathring{l}_{BC}] &= 0 ,\\
[l_{AB}, l_{CD}] &= g_{AC}l_{BD} + g_{BD}l_{AC} - g_{AD}l_{BC} - g_{BC}l_{AD} ,
\end{aligned} \tag{16}
$$

where $\mathring{}$ means the flat limit situation. Therefore $E|_{flat}$ is an invariant subgroup of $G|_{flat}$, isomorphic to the full Poincaré-group. The subgroup $N|_{flat}$ is now completely disconnected from $E|_{flat}$ and apart from this it remains unaltered as a $(p-4)(p-5)/2$-parameter group.

The above contracting property of $E|_S$ allied to its generation of the local isometries of S suggests that $E|_S$ may be considered as a space-time or geometrical symmetry at the same time as $N|_S$ may be considered as an internal symmetry. However, note that such labelling and independence occurs only in the flat limit situation. Before that limit, $E|_S$ and $N|_S$ become completely involved in $G|_S$ which may be labelled as a combined symmetry. In this context, it is worth remembering that the notion of internal symmetry as opposed to that of space-time symmetry is a concept which is applied to the flat space-time

field theory. In a curved space-time situation there is no reason to suppose that such a separation holds. However, in the flat limit the two kinds of symmetries are properly identified and separated.

Assuming the Janet-Cartan limit for the dimension of the embedding space ($\leqslant \frac{n(n+1)}{2}$) than the maximum number of dimensions of $M(m,n)$ is 10 (provided discrete symmetries are not considered. If these symmetries are to be described in terms of pseudo rotations, at least one extra dimension should be added. However, this subject is not dealt with here). Then $5 \leqslant p \leqslant 10$ and the possible internal groups $N|_S$ which emerge, considering all possible signatures of $M(m,n)$, are given in Table 1.

TABLE 1

p	$(p-4)(p-5)/2$	Possible $N\|_{flat}$
5	0	-
6	1	SO(2), SO(1,1)
7	3	SO(3), SO(2,1)
8	6	SO(4), SO(3,1), SO(2,2)
9	10	SO(5), SO(4,1), SO(3,2)
10	15	SO(6), SO(5,1), SO(4,2), SO(3,3)

As it can be seen, really interesting internal symmetries require $p \geqslant 7$. For $p = 5$ no internal group appears SU(4) emerges only for $p = 10$. Other groups such as SU(5) or SO(10) do not appear at all. For this reason, additional internal degrees of freeedom need to be postulated in some cases. Denote by $N'|_S$ the enlarged $N|_S$ with the necessary additional degrees of freedom. For the present purposes, it is unnecessary to specify the enlarged Lie algebra. However, note that the combined symmetry $G|_S$ will also be enlarged to $G'|_S$, through $N'|_S$

III. MASS AND MASS SPLITTING

The next important problem is to construct an appropriate mass operator for the new symmetry such that it reproduces the usual Poincaré-mass operator in the flat limit and also that only in that limit the Lie algebra of $E|_S$ acts nilpotently on that of $G'|_S$. Equivalently, the mass difference between states with the same spin should be distinct from zero but vanish in the flat limit.

Since $G|_S$ is a semi simple group the mass operator should be associated to an invariant operator of that group or, more specifically the second order Casimir operator $\ell_{\alpha\beta}\ell^{\alpha\beta}$. (Actually the mass operator should be derived only from $E|_S$. However becuase this group is not an invariant subgroup of $G|_S$, its universal enveloping algebra contains all the operators ℓ_{ij}, ℓ_{iA} and ℓ_{AB}). Now, in order to restaure the Poincaré-mass operator in the flat limit an appropriate curvature factor must be introduced. Therefore define

$$M^2 = k^2 \sum \left(g_{AB} \frac{1}{\rho^A} \frac{1}{\rho^B} \right) \ell_{\alpha\beta} \ell^{\alpha\beta} , \tag{17}$$

where k is a mass scaling factor. The curvature factor was chosen such that as $E|_S \rightarrow P$, $M^2 \rightarrow k^2 \overset{o}{\pi}_i \overset{o}{\pi}^i$ This can be seen from (denoting $\rho^2 = \Sigma\, g_{AB} \frac{1}{\rho^A} \frac{1}{\rho^B}$),

$$M^2 = \left(\frac{k}{\rho}\right)^2 \left(\ell_{ij}\ell^{ij} + 2\, \ell_{iA}\ell^{iA} + \ell_{AB}\ell^{AB} \right) =$$

$$= \left(\frac{k}{\rho}\right)^2 \ell_{ij}\ell^{ij} + k^2 \pi_i \pi^i + k^2 \sum \frac{1}{\rho^A \rho_A} \left(\sum_{A \neq B} \ell_{iB} \ell^{iB} \right) - k^2 \sum \frac{1}{\rho^A} \ell_{iA} \left(\sum \frac{1}{\rho^B} \ell_{iB} \right) + \left(\frac{k}{\rho}\right)^2 \ell_{AB}\ell^{AB}. \tag{18}$$

Therefore

$$M^2 \Big|_{flat} = k^2 \overset{o}{\pi}_i \overset{o}{\pi}^i .$$

Let $\{\lambda\}$ be a set of eigenvalues of all Casimir operators of $E|_S$. Then, associated to each set $\{\lambda\}$ there is one unitary representation of that group with representation space $H_{\{\lambda\}}$. Taking the direct sum H of these spaces as a representation space of $G'|_S$, completely reducible with respect to $E|_S$ assuming that each Casimir operator of $E|_S$ has in its domain a complete set of of eigenvalues and finally assuming that M^2 is Hermitean in H, then the spectrum of M^2 may contain isolated points [7]. The reason for starting with a representation of $E|_S$ is to compare with the flat limit situation. In this limit $E|_S$ goes into P and the remaining part of $G|_S$ goes into the internal

symmetry which now becomes disjoint of P.

Let J be an operator of $G'|_S$ such that it contains at least one eigenstate $|b\rangle$ of M^2 in its domain. Then a non zero transition probability $\langle a|J|b\rangle$ may be constructed. Assuming that $|a\rangle$, $|b\rangle$ belong to distinct subspaces $H_{\{\lambda\}}$ then it follows that J cannot belong to $E|_S$. Therefore it is an operator of $N'|_S$. Denoting by J_r the base of the Lie algebra of $N'|_S$ (here r, s run through the dimension of $N'|_S$ obviously in the particular case where $N'|_S = N|_S$ the indices r, s are the same or A, B, etc.). Then $J = J = a^{rs}J_{rs}$ where a^{rs} are arbitrary constants. The generators of $G'|_S$ are generically denoted by $G_{\underset{\sim}{a}\underset{\sim}{b}}$ (boldface indices running through the entire Lie algebra of $G'|_S$). Thus in particular $G_{\underset{\sim}{a}\underset{\sim}{b}}$ may be ℓ_{ij}, π_i or J_{rs}. The difference between the eigenvalues of M^2 belonging to two distinct representations a, b of $E|_S$ is given by

$$m_a{}^2 - m_b{}^2 = \langle a|M^2|a\rangle - \langle b|M^2|b\rangle \tag{19}$$

or, in terms of $\langle a|J|b\rangle$,

$$m_a{}^2 - m_b{}^2 = \frac{\langle a|[M^2, J]|b\rangle}{\langle a|J|b\rangle} \tag{20}$$

Now (17) is equivalent to

$$M^2 = \left(\frac{k}{\rho}\right)^2 \left[g^{ik} g^{jl} \ell_{ij} \ell_{kl} + 2 g^{ij} g^{AB} \ell_{iA} \ell_{jB} + g^{AB} g^{CD} \ell_{AC} \ell_{BD} \right] . \tag{21}$$

Therefore

$$\left[M^2, J \right] = \left(\frac{k}{\rho}\right)^2 \Big[g^{ik} g^{jl} \left(\ell_{ij} [\ell_{kl}, J] + [\ell_{ij}, J] \ell_{kl} \right) +$$

$$+ 2 g^{ij} g^{AB} \left(\ell_{iA} [\ell_{jB}, J] + [\ell_{iA}, J] \ell_{jB} \right) + g^{AB} g^{CD} \left(\ell_{AC} [\ell_{BD}, J] + [\ell_{AC}, J] \ell_{BD} \right) \Big] .$$

In terms of structure constants of $G'|_S$,

$$\left[\ell_{\alpha\beta}, J_{rs} \right] = C_{\alpha\beta\, rs}^{\underset{\sim}{a}\underset{\sim}{b}} G_{\underset{\sim}{a}\underset{\sim}{b}}$$

so that

$$[M^2, J] = \left(\frac{k}{\rho}\right)^2 a^{\lambda s}\Big[g^{ik}g^{jl}\, C_{ij\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\, \{G_{\underset{\sim}{a}\underset{\sim}{b}}, \ell_{kl}\} +$$

$$+\, 2 g^{ij} g^{AB}\, C_{iA\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\{G_{\underset{\sim}{a}\underset{\sim}{b}}, \ell_{jB}\} + g^{AB}g^{CD}\, C_{BD\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\{G_{\underset{\sim}{a}\underset{\sim}{b}}, \ell_{AC}\}\Big],$$

where { , } denotes the anticommutator.

Replacing the above expression in (20) and introducing the notation

$$S_{ij}^{a} = \langle a|\ell_{ij}|a\rangle \quad , \quad P_{iA}^{a} = \langle a|\ell_{iA}|a\rangle \quad , \quad K_{AB}^{a} = \langle a|\ell_{AB}|a\rangle \quad ,$$

it follows that

$$m_a^2 - m_b^2 = \left(\frac{k}{\rho}\right)^2 a^{\lambda s}\Big[g^{ik}g^{jl}\, C_{ij\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\,(S_{kl}^{a} + S_{kl}^{b}) +$$

$$+\, 2 g^{ij}g^{AB}\, C_{iA\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\,(P_{jB}^{a} + P_{jB}^{b}) + g^{AB}g^{CD}\, C_{BD\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\,(K_{AC}^{a} + K_{AC}^{b})\Big]\frac{\langle a|G_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|J|b\rangle} \quad , \tag{22}$$

showing the dependence of the mass splitting on the S_{ij}^{a} ("spin") P_{iA}^{a} ("momentum") and K_{AB}^{a} ("internal spin"). Notice that $C^{\underset{\sim}{a}\underset{\sim}{b}}_{\alpha\beta\, rs}$ are known from the above Lie algebra (11), together with the possible additional internal symmetry. Also it is interesting to notice that except for this additional internal symmetry, all the above quantities are derived from the mother group SO(m,n). In fact all space-time dependence in (22) rests only in the expression of ρ. To see this, the structure constants of $G'|_S$ are rewritten in the Cartesian frame as

$$C_{\alpha\beta\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}} = Z^{\mu}_{,\alpha} Z^{\nu}_{,\beta}\, C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\Big|_S \quad ,$$

where

$$[L_{\mu\nu}, J_{\lambda s}] = C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\, G_{\underset{\sim}{a}\underset{\sim}{b}} \quad .$$

Then (22) becomes

$$m_a^2 - m_b^2 = \left(\frac{k}{\rho}\right)^2 a^{\lambda s}\, C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\Big[g^{ik}g^{jl} Z^{\mu}_{,i} Z^{\nu}_{,j} Z^{\rho}_{,k} Z^{\sigma}_{,l} +$$

$$+\, 2 g^{ij}g^{AB} Z^{\mu}_{,i} Z^{\nu}_{,A} Z^{\rho}_{,j} Z^{\sigma}_{,B} +$$

$$+\, g^{AB}g^{CD} Z^{\mu}_{,B} Z^{\nu}_{,D} Z^{\rho}_{,A} Z^{\sigma}_{,C}\Big]\Big|_S \Big(\langle a|L_{\rho\sigma}|a\rangle + \langle b|L_{\rho\sigma}|b\rangle\Big)\Big|_S \frac{\langle a|G_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|J|b\rangle} \; .$$

Intrducing the notation

$$f^{\mu\nu} = g^{ik} Z^{\mu}_{,i} Z^{\nu}_{,k} \quad , \quad h^{\mu\nu} = g^{AB} Z^{\mu}_{,A} Z^{\nu}_{,B}$$

and

$$U^{+-}_{\rho\sigma}(a,b) = \frac{1}{2}\Big(\langle a|L_{\rho\sigma}|a\rangle \pm \langle b|L_{\rho\sigma}|b\rangle\Big) \; ,$$

then

$$m_a^2 - m_b^2 = 2\left(\frac{k}{\rho}\right)^2\Big[(f^{\mu\rho} + h^{\mu\rho})(f^{\nu\sigma} + h^{\nu\sigma})\Big]\Big|_S U^{+}_{\rho\sigma}(a,b)\, \frac{a^{\lambda s} C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\langle a|G_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|J|b\rangle} . \tag{23}$$

Since $G_{\underset{\sim}{a}\underset{\sim}{b}}$ is a generic element of $G'|_S$,

$$C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\, G_{\underset{\sim}{a}\underset{\sim}{b}} = C^{i}_{\mu\nu\lambda s}\,\Pi_i + C^{ij}_{\mu\nu\lambda s}\,\ell_{ij} + C^{pq}_{\mu\nu\lambda s}\, J_{pq}$$

and since $|a\rangle$, $|b\rangle$ belong to distinct representations of $E|_S$, $\langle a\, |E|_S|\, b\rangle = 0$, so that

$$a^{\lambda s} C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\langle a\,|G_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle = a^{\lambda s} C^{pq}_{\mu\nu\lambda s}\langle a|J_{pq}|b\rangle =$$

$$= a^{\lambda s}\langle a\,|[L_{\mu\nu}, J_{\lambda s}]|b\rangle = \langle a\,|[L_{\mu\nu}, J]|b\rangle \quad .$$

Therefore

$$a^{\lambda s} C_{\mu\nu\lambda s}^{\underset{\sim}{a}\underset{\sim}{b}}\frac{\langle a|G_{\underset{\sim}{a}\underset{\sim}{b}}|b\rangle}{\langle a|J|b\rangle} = 2\, U^{-}_{\rho\sigma}(a,b) \quad . \tag{24}$$

On the other hand from (2) and from $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$ it follows that $g^{\alpha\beta} Z^{\mu}_{,\alpha} Z^{\nu}_{,\beta} = \eta^{\mu\nu}$. That is

$$f^{\mu\nu} + 2 g^{iA} Z^{\mu}_{,i} Z^{\nu}_{,A} + h^{\mu\nu} = \eta^{\mu\nu}$$

and since $g^{iA}|_S = 0$,

$$(f^{\mu\nu} + h^{\mu\nu})|_S = \eta^{\mu\nu} \tag{25}$$

Replacing (24) and (25) in (23),

$$m_a{}^2 - m_b{}^2 = 4 \left(\frac{k}{\rho}\right)^2 \eta^{\alpha\rho} \eta^{\nu\sigma} U^+_{\rho\sigma}(a,b)\, U^-_{\rho\sigma}(a,b) \; . \tag{26}$$

(Notice that replacing $U^{+-}_{\rho\Gamma}$ (a,b) by their definitions in (26), expression (20) is recovered).

Since $U^{+-}_{\rho\sigma}$(a,b) are functions which depend only on the representations of SO(m,n) it follows from (26) that the mass splitting dependence of S rests only on the factor $\frac{1}{\rho^2}$. This should not be a surprise because the mass operator also has this kind of dependence only :

$$M^2 = \left(\frac{k}{\rho}\right)^2 \ell_{\alpha\beta}\, \ell^{\alpha\beta} = \left(\frac{k}{\rho}\right)^2 \left(L_{\alpha\beta} L^{\alpha\beta}\right)\Big|_S = \left(\frac{k}{\rho}\right)^2 L_{\mu\nu} L^{\mu\nu} \, . \tag{27}$$

Notice also that the dependence on the additional internal symmetry also disappeared because of the division by $\langle a \mid J \mid b \rangle$ in (24).

In the flat limit $\rho \to \infty$, $E|_S \to \bar{E}|_{flat} \cong P$, $N'|_S \to N'|_{flat}$ and $[E|_S, N'|_S] = 0$ and from (27), $m_a{}^2 - m_b{}^2 \to 0$ in accordance with O'Raifeartaigh's theorem and as it is understood today, this means that in this limit, and only then, the Lie algebra $E|_{flat}$ acts nilpotently on that of $G|_{flat}$.

IV. CONCLUSIONS

The most interesting aspect of expression (27) is that the symmetries $E|_S$ and $N'|_S$ have been completely absorbed into the overall isometry SO(m,n) of M(m,n). Consequently it is sufficient to find the classification of that

group (or of its combination with the additional internal symmetry if needed) to account for a particle spectrum. This conclusion places the theory on the same basis as the other similar high dimensional theories such as Kaluza-Klein's with the difference that a four-dimensional gravitational field has been given a priori and that the higher dimensional space is flat. Also from this conclusion a general procedure for field theory may be set up. Given the space-time S, find M(m,n) and SO(m,n). Then write the field equation f = 0 covariant under that group and project it into S : $f|_S = 0$, together with the representation of the group : $D(SO(m,n))|_S = 0$.

Again in connection with the above conclusion it may be interesting to mention the compactiness of the internal space in the Kaluza-Klein theories. Here the compactness appear in the four de Sitter constant curvature tangent spaces which generate the geometric symmetry $E|_S$. Since these spaces are considered altogether the internal symmetry has to do with the pseudo rotation between these tangent spaces.

The use of more than one de Sitter symmetries to generate a set of internal and external symmetries is not new. In fact it has been suggested that one of them could generate space-time symmetries while three others would generate internal symmetries [20].

The expression (27) is a generalization of an expression derived for a constant curvature cosmological model [12] where $\rho = R = 10^{28}$ cm is the curvature radius of the universe. In this case the condition (14) is identically satisfied so that $E|_S$ is the de Sitter group. Since $U^{+\mu\nu}(a,b)\, U^-_{\mu\nu}(ab) \cong 1$ then in this case

$$m_a{}^2 - m_b{}^2 \Big|_{de\ Sitter} \approx k^2\, 10^{-56} (Mev)^2 \, . \tag{28}$$

This result has been suggested as a means to establish a cosmological mass scale. Taking the local gravitational field of Schwarzschild space-time, the curvature radius ρ is [21]

$$\rho = \frac{r^{3/2}}{(2m)^{1/2}} \; ,$$

where m is the Schwarzschild mass. Then at the surface of the earth $\rho \cong 10^{13}$ cm which gives in (27)

$$m_a{}^2 - m_b{}^2 \Big|_{Sch.} \approx k^2\, 10^{-26} (Mev)^2 \; .$$

Therefore if spin zero mesons are considered then the expected mass difference would be of the order of 10^4(MeV). This gives an estimating value $k^2 \; 10^{30}(\mathrm{MeV})^2$. Replacing in (28), the spin zero mass splitting in the de Sitter cosmological model would be of the order of $10^{-26}(\mathrm{MeV})^2$ which would then be the order of magnitude of the proposed cosmological mass scale.

The fact that a mass splitting does not appeat in the flat limit seems to be a consequence of the second order contact interpretation of the de Sitter spaces. In other words, the flat limit is more restrictive than the constant curvature limit. Both M^2 and the mass splitting expression (27) depend on $1/\rho^2$. However, in the first case a square root can be taken to give a dependence on $1/\rho$. This is not the case of (27) to give $m_a - m_b$. This is another way of saying that mass can be properly defined in the Minkowski space but the mass splitting (for squared masses) can only be attained in a higher order of geometric approximation to S.

The author is aware of the many difficult problems which may arise as a result of any modification of an existing symmetry principle. The present paper intends only to show that there is a Lie-algebraic go ahead possibility for higher dimensional quantum field theory in curved space-times.

ACKNOWLEDGMENTS

The author wants to express his gratitude to Dr. W. Mecklenburg for helful discussions on the subject. He is indebted to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical physics, Trieste.

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