REFERENCE

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS * LIBRARY *
3000/82

IC/82/77
INTERNAL REPORT
(Limited distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A PRESCRIPTION FOR n-DIMENSIONAL VIERBEINS *

Ashfaque H. Bokhari
Mathematics Department, Quaid-i-Azam University, Islamabad, Pakistan,

and

Asghar Qadir **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Recent developments in supergravity have brought the n-dimensional Vierbein formalism into prominence. Here we provide a prescription for writing down a Vierbein given an arbitrary (in general non-diagonal) metric tensor in a Riemannian or pseudo-Riemannian space.

MIRAMARE - TRIESTE
June 1982

* To be submitted for publication.

** Permanent address: Mathematics Department, Quaid-i-Azam University, Islamabad, Pakistan. Also, the Centre of Basic Sciences (UGC), Islamabad, Pakistan.

With the advent of supergravity theories [1] the Vierbein formalism [2] has come into extensive use. Basically a Vierbein, $e_a{}^i$, converts a flat metric in Cartesian co-ordinates, $\tilde{g}_{ij}$, into a general metric, g_{ab}, according to the equation

$$e_a{}^i \; e_b{}^j \; \tilde{g}_{ij} = g_{ab} \quad . \tag{1}$$

In some sense, then, the Vierbein may be regarded as a "four-dimensional square root" of the general metric tensor. In fact this applies in more or less the same sense that the Dirac spinor is regarded as "the square root of the Minkowski 4-vector". The curved space-time generalization of the flat space-time Dirac matrices, $\tilde{\gamma}_i$, given by the commutation relations,

$$[\gamma_a, \gamma_b] = 2 g_{ab} \tag{2}$$

transform according to the rule

$$\gamma_a = e_a{}^i \tilde{\gamma}_i \quad . \tag{3}$$

These generalized γ-matrices are frequency used when dealing with spinor fields in general relativity [3]. It would be useful to be able to write down Vierbeins in an arbitrary space-time. Further, higher dimensional Vierbeins would be useful when dealing, for example, with the eleven-dimensional supergravity which reduces to SU(8) supergravity [4], or in higher dimensional theories of the Kaluza-Klein variety such as those of Chodos and Detweiler or Halpern [5]. In this paper we provide a prescription to be able to write down a Vierbein given a metric in any number of dimensions.

Before going on we need to introduce certain notation. To make that notation clear we shall first discuss the Vierbein prescription in 2-dimensions in full detail, in 3-dimensions in somewhat less detail, and in 4-dimensions in much less detail. First we notice that for a diagonal metric tensor in an arbitrary number of dimensions the Vierbein can always be written as

$$e_a{}^i = (g_{aa})^{1/2} \delta_a{}^i \quad , \tag{4}$$

where we have dropped the summation convention for the present. Notice that we could have other solutions in which the Vierbein need not become the identity when we revert to flat space in Cartesian co-ordinates. We shall not be much concerned with such prescriptions, but shall require that our general prescription reduces to Eq.(4) in the case that the metric tensor tends to the diagonal form.

For a two-dimensional Riemannian metric there are two prescriptions for the Vierbein which satisfy our criteria

$$e_a^{\ i} = \begin{bmatrix} \pm\sqrt{g_{11}} & \pm g_{12}/\sqrt{g_{11}} \\ 0 & \pm\sqrt{(g_{11}g_{22} - g_{12}^2)/g_{11}} \end{bmatrix} \quad (i,a = 1,2) \qquad (5)$$

and the equivalent expression obtained by interchanging g_{11} and g_{22}. An example of the type of Vierbein which we are not bothering about here, is

$$e_1^{\ 1} = e_2^{\ 2} = 0; \quad e_2^{\ 1} = \pm\sqrt{g_{11}}, \quad e_1^{\ 2} = \pm\sqrt{g_{22}} \quad . \qquad (5')$$

Notice that the diagonal elements of the metric tensor cannot be zero here. Notice, also, that the signs of each row must go together (positive with positive and negative with negative) but for separate rows are independent.

If the metric is non-Riemannian (writing the co-ordinates as 0 and 1 co-ordinates), the Vierbein will now be written as

$$e_a^{\ i} = \begin{bmatrix} \pm\sqrt{g_{00}} & \pm g_{01}/\sqrt{g_{00}} \\ 0 & \pm\sqrt{(g_{01}^2 - g_{00}g_{11})/g_{00}} \end{bmatrix} \quad (i,a = 0,1) \qquad (5'')$$

In this case the diagonal elements of the metric tensor <u>can</u> be zero. If $g_{11} = 0$ no problem arises, but if $g_{00} = 0$ we have to interchange g_{00} and $-g_{11}$ in Eq.(5"). (Here we have taken the convention that the metric is positive for a time-like vector). In the case that both the diagonal components are zero, the Vierbein components must satisfy the equations

$$e_0^{\ 0} = \pm e_0^{\ 1}; \quad e_1^{\ 0} = \mp e_1^{\ 1}; \quad e_0^{\ 0}\, e_1^{\ 0} = g_{01}/2 \qquad (5''')$$

none of whose infinitely many solutions can satisfy our earlier requirements.

We now present some notation which will be used shortly. Consider the entire (symmetric) matrix g_{ab} $(a,b = 1,\ldots,n)$. We shall denote the determinant of the submatrix composed of the first m rows and m columns by $D(m)$. The determinant of the submatrix obtained by replacing the p^{th} row $(p \leq m)$ by the corresponding part of the j^{th} row of the entire matrix will be denoted by $\underset{p\to j}{D(m)}$, and the determinant of the submatrix obtained by

replacing the q^{th} column $(q \leq m)$ by the corresponding part of the k^{th} column will be denoted by $\underset{q\to k}{D(m)}$. We are now in a position to generalize the two-dimensional prescription to higher dimensions.

The required three-dimensional Vierbeins in a Riemannian space are

$$e_a^{\ i} = \begin{pmatrix} \dfrac{\pm \underset{1\to 1}{D(1)}}{\sqrt{D(1)}} & \dfrac{\pm \underset{1\to 2}{D(1)}}{\sqrt{D(1)}} & \dfrac{\pm \underset{1\to 3}{D(1)}}{\sqrt{D(1)}} \\ 0 & \dfrac{\pm \underset{2\to 2}{D(2)}}{\sqrt{D(1)\cdot D(2)}} & \dfrac{\pm \underset{2\to 3}{D(2)}}{\sqrt{D(1)\cdot D(2)}} \\ 0 & 0 & \dfrac{\pm \underset{3\to 3}{D(3)}}{\sqrt{D(2)\cdot D(3)}} \end{pmatrix} \qquad (6)$$

provided that $D(1)$, $D(2)$, $D(3) \neq 0$. Clearly $D(3) \neq 0$ if the metric is non-singular. We also have prescriptions which interchange the 1,2,3 co-ordinates. Even if one of the prescriptions goes singular, one of the others must be non-singular in the Riemannian space. For a pseudo-Riemannian space this need not be true. Nevertheless, there will exist the solutions which we are discarding on account of our criterion. For example

$$\left.\begin{aligned} e_0^{\ 0} &= 2g_{01}\, g_{02}/g_{03} = \pm e_0^{\ 1}; \quad e_0^{\ 2} = e_1^{\ 1} = e_2^{\ 1} = 0; \\ e_1^{\ 0} &= g_{03}/2g_{02} = \pm e_1^{\ 2}; \quad e_2^{\ 0} = g_{03}/2g_{01} = \mp e_2^{\ 2} \end{aligned}\right\} \quad . \qquad (6')$$

where $g_{00} = g_{11} = g_{22} = 0$ and the Cartesian metric signature is $(+,-,-)$. We will not discuss such solutions further.

For the four-dimensional Vierbeins in a Riemannian space the required prescription is

$$e_a^{\ i} = \begin{pmatrix} \dfrac{\underset{1\to 1}{D(1)}}{\sqrt{D(1)}} & \dfrac{\underset{1\to 2}{D(1)}}{\sqrt{D(1)}} & \dfrac{\underset{1\to 3}{D(1)}}{\sqrt{D(1)}} & \dfrac{\underset{1\to 4}{D(1)}}{\sqrt{D(1)}} \\ 0 & \dfrac{\underset{2\to 2}{D(2)}}{\sqrt{D(1)\cdot D(2)}} & \dfrac{\underset{2\to 3}{D(2)}}{\sqrt{D(1)\cdot D(2)}} & \dfrac{\underset{2\to 4}{D(2)}}{\sqrt{D(1)\cdot D(2)}} \\ 0 & 0 & \dfrac{\underset{3\to 3}{D(3)}}{\sqrt{D(2)\cdot D(3)}} & \dfrac{\underset{3\to 4}{D(3)}}{\sqrt{D(2)\cdot D(3)}} \\ 0 & 0 & 0 & \dfrac{\underset{4\to 4}{D(4)}}{\sqrt{D(3)\cdot D(4)}} \end{pmatrix} , \qquad (7)$$

where the sign ambiguity is left implicit for convenience. Again, the required prescriptions will be given by Eq.(7) and all possible relabellings of the co-ordinates. We assume that there will exist one prescription, at least, which is non-singular, i.e. we are not dealing with the cases where this assumption does not hold.

To be able to write the formulae for the Vierbeins more compactly we define $D(0) = +1$ in a Riemannian space and -1 in a space-time. More general cases can be similarly dealt with. The pattern emerging leads us to expect that, given an n-dimensional metric tensor, g_{ab}, such that for some suitable numbering of co-ordinates $D(m) \neq 0$ for all $m \leqslant n$, our Vierbeins will be given by

$$e_a{}^i = \left.\begin{array}{ll} \underset{i\to a}{D(i)} / \sqrt{D(i-1)\, D(i)} & i \leqslant a \\ = 0 & i > a \end{array}\right\} . \tag{8}$$

To verify this conjecture consider the expression

$$G_{ab} = e_a{}^i\, e_b{}^j\, \tilde{g}_{ij} \quad , \tag{9}$$

where $\tilde{g}_{ij} = 1$ if $i = j$ and 0 otherwise (or with -1 instead of 1 for the space co-ordinates in a space-time metric). We want to verify that $G_{ab} = g_{ab}$. This would be done by expanding Eq.(9) using Eq.(8) and verifying that the coefficient of g_{ab} is unity and of all g_{cd}, $c \neq a$, $d \neq b$ is zero. Now, from Eq.(8), the summation is over all $i \leqslant a$ and $j \leqslant b$. Thus

$$G_{ab} = \sum_{i=1}^{\inf(a,b)} \frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)} \quad . \tag{10}$$

Let $a = \inf.(a,b)$. Then, expanding out for $i = a$ we have

$$G_{ab} = \frac{\underset{a\to b}{D(a)}}{D(a-1)} + \sum_{i=1}^{a-1} \frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)} \quad . \tag{11}$$

Now the first term on the right-hand side of Eq.(11) can be expanded to give

$$\underset{a\to b}{D(a)} = g_{ab}\cdot D(a-1) + \sum_{i=1}^{a-1} g_{ib}\, \overset{i\to a}{D(a-1)} \cdot (-1)^{a-i} \quad . \tag{12}$$

Inserting Eq.(12) into Eq.(11) we see that

$$G_{ab} = g_{ab} + \sum_{i=1}^{a-1} \left[(-1)^{a-i} \cdot g_{ib} \cdot \frac{\underset{i\to a}{D(a-1)}}{D(a-1)} + \frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)} \right] \quad . \tag{13}$$

We now have to demonstrate that the term in the brackets, call it H_{ab}, is zero. Again expanding for $i = a - 1$

$$H_{ab} = -g_{a-1,b} \frac{\underset{a-1\to a}{D(a-1)}}{D(a-1)} + \frac{\underset{a-1\to a}{D(a-1)} \cdot \underset{a-1\to b}{D(a-1)}}{D(a-2)\cdot D(a-1)}$$
$$+ \sum_{i=1}^{a-2} \left[g_{ib} \cdot (-1)^{a-i} \frac{\underset{i\to a}{D(a-1)}}{D(a-1)} + \frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\cdot D(i)} \right] \quad . \tag{14}$$

Using Eq.(12) with a replaced by $a - 1$ for the second term in Eq.(14), we see that the coefficients of $g_{a-1,b}$ cancel and we get

$$H_{ab} = \sum_{i=1}^{a-2} \left[\frac{\underset{i\to a}{D(i)} \cdot \underset{i\to b}{D(i)}}{D(i-1)\; D(i)} + (-1)^{a-i} \cdot g_{ib} \cdot \frac{\underset{i\to a}{D(a-1)}}{D(a-1)} \right.$$
$$\left. + (-1)^{a-i} g_{ib} \cdot \frac{\underset{a-1\to a}{D(a-1)} \cdot \underset{i\to a-1}{D(a-2)}}{D(a-2)\cdot D(a-1)} \right] \quad . \tag{15}$$

This procedure can be repeated for the coefficients of $g_{a-2,b}$, $g_{a-3,b}$ and so on by expanding out $i = a - 2$, $i = a - 3$, and so on, successively. In the first case we get a determinant which is zero, in the next two determinants which cancel, in the next a determinant of determinants which is zero, and so on. We can continue this procedure till we reach g_{3b}, for which we already know the formula works, from Eq.(7). Thus we see that $G_{ab} = g_{ab}$ and so the Vierbeins are given by Eq.(8).

Apart from the main theme of this paper there are two points of mathematical interest in a prescription for writing down Vierbeins. The first point is that we have provided a procedure for evaluating the "square roots" of a symmetric matrix. It is immediately apparent that there are many linearly independent square roots of symmetric matrices. It would be interesting to find out the number of linearly independent Hermitian square roots of unity for an $n \times n$ identity matrix. For the 2×2 case there are the four Pauli spin matrices. What do we have for higher dimensions?

The second point of interest is the fact that the Vierbein may be regarded as a co-ordinate transformation. Thus we can write

$$e_a{}^i = \partial u^i / \partial x^a \tag{16}$$

Reading Eq.(16) with Eq.(1), we would have a set of $n(n+1)/2$ non-linear, partial differential equations of the n functions u^i, of the n variables x^a, in terms of the $n(n+1)/2$ functions $g_{ab}(x^c)$ $(b \geqslant a)$. In the case of a Riemannian metric we get elliptical equations

$$\sum_{i=1}^{n} (\partial u^i/\partial x^a)\,(\partial u^i/\partial x^b) = g_{ab}(x^c) \tag{17}$$

which have the solutions

$$u^i(x^c) = \sum_{i \leqslant a} \int dx^a \; D(i)/\sqrt{D(i-1)\,D(i)}_{\;i \to a} \;. \tag{18}$$

Eqs.(17) can be further generalized in two ways. By making the $g_{ab}(x^c)$ pseudo-Riemannian we get more general equations having a sum up to some value, $i = m$, and a difference from $i = m + 1$ to $i = n$. A further generalization is by taking the $\tilde{g}_{ij}$ in Eq.(1) to be a diagonal metric tensor. This introduces a coefficient $(\tilde{g}_{ii})^{1/2}$ into the summation in Eq.(17). These generalized equations are solved by dividing the integrand in Eq.(18) by $(\tilde{g}_{ii})^{1/2}$.

ACKNOWLEDGMENTS

We would like to thank Faheem Hussain for some useful discussions. One of us (A.Q.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

1) P. van Nieuwenhuizen and D.Z. Freeman, Supergravity (North-Holland, Amsterdam 1979);
S.W. Hawking and M. Roček, Supergravity and Superspace (Cambridge University Press, 1981);
P. van Nieuwenhuizen, Phys. Rep. 68, 192 (1981).

2) W.L. Bade and H. Jehle, Rev. Mod. Phys. 25, 714 (1953).

3) D. Brill and J.A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).

4) T.L. Curtwright and P.G.O. Freund in Supergravity (Ref.1);
W. Nahm, Nucl. Phys. B135, 149 (1978);
See also P. van Nieuwenhuizen in Ref.1.

5) Th. Kaluza, Sitzsunger Preuss. Akad. Wiss. Phys. Math. Kl.LIV, 966 (1921);
O. Klein, Z. Phys. 37, 875 (1926);
E. Witten, "Search for a realistic Kaluza-Klein theory", Princeton preprint;
A. Chodos and S. Detweiler, Phys. Rev. D21, 2167 (1980);
L. Halpern, Physics and Contemporary Needs, Vol.5, Eds. A. Qadir and Riazuddin (Plenum publishers, in press);
See also Gen. Rel. Grav. 8, 623 (1977).

The operators a and $a^\dagger$ are obtained by inverting Eq.(21) and using the orthonormality relations, Eq.(15), so that

$$a_{kqm} = -i \int_\Sigma \psi^*_{kqm} \overleftrightarrow{f}^\mu \Phi \, d\Sigma_\mu \qquad (27a)$$

$$a^+_{kqm} = i \int_\Sigma \psi_{kqm} \overleftrightarrow{f}^\mu \Phi \, d\Sigma_\mu \qquad (27b)$$

where the integral is performed over the t = 0 space-like hypersurface.

Quantization of the scalar field in Minkowski co-ordinates is well known. The field operator Φ is expanded in the Minkowski modes $\bar{\psi}_{kqm}$ (denoted by a bar) for $\omega > 0$

$$\Phi = \sum_{\substack{m=-\infty \\ \omega>0}}^{\infty} \int \left(\bar{a}_{kqm} \bar{\psi}_{kqm} + \bar{a}^+_{kqm} \bar{\psi}^*_{kqm} \right) q \, dq \, dk \qquad (28)$$

with

$$\bar{a} = -i \int_{\Sigma'} \bar{\psi}^* \overleftrightarrow{f}^\mu \Phi \, d\Sigma'_\mu \qquad (29a)$$

$$\bar{a}^+ = i \int_{\Sigma'} \bar{\psi} \overleftrightarrow{f}^\mu \Phi \, d\Sigma'_\mu \qquad (29b)$$

where the $\bar{\psi}$'s are normalized over a t' = 0 hypersurface. Note that because of the co-ordinate transformation, Eq.(7), the t' = 0 hypersurface coincides with the t = 0 hypersurface.

If we now substitute Eq.(22) into Eqs.(29) we obtain the Bogoliubov transformations

$$\bar{a} = a \quad , \quad \bar{a}^+ = a^+ \qquad (30)$$

so that there is no mode mixing and hence no radiation observed by the rotating observer.

IV. CONCLUSIONS

We have studied the quantization of a scalar field in the metric of a relativistic rotating observer. We are able to unambiguously define positive energy modes in this metric, which is neither static ($g_{\mu\nu}$ independent of time) nor stationary ($g_{0\mu} = 0$ for $\mu \neq 0$)! The vacuum in this metric does not differ from the Minkowski vacuum, a result identical to that obtained by quantizing in a Galilean rotating frame. It appears that the mere existence of a surface on which the metric is singular is not sufficient to produce a vacuum which is different from the Minkowski vacuum. To produce a different vacuum it seems that it may be necessary to have an event horizon [13].

ACKNOWLEDGMENTS

The authors would like to thank Professor G. Denardo for useful discussions and Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.