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QUANTIZATION IN ROTATING CO-ORDINATES REVISITED *

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ABSTRACT

Recent work on quantization in rotating co-ordinates showed that no radiation would be seen by an observer rotating with a constant angular speed. This work used a Galilean-type co-ordinate transformation. We show that the same result holds for a Lorentz-type co-ordinate system, in spite of the fact that the metric has a co-ordinate singularity at $r\Omega = 1$. Further, we are able to define positive and negative energy modes for a particular case of a non-static, non-stationary metric.

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I. INTRODUCTION

There has been much work done on the problem of identifying the correct relativistic transformation to rotating co-ordinates [1)-7)] and quantizing in the background metric so obtained [5),8),9)]. The original argument was given by Einstein [1)]. He demonstrated that the spatial metric would be non-Euclidean. The demonstration relied on a simple thought experiment, where Einstein showed that the number of rods set along the circumference of a rotating disc would not be π times the number along the diagonal. He did not, however, appeal to any specific form for the metric. Later authors [2),3)] demonstrated Einstein's result using, without providing any justification, the following transformation to a frame rotating at a constant angular speed, Ω , in cylindrical co-ordinates;

$$t = t' , \quad r = r', \quad \varphi = \varphi' - \Omega t', \quad z = z' , \tag{1}$$

where (t',r',φ',z') are the cylindrical Minkowski co-ordinates and (t,r,φ,z) are the rotating co-ordinates. As is obvious, this transformation is the cylindrical equivalent of the Galilean co-ordinate transformation. They then derive the metric in rotating co-ordinates as

$$ds^2 = (1-r^2\Omega^2)dt^2 - 2\Omega r^2 dt\,d\varphi - dr^2 - r^2 d\varphi^2 - dz^2 \tag{2}$$

(in units with $c = 1$) and argue that it gives the time dilation factor

$$\gamma = (1-r^2\Omega^2)^{-1/2} \tag{3}$$

which is expected to arise due to relative motion. This argument is obviously false. An analogous argument could be used for the usual Galilean transformation and it would then be argued that Galilean transformations give time dilation. In fact, by taking two different times in Minkowski space, t_1' and t_2', we see that $\Delta t = \Delta t'$, i.e. there is no time dilation. The γ^{-2} factor appearing in Eq.(2) is the "classical" one which is cancelled by the relativistic γ^2 factor so as to have the Lorentz metric invariant. Remember that <u>no</u> γ factors appear in the Lorentz metric but only in the co-ordinate transformations. This fact was pointed out by various authors [4),5)] who derived the transformation

$$t = \gamma(t' - r'^2\Omega\varphi'), \quad r = r', \quad \varphi = \gamma(\varphi' - \Omega t'), \quad z = z' \tag{4}$$

by direct analogy with the uniform linear Lorentz transformations. This transformation leads to the metric

$$ds^2 = dt^2 + 2A\,dt\,dr - (1 - A^2 + B^2)dr^2 + 2Br\,dr\,d\varphi - r^2 d\varphi^2 - dz^2 \tag{5}$$

where

$$A = r\Omega\gamma^2(\Omega t + 2\varphi), \quad B = r^2\Omega^2\gamma^2\varphi \tag{6}$$

A debate arose [4)-6)] about the general relativistic aspects of the transforamtions on two basic points. In answer to the objection that Eq.(1) is a Galilean transformation, it was argued that general relativity allows all co-ordinate transformations. Against the transformation given by Eq.(4) it was argued that the spatial geometry on the rotating cylinder, given by Eq.(5) on putting $dr = 0$, corresponds to an Euclidean geometry, and hence it will be in conflict with the observed Sagnac effect [10)]. As regards the first point there is no doubt that all co-ordinate transformations are allowed by general relativity. The caveat is that not every co-ordinate transformation should correspond to uniform motion. Thus the Galilean transformation, while perfectly valid, does not apply to uniform linear motion. Similarly, Eq.(1) does not apply to uniform circular motion. Thus the transformations given by Eq.(1) are unacceptable. As regards the second point, however, the objection is sound [6)]. Hence, the transformations given by Eq.(4) are also unacceptable.

A much earlier paper by Post [7)] already provides the correct transformation

$$t = \gamma^{-1}t', \quad r = r', \quad \varphi = \varphi' - \Omega t', \quad z = z' \tag{7}$$

by simply applying the complete Lorentz transformations

$$\left.\begin{aligned} t' &= \gamma(t + \mathbf{v}\cdot\mathbf{r}) \\ \mathbf{r}' &= \mathbf{r} - \mathbf{v}\left[(1-\gamma)\left(\frac{\mathbf{v}\cdot\mathbf{r}}{v^2}\right) - \gamma t\right] \end{aligned}\right\} \tag{8}$$

to uniform circular motion, remembering that for circular motion $\mathbf{v}\cdot\mathbf{r} = 0$ at each point. The time dilation is seen readily from Eq.(7). This extremely elegant paper seems to have gone unnoticed by all except Grøn [6)] who has misunderstood it. He quotes Post as having agreed with the transformations given by Eq.(1). In fact, Post had merely pointed out [11)] that there was no experiment available at the time to distinguish between the Sagnac effect obtained from Eq.(1) and that obtained from Eq.(7), as these are equivalent to lowest order in Ωr. Post had also noted that the transformation Eq.(1) does not give the required time dilation. Post gives extremely convincing consistency arguments to believe the transformation given by Eq.(7). The above co-ordinates are only locally defined as would be expected for accelerated observers [12)]. Incidentally, many of Grøn's arguments are fallacious. For example, in Einstein's thought experiment, he imagines actual rods as shrinking, while the ideal measuring rods do not.

On quantizing the scalar field in the frame of a rotating observer, it has been shown [8),9)] that no radiation would be seen by an observer whose co-ordinates are given by Eq.(2), while if Ω is time varying in Eq.(1) there would be radiation [8)]. As pointed out earlier, this analysis, while perfectly valid for the co-ordinate frame represented by Eq.(1), simply does <u>not</u> apply to an actual observer in uniform circular motion. In the next section, we briefly consider some of the properties of the metric obtained from the correct co-ordinate transformation, Eq.(7). In Sec.III we quantize the massive scalar field in this background metric, and in the final section we present a brief discussion of our results.

II. THE METRIC IN ROTATING CO-ORDINATES

From Eq.(7) the line element for a rotating observer becomes

$$ds^2 = dt^2 + 2r\Omega^2\gamma^2 t\,dt\,dr - 2r^2\Omega\gamma\,dt\,d\varphi - (1 - r^2\Omega^4\gamma^4 t^2)dr^2 - 2r^3\Omega^3\gamma^3 t\,dr\,d\varphi - r^2 d\varphi^2 - dz^2 . \tag{9}$$

The determinant of the metric tensor is $-r^2\gamma^2$. We can see, immediately, that this metric has a co-ordinate singularity at $r\Omega = 1$, in contrast to the metric given by Eq.(2), which has a determinant $-r^2$.

This metric admits four Killing vector fields, one time-like and three space-like. The covariant and contravariant forms of the Killing vectors are

$$\left.\begin{aligned} K_\mu &= (\gamma, r\Omega^2\gamma^3 t, 0, 0) \quad , & K^\mu &= (\gamma^{-1}, 0, -\Omega, 0) \\ L_\mu &= (0, 1, 0, 0) \quad , & L^\mu &= (0, -1, 0, 0) \\ M_\mu &= (r^2\Omega\gamma, r^2\Omega^3\gamma^3 t, 0, 0) \quad , & M^\mu &= (0, 0, -1, 0) \\ N_\mu &= (0, 0, 0, 1) \quad , & N^\mu &= (0, 0, 0, -1) \end{aligned}\right\} \tag{10}$$

which have magnitudes $1, -1, -r^2$ and -1 respectively. Notice that the hypersurface orthogonal to the time-like Killing vector, K^μ, is given by γt = constant, or t' = constant. It is clear that we have Killing vectors defined for all four dimensions throughout the allowable region $0 < r < \Omega^{-1}$, but the co-ordinates used are only locally defined. Note that the surface $r = \Omega^{-1}$ is a singular space-like surface for the observer and is not a horizon.

The geodesic equations for this metric are quite complicated. However, if we only consider instantaneous radial motion, $\dot{\varphi} = 0$, and restrict ourselves to $t = 0$, we find that we obtain the usual centrifugal and Coriolis forces

$$\ddot{r} = r\Omega^2 \quad , \quad r\ddot{\varphi} + 2\Omega\dot{r} = 0 \tag{11}$$

unlike the case when Eq.(5) is used, when γ factors come in [5),6)].

III. QUANTIZATION OF THE SCALAR FIELD

In rotating co-ordinates the Klein-Gordon equation becomes

$$\Big[(\gamma^{-2} - r^2\Omega^4\gamma^4t^2)\frac{\partial^2}{\partial t^2} + (2\Omega^2\gamma^2 t + r^2\Omega^4\gamma^4 t)\frac{\partial}{\partial t}$$

$$+ 2r\Omega^2\gamma^2 t\frac{\partial^2}{\partial r\partial t} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} - 2\Omega\gamma^{-1}\frac{\partial^2}{\partial t\partial\varphi} - \frac{1}{r^2\gamma^2}\frac{\partial^2}{\partial\varphi^2} - \frac{\partial^2}{\partial z^2}\Big]\psi = -M^2\psi. \tag{12}$$

At first sight this equation is not easily separable. However, one can easily check that this equation has the simple solutions

$$\psi_{kqm;\omega} = N e^{-i\gamma t(\omega - m\Omega)} e^{im\varphi} e^{ikz} J_m(qr) \tag{13}$$

with integer m, $q \geqslant 0$, $\omega^2 = k^2 + q^2 + M^2$ and N a normalization factor to be obtained below. This solution is obtained from the Minkowski solution by substituting for t', φ' from the inverse of the transformations given by Eq.(7), i.e.

$$t' = \gamma t \quad , \quad \varphi' = \varphi + \gamma\Omega t \tag{14}$$

The normalization factor N is obtained by normalizing the mode functions over a $t = 0$ hypersurface Σ ;

$$i\int_\Sigma \psi^*_{k'q'm';\omega'} \overleftrightarrow{f}^\mu \psi_{kqm;\omega}\, d\Sigma_\mu = \delta_{mm'}\,\delta(k-k')\frac{1}{q}\delta(q-q'), \tag{15}$$

where

$$\overleftrightarrow{f}^\mu = |g|^{1/2} g^{\mu\nu}\overrightarrow{\frac{\partial}{\partial x^\nu}} - \overleftarrow{\frac{\partial}{\partial x^\nu}}|g|^{1/2} g^{\mu\nu} . \tag{16}$$

This yields the normalization constant

$$N = \frac{1}{2\pi |2\omega|^{1/2}} \tag{17}$$

so that the normalized mode functions are

$$\psi_{kqm;\omega}(t, \underline{x}) = \frac{1}{2\pi |2\omega|^{1/2}} e^{-i\gamma t(\omega - m\Omega)} e^{im\varphi} e^{ikz} J_m(qr) \tag{18}$$

Let us now consider the distinction between positive and negative energy modes. The Lie derivative with respect to the time-like Killing vector K^μ is

$$\mathcal{L}_{\underline{K}} \equiv K^\mu \nabla_\mu = \gamma^{-1}\frac{\partial}{\partial t} - \Omega \frac{\partial}{\partial \varphi} \tag{19}$$

We see immediately that the mode functions, Eq.(19), are precisely the required energy eigenfunctions

$$\mathcal{L}_{\underline{K}} \, \psi_{kqm;\omega} = -i\omega \, \psi_{kqm;\omega} \tag{20a}$$

$$\mathcal{L}_{\underline{K}} \, \psi^*_{kqm;\omega} = i\omega \, \psi^*_{kqm;\omega} \tag{20b}$$

Hence $\psi_{kqm;\omega}$ and $\psi^*_{kqm;\omega}$ are the positive and negative energy solutions defined by $\omega > 0$ in contrast to the positive energy defined by $\omega - m\Omega > 0$ when one used either the Galilean transformation [8),9)] to rotating frames, Eq.(1), or the incorrect relativistic transformation [5)], Eq.(4). We can now drop the subscript ω on the mode functions. Here the conditions for positive energy and positive norm are identical [13)].

As a consequence of Eqs.(20) a positive energy mode in Minkowski cylindrical co-ordinates remains a positive energy mode in rotating co-ordinates and there is no mode mixing. Hence no radiation from the Minkowski vacuum would be observed by a rotating observer. We now demonstrate this explicitly by quantizing the scalar field in both co-ordinate systems and then by obtaining the Bogoliubov transformations.

The field operator Φ can now be written in terms of rotating positive energy modes

$$\Phi = \sum_{\substack{m=-\infty \\ \omega>0}}^{\infty} \int \left(a_{kqm}\psi_{kqm} + a^+_{kqm}\psi^*_{kqm}\right) q\,dq\,dk \, . \tag{21}$$

Following the canonical procedure we can define a conjugate momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} \tag{22}$$

so that

$$\Pi(0,\underline{x}) = -i \sum_{m=-\infty}^{\infty} \int \omega r \big(a_{kqm}\psi_{kqm}(0,\underline{x}) - a^+_{kqm}\psi^*_{kqm}(0,\underline{x})\big) q\,dq\,dk \tag{23}$$

and postulate the canonical equal-time commutation relations

$$[\Phi(0,\underline{x}), \Pi(0,\underline{y})] = i\,\delta^3(\underline{x}-\underline{y})$$
$$[\Phi(0,\underline{x}), \Phi(0,\underline{y})] = [\Pi(0,\underline{x}), \Pi(0,\underline{y})] = 0 \, . \tag{24}$$

Eqs.(24) lead to the canonical commutation relations for the operators a and a^+

$$[a_{kqm}, a^+_{k'q'm'}] = \delta_{mm'}\,\delta(k-k')\frac{1}{q}\delta(q-q')$$
$$[a,a] = [a^+,a^+] = 0 \, . \tag{25}$$

This result is again in contrast to that obtained by Letaw and Pfautsch [9)]. The vacuum here is defined for all $\omega > 0$, by

$$a|0\rangle_R = 0 \tag{26}$$