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THE MODIFIED GAUSS DIAGONALIZATION OF POLYNOMIAL MATRICES

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ABSTRACT

The Gauss algorithm for diagonalization of constant matrices is modified for application to polynomial matrices. Due to this modification the diagonal elements become pure polynomials rather than rational functions.

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INTRODUCTION

The application of numerical analysis methods for elimination and diagonalization [1-3] is usually used for constant matrices. Such a method has been applied on polynomial matrices [4]. The synthesis of electrical networks, especially in testing for positive real matrices, needs, for convenience and simplicity, diagonalization in such a way that the elements of the matrix obtained are polynomials instead of rational functions. This cannot be achieved using either the original Gauss alogorithm or the modified method given in the Gantmacher book [4]. The present work may be considered as a trial to reach the required procedure. The suggested procedure leads to a diagonalized matrix with polynomial elements. This is a good advantage with respect to Gantmacher's method which leads to a diagonalized matrix of rational functions. It also avoids the long procedures of testing methods for non-negativity of real rational matrices [5-7] as well as singularities of such methods and their criteria.

Consider the following criterion.

CRITERION OF DIAGONALIZATION

Suppose that we have the $n \times n$ hermitian polynomial matrix $A(s)$ of the complex variable $s = \sigma + j\omega$:

$$A(s) = \begin{bmatrix} a_{11}^{(0)}(s) & a_{12}^{(0)}(s) & a_{13}^{(0)}(s) & \cdots & a_{1n}^{(0)}(s) \\ a_{21}^{(0)}(s) & a_{22}^{(0)}(s) & a_{23}^{(0)}(s) & \cdots & a_{2n}^{(0)}(s) \\ a_{31}^{(0)}(s) & a_{32}^{(0)}(s) & a_{33}^{(0)}(s) & \cdots & a_{3n}^{(0)}(s) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{(0)}(s) & a_{n2}^{(0)}(s) & a_{n3}^{(0)}(s) & \cdots & a_{nn}^{(0)}(s) \end{bmatrix} \qquad (1)$$

where the a's are polynomials. It is necessary to diagonalize (1) in order to have the final form:

$$D^{(r)}(s) = \begin{bmatrix} a_{11}^{(0)}(s) & & & & & & 0 \\ & a_{22}^{(1)}(s) & & & & & \\ & & a_{33}^{(2)}(s) & & & & \\ & & & \ddots & & & \\ & & & & a_{rr}^{(r-1)}(s) & & \\ & & & & & 0 & \\ 0 & & & & & & \ddots \; 0 \end{bmatrix} \qquad (2)$$

instead of the form:

$$C^{(r)}(s) = \begin{bmatrix} a_{11}^{(0)}(s) & & & & & 0 \\ & \dfrac{a_{22}^{(1)}(s)}{a_{11}^{(0)}(s)} & & & & \\ & & \dfrac{a_{33}^{(2)}(s)}{a_{11}^{(0)}(s)\, a_{22}^{(1)}(s)} & & & \\ & & & \ddots & & \\ & & & & \dfrac{a_{rr}^{(r-1)}(s)}{a_{11}^{(0)}(s)\, a^{(1)}(s) \ldots a_{r-1,r-1}^{(r-2)}(s)} & \\ & 0 & & & & 0 \\ & & & & & \ddots \; 0 \end{bmatrix} \qquad (3)$$

obtained by using the original Gauss elimination algorithm [1] on polynomial matrices. r is the rank of $A(s)$, and the elements of (3) are rational polynomials, as can be seen

To achieve (2) we make use of the polynomial matrices

$$B_i(s), \qquad i = 1,2,\ldots,r$$

and their conjugate transpose

$$B_i^t(-s), \qquad i = 1,2,\ldots,r$$

with $\det B_i^t(-s) \neq 0$.

PROCEDURE

Take

$$B_1 = \begin{bmatrix} 1 & & & & \\ -a_{21}^{(0)}(s) & a_{11}^{(0)}(s) & & 0 & \\ -a_{31}^{(0)}(s) & & a_{11}^{(0)}(s) & & \\ \vdots & & 0 & \ddots & \\ -a_{n1}^{(0)}(s) & & & & a_{11}^{(0)}(s) \end{bmatrix} \qquad (4)$$

then define

$$D_L^{(1)}(s) = B_1(s)\, A(s) = \left[\begin{array}{c|cccc} a_{11}^{(0)}(s) & a_{12}^{(0)}(s) & \ldots\ldots & a_{1n}^{(0)}(s) \\ \hline 0 & & & \\ \vdots & & A^{(1)}(s) & \\ 0 & & & \end{array}\right] \qquad (5)$$

where

$$A^{(1)}(s) = \begin{bmatrix} a_{22}^{(1)}(s) & a_{23}^{(1)}(s) & \ldots\ldots & a_{2n}^{(1)}(s) \\ a_{32}^{(1)}(s) & a_{33}^{(1)}(s) & \ldots\ldots & a_{3n}^{(1)}(s) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n2}^{(1)}(s) & a_{n3}^{(1)}(s) & \ldots\ldots & a_{nn}^{(1)}(s) \end{bmatrix} \qquad (6)$$

with

$$a_{ij}^{(1)}(s) = a_{ij}^{(0)}(s)\, a_{11}^{(0)}(s) - a_{i1}^{(0)}(s)\, a_{1j}^{(0)}(s) \; . \qquad (7)$$

Now, $B_1^t(-s)$ will have the following form:

$$B_1^{t}(-s) = \begin{bmatrix} 1 & -a_{21}^{(0)}(s) & -a_{31}^{(0)}(s) & \cdots & -a_{11}^{(0)}(s) \\ & a_{11}^{(0)}(s) & & & \\ & & a_{11}^{(0)}(s) & & 0 \\ & 0 & & \ddots & \\ & & & & a_{11}^{(0)}(s) \end{bmatrix} \cdot \tag{8}$$

Hence

$$D^{(1)}(s) = D_L^{(1)}(s)\; B_1^{t}(-s) = \left[\begin{array}{c|c} a_{11}^{(0)}(s) & 0 \\ \hline 0 & A^{(1)}(s) \end{array}\right] \cdot \tag{9}$$

Now, define

$$B_2(s) = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & 0 & \\ 0 & -a_{32}^{(1)}(s) & a_{22}^{(1)}(s) & & \\ \vdots & \vdots & & \ddots & \\ 0 & -a_{n2}^{(1)}(s) & 0 & & a_{22}^{(1)}(s) \end{bmatrix} \tag{10}$$

and similarly as before,

$$D_L^{(2)}(s) = B_2(s)\; D^{(1)}(s) = \left[\begin{array}{cc|ccc} a_{11}^{(0)}(s) & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^{(1)}(s) & a_{23}^{(1)}(s) & \cdots & a_{2n}^{(1)}(s) \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{3}{c}{A^{(2)}(s)} \end{array}\right] \tag{11}$$

where

$$A^{(2)}(s) = \begin{bmatrix} a_{33}^{(2)}(s) & a_{34}^{(2)}(s) & \cdots & a_{3n}^{(2)}(s) \\ a_{43}^{(2)}(s) & a_{44}^{(2)}(s) & \cdots & a_{4n}^{(2)}(s) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n3}^{(2)}(s) & a_{n4}^{(2)}(s) & \cdots & a_{nn}^{(2)}(s) \end{bmatrix} \tag{12}$$

with

$$a_{ij}^{(2)}(s) = a_{ij}^{(1)}(s)\; a_{22}^{(1)}(s) - a_{i2}^{(1)}(s)\; a_{2j}^{(1)}(s) \quad . \tag{13}$$

Similarly, we find $B_2^{t}(-s)$ and multiply $D_L^{(2)}(s)$ by it to get

$$D^{(2)}(s) = D_L^{(1)}(s)\; B_2^{t}(-s) = \left[\begin{array}{cc|c} a_{11}^{(0)}(s) & 0 & \\ 0 & a_{22}^{(1)}(s) & 0 \\ \hline \multicolumn{2}{c|}{0} & A^{(2)}(s) \end{array}\right] \cdot \tag{14}$$

We continue this procedure and find $B_3(s)$ and $B_3^{t}(-s)$, $B_4(s)$ and $B_4^{t}(-s)$,... until reaching $A^{(r-1)}(s)$ after which $A^{(r)}(s) = 0$ and hence (2) will soon be obtained, written as

$$D^{(r)}(s) = B_r(s)\; (B_{r-1}(s)\; (B_{r-2}(s)\; (\ldots(B_2(s)\; (B_1(s)\; A(s)\; B_1^{t}(-s))\; B_2^{t}(s)) \ldots)\; B_{r-2}^{t}(-s))\; B_{r-1}^{t}(-s))\; B_r^{t}(-s) \tag{15}$$

or

$$D^{(r)}(s) = \left[\begin{array}{cccc|c} a_{11}^{(0)}(s) & & 0 & & \\ & a_{22}^{(1)}(s) & & & 0 \\ & & \ddots & & \\ 0 & & & a_{rr}^{r-1}(s) & \\ \hline \multicolumn{4}{c|}{0} & A^{(r)}(s) = 0 \end{array}\right] \tag{16}$$

REMARK

Note that since $B_i^{t}(-s)$ does not affect the diagonal polynomials of the modified matrix, the diagonalization can be made only from the left side. This is significant for programming on digital computers [5].

DIRECT APPLICATION TO THE ALGORITHM

This algorithm has many applications in network theory, for example, and in other fields where the diagonalization of polynomial matrices is needed. The following theorem which was worked out by the author [5] uses this alogorithm directly.

THEOREM [5]

Let (1) be an $n \times n$ hermitian polynomial matrix of the complex variable $s = \sigma + j\omega$. It is non-negative at the $j\omega$-axis for all real values of ω, if and only if all the diagonal polynomials in (16) are non-negative at $s = j\omega$ for all real ω.

CONCLUSIONS

The Gauss diagonalization is applied successfully to polynomial matrices. The elements of the obtained diagonalized matrix are polynomials. This helps in solving the problem of testing rational matrices for real non-negativeness.

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