ITEP - 102

INSTITUTE OF THEORETICAL
AND EXPERIMENTAL PHYSICS

D.R.Lebedev, A.O.Radul*)

GENERALIZED INTERNAL LONG WAVES EQUATIONS: CONSTRUCTION, HAMILTONIAN STRUCTURE AND CONSERVATION LAWS

MOSCOW 1982

*) Moscow State University

УДК 517.2 М-16

A general class of the ILW type equations is constructed. Hamiltonian structure and infinite numbers of conservation laws are introduced.

0. INTRODUCTION. Recent studies [1 - 4] have shown that the following equation

$$u_t + \delta^{-1} u_x + 2 u u_x + T[u_{xx}] = 0 \tag{1}$$

where $T[u](x) = \int_{-\infty}^{\infty} (1/2\delta)\, cth(\pi(y-x)/2\delta)\, u(y)\, dy$ is of mathematical and physical interest. This equation has many important mathematical features similar to those of the Korteweg de Vries equation (KdV). Physically it represents internal long waves (ILW) in a stratified fluid of finite depth characterized by the real parameter δ [5 - 6]. The limiting cases of ILW are: the KdV ($\delta \to 0$) and the Benjamin-Ono (BO) ($\delta \to \infty$) equations [3].

For a long time the outward similarity of the ILW (BO) equation with the KdV equation, the existence of infinite number of the conservation laws, the Bäclund transformation and so on have stimulated creation of a general theory where ILW (BO) would be the simplest example (like KdV for the general Lax equations). As early as in October 1978 at the Leningrad soliton conference L.D.Faddeev has emphasized the importance of studying the BO equation by making use of the group-theoretical methods, as it was the case with KdV.

In this paper we consider some aspects of the theory of the ILW type equations. Let us state the main results of this paper.

The first result is the construction of the general class of the ILW type equations by means of the formal Zakharov-Shabat "dressing" method [7] (Zakharov-Shabat's technique for ILW was disovered in [4]). Let L_0 be the symbol of skew Hermitian differential operator with constant coefficients, K be the symbol of the Volterra operator with the coefficients holomorphic in the strip, $-i\delta < \operatorname{Im} z < i\delta$ satisfying some additional conditions (see item 3 below). It is required that K should satisfy the following condition: the symbol $L = (1+K^-)L_0(1+K^+)^{-1}$ is purely differential. (Here $K^\pm$ denotes the boundary values of the symbol K, i.e. if $K = \sum_{j=0}^{\infty} K_j(z)\xi^{-j-1}$ then $K^\pm = \sum_{j=0}^{\infty} K_j(x \pm i\delta)\xi^{-j-1}$. The difinitions of symbols, symbol multiplication and so on are given below.) Dress another skew Hermitian operator M_0 with the constant coefficients by the symbol K and denote the differential part of $(1+K)M_0(1+K)^{-1}$ by M. Then the equation

$$L_t = LM^+ - M^-L \tag{2}$$

is well defined i.e. the number of the unknown coefficients of the operator L is equal to the number of nonlinear equations. (For the ILW equation $L_0 = -\xi + 1/2i\delta$, $M_0 = i\xi^2$ [4].)

As the second result of our paper we show that the equations (2) are Hamiltonian in the so called second Hamiltonian structure [8 - 10].

The third result is the involution statement for the corresponding Hamiltonians.

We do not discuss the limits $\delta \to 0$ and $\delta \to \infty$ of (2) as it was done in [3] because of brevity of our note.

Finally we would like to point out that our work has been inspired to by our thinking out the relation between Zakharov-Shabat's technique for ILW discovered in [4] and the group-theoretical methods of the works [11 - 12].

1. SYMBOL ALGEBRAS. Let $\mathcal{B}$ be a differential ring of complex valued functions from the Schwarz space (smoth functions rapidly vanishing at $\pm\infty$ with their derivatives). Such a ring is endowed with a derivation $\partial_x : \mathcal{B} \to \mathcal{B}$. $\mathcal{B}((\xi^{-1}))$ means the ring of formal Laurent series $X = \sum_{j=-\infty}^{N} X_j (i\xi)^j$ over $\mathcal{B}$ with the finite number of positive terms. There are two derivations on $\mathcal{B}((\xi^{-1}))$: $\partial_\xi = \partial/\partial_\xi$ and ∂_x : $\sum_j X_j (i\xi)^j \to \sum_j \partial_x X_j (i\xi)^j$. Using them one may define new associative multiplication on $\mathcal{B}((\xi^{-1}))$ which is called symbol multiplication $X \circ Y = \sum_{\alpha \geq 0} (i)^{\alpha} X_\xi^{(\alpha)} Y_x^{(\alpha)} / \alpha!$, where $X_\xi^{(\alpha)} = \partial_\xi^\alpha X$, $Y_x^{(\alpha)} = \partial_x^\alpha Y$. Usually we will omit the sign $\circ$ always implying equalities such as $\xi\alpha = \xi\circ\alpha$. The inverse element for X (if it exists) is defined with respect to the symbol multiplication.

Using the symbol multiplication one can construct the complex Lie algebra $\mathfrak{g} = \{ \sum_{j=-\infty}^{N} X_j (i\xi)^j \}$ with the brackets $a\circ b - b\circ a$ in $\mathcal{B}((\xi^{-1}))$. Our Lie algebra can be splitted into a direct sum of two complex subspaces $\mathfrak{g} = \mathfrak{g}_+ \dot{+} \mathfrak{g}_-$ where $\mathfrak{g}_+ = \{ \sum_{j=0}^{N} X_j (i\xi)^j \}$ and $\mathfrak{g}_- = \{ \sum_{j=-\infty}^{-1} X_j (i\xi)^j \}$. $\mathfrak{g}_+ (\mathfrak{g}_-)$ means the Lie algebra of the differential (Volterra) operators. By the $X_+ (X_-)$ we will denote the projection of $X \in \mathfrak{g}$ on $\mathfrak{g}_+ (\mathfrak{g}_-)$ which will be referred to as the differential (integral) part of X.

Let $\operatorname{res}(\sum X_j (i\xi)^j) = X_{-1}$. Then, as usual, for any $X \in \mathcal{B}((\xi^{-1}))$, $\operatorname{tr} X = \int_{-\infty}^{\infty} \operatorname{res} X \, dx$. The main property of the trace is equality $\operatorname{tr} [X, Y] = 0$, so there is invariant, non-degenerate scalar product $\langle X, Y \rangle = \operatorname{tr}(XY)$ on Lie algebra $\mathfrak{g}$ (see [11 - 13] for further details). Identify $\mathfrak{g}_+$ with the dual space to $\mathfrak{g}_-$ via the scalar product.

There are two operations on $\mathfrak{g}$, the transposition and complex conjugation: for any $X = \sum_{j=-\infty}^{N} X_j (i\xi)^j$: ${}^tX = \sum_{j=-\infty}^{N} (-1)^j (i\xi)^j X_j$ and $X^* = \sum_{j=-\infty}^{N} (-1)^j X_j^* (i\xi)^j$. $X \in \mathfrak{g}$ is called skew Hermitian if ${}^tX^* = -X$. The skew Hermitian symbols form the real Lie subalgebra $u\mathfrak{g}$ in $\mathfrak{g}$. The restriction of $\langle , \rangle$ on $u\mathfrak{g}$ gives the invariant, real, non-degenerate scalar product on $u\mathfrak{g}$. The subalgebra $u\mathfrak{g}_+ = u\mathfrak{g} \cap \mathfrak{g}_+$ may be regarded as dual space to $u\mathfrak{g}_-$ via this scalar product. (The reality od restriction of $\langle , \rangle$ on $u\mathfrak{g}$ means $\langle X, Y\rangle^* = \langle X, Y\rangle$ for any X, Y).

Any $X \in u\mathfrak{g}$ can be written as $X = i\sum_{j=-\infty}^{N} X_j (i\xi)^j$. From the relation ${}^tX^* = -X$ it is easy to show that $Re\, X_j$ can be chocsen arbitrary, and $Im\, X_j$ can be expressed as linear functions of $Re\, X_k$ with $k > j$

$$Im\, X_j = \Lambda_{k>j}(Re\, X_k) = \sum_{k,d} \lambda_{k,d} (Re\, X_k)_x^{(d)}, \quad \lambda_{k,d} \in R$$

2. ANALYTICAL PROPERTIES OF THE OPERATOR T. Denote by S_C (S_R) the Schwarz space of smoth complex (real) valued functions on the real axis which are rapidly vanishing at infinity with their derivatives. (This space will be denoted by S without index C or R if the index is not important). The Fourier transformation converts the operator T into the operator of multiplication times the function $-i\,cth(k\delta)$. This allows to compute the asymptotic behavior of $Tu(x)$ at $|x| \to \infty$: $Tu(x) \sim -(1/2\delta)\, sign(x) \int_{-\infty}^{\infty} u(y)\, dy$. So the operator T maps the space S into the space of C^∞ functions approaching to a constant at $\pm\infty$. One can also show that the operator $T\, d/dx$ maps S into S.

Proposition 1. [4] a) For any function $u(x) \in S$ there

exist unique fuction $U(z)$ possessing the following properties:
(i) $U(z)$-holomorphic, bounded and continuos up to the boundary in the strip $-i\delta < \mathrm{Im}\, z < i\delta$, (ii) $(U^- - U^+)(x) = i\, u(x)$
(iii) $T u(x) = (U^- + U^+)(x)$; $U^\pm$ denotes the boundary values $U(x \pm i\delta)$ of the function $U(z)$.; b) The function

$$U(z) = \int_{-\infty}^{\infty} (1/(2\delta))\, \mathrm{cth}\, (\pi(y-z))/(2\delta)\,)\, u(y)\, dy \; ; \qquad (3)$$

c) If $W(z)$ possesses the properties (i), (ii), then $W(z) - U(z) = \mathrm{const}$;
d) If $W(z)$ possesses the property (i) and $(W^- - W^+)(x) \in S$ then

$$T(W^- - W^+) = i(W^- + W^+) + \mathrm{const}\,.$$

Remark. The constant in (3) can be computed by compairing asymptotic behavior of the right and left hand sides in eq. (4). For example if $U(z)$ is constructed from $u(x) \in S$ (as given by eq. (3)) and $V(z)$ – by $v(x) \in S$ then $U(z)V(z)$ satisfies conditions c) of Prop. 1. So

$$T(U^- V^- - U^+ V^+) = i\,(U^- V^- + U^+ V^+) - (i/(8\delta^2)) \int_{-\infty}^{\infty} u\, dy \cdot \int_{-\infty}^{\infty} v\, dy\,.$$

From this one can obtain the formula:

$$Tu\, Tv - uv = T(u\, Tv + v\, Tu) + (1/(4\delta^2)) \int_{-\infty}^{\infty} u\, dy \int_{-\infty}^{\infty} v\, dy$$

In conclusion let us point out the following evident properties of operator T : a) $T\, d/dx - d/dx\, T = 0$; b) $\int_{-\infty}^{\infty} u_1\, T u_2\, dy = -\int_{-\infty}^{\infty} (T u_1)\, u_2\, dy$ for any $u_1, u_2 \in S$;
c) $(Tu)^* = T(u^*)$.

3. THE FORMAL VERSION OF ZAKHAROV-SHABAT "DRESSING" METHOD

Proposition 2. Let L_0 be the symbol of skew Hermitian differential operator with constant coefficients: $L_0 = i\sum_{k=0}^{n} C_k (i\xi)^k$, $C_k \in R[\delta, \delta^{-1}]$, $C_n = 1$; $\ell_j(x)$ ($j = 0, \dots, n-1$) – is a set of n functions from the Schwarz space S_C such that the symbol $L_1 = i\sum_{j=0}^{n-1} \ell_j (i\xi)^j$ is skew Hermitian. Then a) there exists a unique symbol $K = \sum_{j=-\infty}^{-1} K_j(z)(i\xi)^j$ satisfying the conditions (i) $K_j(z)$ are holomorphic, bounded and continuous up to the boundary in the strip $-i\delta < \operatorname{Im} z < i\delta$; (ii) $(1+K^-)L_0(1+K^+)^{-1} = L$ is a differential skew Hermitian symbol; (iii) $L - L_0 = L_1$; (iv) $T(K^- - K^+) = \sum_{j=-\infty}^{-1} T(K_j^- - K_j^+)(i\xi)^j = i(K^- + K^+)$. b) The symbol K satisfies the conditions ${}^t(1+K^+)^* = (1+K^-)^{-1}$, ${}^t(1+K^-)^* = (1+K^+)^{-1}$, which we will call the reality conditions. c) The coefficients $K_j^\pm$ of the symbols $K^\pm$ lie in the ring engendered by ℓ_j and operators T and d/dx (see item 5 below).

Let $W = (1+K)M_0(1+K)^{-1}$, $M = W_+$, $M_0 = i\sum_{j=0}^{N} m_j (i\xi)^j$, $m_j \in R[\delta, \delta^{-1}]$, where $1+K$ is the above constructed symbol. The coefficients of $W^\pm$ and consequently of $M^\pm$ are expressed in terms of ℓ_j. This allows one to write the system of equations for functions ℓ_j:
$L_t = LM^+ - M^-L$. As $(M^\pm)^* = (M^*)^\mp$ and ${}^t(M^\pm) = ({}^tM)^\mp$ then $LM^+ - M^-L \in \mathfrak{m}_{j+}$ and consequently equations for $\operatorname{Im}\ell_j$ are linear differential combinations of equations on $\operatorname{Re}\ell_j$. Moreover as $LW^+ - W^-L = (1+K^-)(L_0M_0 - M_0L_0)(1+K^+)^{-1} = 0$ then our equation can be written in the form $L_t = -LW_-^+ + W_-^-L$. It shows that the order of the r.h.s. of (2) is $n-1$ with respect to ξ i.e. the number of equations is equal to the number of unknown functions.

Thus the system of equations on $Re\, \ell_j$ is well defined. In the case $L_0 = -\xi + 1/2 i\delta$, $M_0 = i\xi^2$ we come to the ILW equation.

4. GELFAND-DIKII SYMPLECTIC STRUCTURE. Here we will briefly review the results of [9] in the form we need. Let now $\mathcal{B}$ be the ring $S + C = \{\varphi = \psi + c,\ \psi \in S,\ c \in C\}$. Let $L_0 = i\sum_{k=0}^{z} C_k (i\xi)^k \in \mathfrak{u}\mathfrak{g}_+$, $C_k \in R[\delta, \delta^{-1}]$, $C_z = 1$. Consider a subspace $\mathfrak{u}\mathfrak{g}_+ - \mathcal{N} = \{L = L_0 + i\sum_{k=0}^{z-1} \ell_k (i\xi)^k$. For any $X \in \mathfrak{u}\mathfrak{g}_-$ we can construct vector field V_X on $\mathcal{N}$. It is given by the formula $V_X(L) = -i(L(XL)_+ - (LX)_+ L)$. The field V_X is uniquely defined by the initial part of X - $X^z \in \mathfrak{u}\mathfrak{g}_- / \mathfrak{u}\mathfrak{g}_-^z$ where $\mathfrak{u}\mathfrak{g}_-^z$ means an ideal in $\mathfrak{u}\mathfrak{g}_-$ consisting of the elements $Y = \sum_{j=-\infty}^{-(r+1)} Y_j (i\xi)^j$. The fields V_X form Lie algebra with the Lie bracket $[V_X(L), V_Y(L)] = V_{[X,Y]_L}$ where $[X,Y]_L = i(X(LY)_+ - (XL)_- Y) - (X \leftrightarrow Y)$, $[X,Y]_L \in \mathfrak{u}\mathfrak{g}_-$. There is 2-form

$$\omega(V_X, V_Y)(L) = \langle V_X(L), Y \rangle = 1/2 \langle L, [X,Y]_L \rangle$$

on the vector fields V_X . It is skew simmetric and closed.

5. RING OF FUNCTIONALS. Now we construct the ring of functions $\tilde{\mathcal{R}}_k(u_0, \dots, u_{n-1})$, $u_i \in S_k$. k denotes the field of real R or complex C numbers. (We will omit the index k if it is not important.) An element of the ring $\tilde{\mathcal{R}}_k(u_0, \dots, u_{n-1})$ is the linear combination of monomials with coefficients in k. The definition of monomials is given by induction. The degree and the way of construction are defined simultaneously. 1) λu_i, $\lambda \in k$ is the monomial, $\gamma \lambda u_i = (\lambda u_i)$ is the way,

$\deg_{\gamma\lambda u_i} \lambda u_i = 0$; 2) if Q is the monomial and $\deg_{\gamma Q} Q \geq 0$ then TQ is the monomial, $\gamma TQ = (T\gamma Q)$ is the way, $\deg_{\gamma TQ} TQ = \deg_{\gamma Q} Q - 1$, 3) if Q is the monomial then $\partial_x Q$ is the monomial, $\gamma \partial_x Q = (\partial_x \gamma Q)$ is the way, $\deg_{\gamma \partial_x Q} \partial_x Q = \deg_{\gamma Q} Q + 1$; 4) if Q is the monomial and P is the monomial then $Q \cdot P$ is the monomial, $\gamma Q \cdot P = (\gamma Q \cdot \gamma P)$ is the way, $\deg_{\gamma Q \cdot P} Q \cdot P = -1$ if $\deg_{\gamma Q} Q = \deg_{\gamma P} P = -1$ and 0 in the other cases. For any monomial G define $\deg G = \max_{\gamma G} \deg_{\gamma G} G$. The number of ways of construction is finite so that maximum exists. For example let $G = u u_x$, $\gamma G = ((u)(\partial_x(u)))$, $\deg_{\gamma G} G = 0$. There exists another way $\gamma' G = (\partial_x(u/2(u)))$, for which $\deg_{\gamma' G} G = 1$. So $\deg G = 1$. There is a filtration $\mathcal{R}_i = \{ Q \mid \deg Q \geq i \}, i \geq -1$ on the set $\mathcal{R}$ of monomials. The Schwartz space S also has a filtration $S_i = \{ \psi \mid \psi = (d/dx)^i \varphi , \varphi \in S \}$. Define $S_{-1} = \{ \psi \mid \psi = \int_{-\infty}^{x} \varphi(y)\, dy + \mathrm{const} , \varphi \in S \}$. It is easy to check that $\mathcal{R}_i \subset S_i$, but $\mathcal{R}_i \neq \mathcal{R} \cap S_i$. For example $T(uTu) \in S_0$, but $T(uTu) \in \mathcal{R}_{-1} \setminus \mathcal{R}_0$. Define the degree on the ring $\tilde{\mathcal{R}}(u_0, \ldots, u_{n-1})$ by $\deg F = \min_{\alpha} \deg f_\alpha$ where $F = \sum c_\alpha f_\alpha$, $f_\alpha \in \mathcal{R}(u_0, \ldots, u_{n-1})$. There is a filtration on $\tilde{\mathcal{R}}(u_0, \ldots, u_{n-1})$: $\tilde{\mathcal{R}}_j = \{ F \in \tilde{\mathcal{R}} \mid \deg F \geq j \}$, $j \geq -1$.

Let $L = L_0 + L_1$ is skew Hermitian symbol, $L_0 = i \sum_{k=0}^{n} c_k (i\xi)^k$, $c_k \in R$, $c_n = 1$, $L_1 = i \sum_{j=0}^{n-1} \ell_j (i\xi)^j$, $\ell_j \in S_C$. Then $\mathrm{Re}\, \ell_j = v_j \in S_R$ and $\ell_k = v_k + i \Lambda_{s>k}(v_s)$ where $\Lambda_{s>k}(v_s)$ is a linear combination of v_j and their derivatives. The ring of functionals $\mathfrak{F}_R(v_0, \ldots v_{n-1})$ consists of linear combinations with real coefficients of elementary functionals. The elementary functionals are: $F_\alpha = \prod_{i=1}^{n} \int_{-\infty}^{\infty} P_i \, dx$ $P_i \in \widetilde{\mathcal{R}}_{0,R}(v_0, \dots, v_{n-1})$. The space $\widetilde{\mathcal{R}}_{0,R}(v_0, \dots, v_{n-1})$ can be embeded into the $\widetilde{\mathcal{R}}_{0,C}(\ell_0, \dots, \ell_{n-1})$ as subspace $\widetilde{\mathcal{R}}^{\sigma}_{0,C}(\ell_0, \dots \ell_{n-1})$ of functions invariant under operation of complex conjugation σ.

The functional $F \in \widetilde{\mathcal{F}}_R(v_0, \dots, v_{n-1})$ has the variational derivative if $\delta F = \sum A_i \delta v_i$. In this case the coefficients A_i are given by $A_i = \delta F/\delta v_i = A_i^1 \cdot A_i^2$ where $A_i^1 \in \widetilde{\mathcal{F}}_R(v_0, \dots, v_{n-1})$, $A_i^2 \in \widetilde{\mathcal{R}}_R(v_0, \dots, v_{n-1})$. For example

$$\delta I = \delta[(\int v) \cdot (\int v T v_x)] = \int ((\int v T v_x + (\int v) \cdot 2 T v_x) \delta v$$

then $\delta I/\delta u = (\int v T v_x) + (\int v) T v_x$.

Rewrite $F \in \widetilde{\mathcal{F}}_R(v_0, \dots, v_{n-1})$ in variables ℓ_j. Then $\delta F = \int \sum B_j \delta \ell_j$ and $B_j = \delta F/\delta \ell_j = B_j^1 \cdot B_j^2$. $B_j^1 \in \widetilde{\mathcal{F}}_C(\ell_0, \dots, \ell_{n-1})$, $B_j^2 \in \widetilde{\mathcal{R}}_C(\ell_0, \dots, \ell_{n-1})$. Define $\nabla F \in \mathcal{U}_{\mathcal{O}_-}/\mathcal{U}_{\mathcal{O}_-}^{n}$ from the relation $\delta F = \int \operatorname{res} \nabla F \, \delta L$. We can take $-i \sum_{j=0}^{n-1} (i\xi)^{-j-1} \delta F/\delta \ell_j$ as representative of ∇F.

Proposition 3. a) Any $F \in \widetilde{\mathcal{F}}_R$ has a variational derivative; b) $\widetilde{\mathcal{F}}_R$ is closed under operation $\{,\}$:

$$\{F_1, F_2\} = \omega(V_{\nabla F_1}, V_{\nabla F_2}) = \int \operatorname{res} -i(L(\nabla F_1 L)_+ - (L \nabla F_1)_+ L) \nabla F_2.$$

Remark. The ring of symbols $\widetilde{\mathcal{R}}_R(v_0, \dots v_{n-1})((\xi^{-1}))$ has an unusual property: the formula $\operatorname{tr}[\sigma_1, \sigma_2] = 0$ is false. The true formula is $\operatorname{tr}[\sigma_1, \sigma_2] = \int d/dx \operatorname{res}(\sigma_{1,\xi} \sigma_2)$. For example $\operatorname{tr}[T v_0 \xi, (T v_0)^2 \xi^{-1}] = -(1/(4\delta^3))(\int v_0)^3 \neq 0$. So the operation $\{,\}$ even is not skew symmetric on $\widetilde{\mathcal{F}}_R(v_0, \dots v_{n-1})$.

Consider the subspace $\mathcal{F}_R \subset \widetilde{\mathcal{F}}_R(v_0, \dots, v_{n-1})$ consisting of functionals satisfying the additional condition

$$\nabla F = X_1 + X_2 + Y, \quad X_1 = \sum X_{1j} (i\xi)^j, \quad X_{1j} = X^1_{1j} \cdot X^2_{1j},$$

$X^1_{1j} \in \widetilde{\mathcal{F}}_C(\ell_0,\dots,\ell_{n-1})$, $X^2_{1j} \in S_C$ and Y satisfies the condition $L(YL)_+ - (LY)_+L = 0$.

Proposition 4. a) The subspace $\mathcal{F}_R$ is closed under the operation $\{,\}$. b) The operation $\{,\}$ defines Lie algebra structure on $\mathcal{F}_R$.

6. THE ILW-TYPE EQUATIONS ARE HAMILTONIANIAN. Rewrite (2) in the form:

$$L_t = -i\left(L(XL)_+ - (LX)_+L\right) \qquad (5)$$

If $M_0 = i\sum_{r=0}^{m} m_r (i\xi)^r$, $L_0 = i\sum_{k=0}^{n} c_k (i\xi)^k$, $c_n = 1$; $m_r, c_k \in R$ then $X = [(1+K^+)X_0(1+K^-)^{-1}]_-$ where $X_0 = i\sum_{j=-n}^{m-n} \lambda_j (i\xi)^j$ and $\lambda_j \in R$ are defined from the expansion: $M_0/L_0 = \sum_{j=-\infty}^{m-n} \lambda_j (i\xi)^j$.

Now it is naturally to suppose that $[(1+K^+)i(i\xi)^s(1+K^-)^{-1}]_-$ is gradient of some functional $H_s \in \mathcal{F}_R$, i.e.

$$\delta H_s = \int \mathrm{res}\,(1+K^+)\,i(i\xi)^s(1+K^-)^{-1}\,\delta L \qquad (6)$$

Let us formulate some statements necessary to transform the r.h.s. (6).

Proposition 4. a) If $\sigma_1, \sigma_2 \in \widetilde{\mathcal{R}}(\ell_0,\dots,\ell_{n-1})((\xi^{-1}))$ then $\int d/dx\,\mathrm{res}\,(\sigma_1 \circ \sigma_2) = \int d/dx\,\mathrm{res}\,\sigma_1 \cdot \sigma_2$. In the other words the symbols are multiplied as series under the sigh of $\int d/dx\,\mathrm{res}$.
b) Let $\sigma_1, \sigma_2 \in \widetilde{\mathcal{R}}(\ell_0,\dots,\ell_{n-1})((\xi^{-1}))$ then $\int \mathrm{res}\,[\sigma_1,\sigma_2] = \int d/dx\,\mathrm{res}\,(\sigma_{1\xi}\,\sigma_2)$. Denote $V = -i(K^- - K^+)$.
c) Let K satisfy the conditions of Prop.2. Then $T((K^-)^r - (K^+)^r) = i((K^-)^r + (K^+)^r) + C_r$ where $C_r = \{0$ if $r = 2k+1$; $-i(\int V)^r/(2^{2r-1}\delta^r)$, $r = 2k\}$. d) $\int d/dx (K^-)^r \delta K^- = 0$ if $r = 2k+1$, $(-1/(2^{2r+1}\delta^{r+1}))(\int V)^r(\int \delta V)$ if $r = 2k$.

A direct computation allows to establish our main results.

Theorem. 1) $H_s = \operatorname{res}\,(i L_0 + L_{0,\xi}/(2\delta))\, i(i\xi)^s \cdot$
$\cdot \sum_{k=0}^{\infty} (1/(4\delta))^{2k} (\int V)^{2k+1}/(2k+1)$

2) $\{H_s, H_p\} = 0$.

The formula for H_s is worth to be commented. The reality conditions for the symbol K show that $H_s \in \tilde{\mathfrak{F}}_R(v_0, \dots, v_{n-1})$. As $\nabla H_s = [(1+K^+)\, i(i\xi)^s (1+K^-)^{-1}]_-$ then $H_s \in \mathfrak{F}_R$ and in fact $H_s = \int P_s(v_0, \dots, v_{n-1})$, $P_s \in \tilde{\mathfrak{R}}_{0,R}(v_0, \dots, v_{n-1})$. The terms in the formula for H_s play different roles. The term with $\int V$ contains the densities P_s and the terms with $(\int V)^{2k+1}$, $k \geq 1$ compensate the additional part of $\int V$ arising from use the formula (4). So ILW-type equations (5) are Hamiltonian with respect to Gelfand-Dikii symplectic structure with Hamiltonian $H = \sum_{j=-n}^{n-m} \lambda_j H_j$; and have infinite number of conservation laws in involution. In the case of ILW equation $L_0 = -\xi + 1/(2i\delta)$ and

$$H_s = \operatorname{res}\,[-i(i\xi)^{s+1} \sum_{k=0}^{\infty} (1/(4\delta))^{2k} (\int V)^{2k+1}/(2k+1)].$$

Some initial Hamiltonians are equal: $H_{-1} = \int u$, $H_0 = \int u^2/2$,
$H_1 = \int (u^3/3 + uTu_x/2 + u^2/4\delta)$, $H_2 = \int (u^4/4 + (3/4)u^2 T u_x +$
$+ (3/8)(Tu_x)^2 + (u_x)^2/8 + u^3/(3\delta) + uTu_x/(2\delta) + u^2(8\delta^2))$.
The Hamiltonian H for ILW $= -(H_1 + H_0/(2\delta)) = -\int (u^3/3 + uTu_x/2 + u^2/2\delta)$.
For the ILW equation (5) has the form: $u_t = (d/dx)\,\delta H/\delta u$.
The involution statement means $\int (\delta H_p/\delta u)(d/dx)(\delta H_s/\delta u) = 0$.
For BO equation this result is well known [14] .

The authors would like to thank Yu.I.Manin for stimulating our studies in ILW equation and I.V.Cherednik, I.M.Gelfand, M.A.Shubin and M.B.Voloshin for useful discussions.

APPENDIX.

The proof of the Theorem.

1). Let us rewrite consiquently the r.h.s. of eq. (6). We will write $\int f$ insted of $\int_{-\infty}^{\infty} f(x)\,dx$ for brevity

$$\int \operatorname{res}(1+K^+)i(i\xi)^3(1+K^-)^{-1}\,\delta L =$$

$$= \int \operatorname{res}(1+K^+)i(i\xi)^3(1+K^-)^{-1}\delta((1+K^-)L_0(1+K^+)^{-1}) =$$

$$= \int \operatorname{res}(1+K^+)i(i\xi)^3(1+K^-)^{-1}\delta K^- L_0(1+K^+)^{-1} -$$

$$- \int \operatorname{res}(1+K^+)i(i\xi)^3(1+K^-)^{-1}(1+K^-)L_0(1+K^+)^{-1}\delta K^+(1+K^+)^{-1} =$$

$$\overset{\text{Pr. 4 b)}}{=} \int \operatorname{res} L_0\, i(i\xi)^3(1+K^-)^{-1}\delta K^- - \int \operatorname{res} i(i\xi)^3 L_0(1+K^+)^{-1}\delta K^+ -$$

$$- \int (d/dx)\operatorname{res}(1+K^+)i(i\xi)^3(1+K^-)^{-1}\delta K^- L_{0,\xi}(1+K^+)^{-1} +$$

$$+ \int (d/dx)\operatorname{res}(1+K^+)i(i\xi)^3(1+K^-)^{-1}\delta K^- L_0(1+K^+)^{-1}K^+_\xi(1+K^+)^{-1} -$$

$$- \int (d/dx)\operatorname{res} K^+_\xi\, i(i\xi)^3 L_0(1+K^+)^{-1}\delta K^+(1+K^+)^{-1}.$$

Let us expand $(1+K^+)^{-1} = 1 - K^+ + K^+K^+ - K^+K^+K^+ + \ldots$ then

r.h.s. (6) =

$$\overset{\text{Pr. 4 a)}}{=} \int \operatorname{res} L_0 i(i\xi)^3(\delta K^- - \delta K^+) + \int \operatorname{res} L_0 i(i\xi)^3\Big(\sum_{\alpha=1}^{\infty}(-1)^\alpha (K^-)^\alpha\Big)\delta K^- -$$

$$- \int \operatorname{res} L_0\, i(i\xi)^3\Big(\sum_{\alpha=1}^{\infty}(-1)^\alpha (K^+)^\alpha\Big)\delta K^+ -$$

$$- \int d/dx \operatorname{res} L_{0,\xi}\, i(i\xi)^{s}(1+K^-)^{-1}\,\delta K^- +$$

$$+ \int d/dx \operatorname{res} i(i\xi)^{s} L_0 (1+K^+)^{-1} K^+_{\xi}[(1+K^-)^{-1}\delta K^- - (1+K^+)^{-1}\delta K^+].$$

As $K^- = (TV+iV)/2$ and $K^+ = (TV-iV)/2$ so $(1+K^-)^{-1}\delta K^- - (1+K^+)^{-1}\delta K^+ \in \tilde{\mathcal{R}}_0((\xi^{-1}))$ and the last term vanishs. Thus

$$\text{r.h.s. } (6) =$$

$$= \int \operatorname{res} L_0\, i(i\xi)^{s}\, i\,\delta V + \int \operatorname{res} L_0\, i(i\xi)^{s} \sum_{d=1}^{\infty} (-1)^d (K^-)^d (\delta TV + \delta iV)/2 -$$

$$- \int \operatorname{res} L_0\, i(i\xi)^{s} \Big(\sum_{d=1}^{\infty}(-1)^d (K^+)^d\Big)(\delta TV - i\delta V)/2 -$$

$$- \int (d/dx) \operatorname{res} L_{0,\xi}\, i(i\xi)^{s} \sum_{d=0}^{\infty} (-1)^d (K^-)^d\, \delta K^- =$$

$$= \int \operatorname{res} L_0\, i(i\xi)^{s}\, i\,\delta V + \int \operatorname{res} L_0\, i(i\xi)^{s} \sum_{d=1}^{\infty} (-1)^d ((K^-)^d - (K^+)^d)\, T\delta V/2 +$$

$$+ \int \operatorname{res} L_0\, i(i\xi)^{s} \sum_{d=1}^{\infty} (-1)^d ((K^-)^d + (K^+)^d)\, i\,\delta V/2 +$$

$$+ \operatorname{res} L_{0,\xi}\, i(i\xi)^{s} \sum_{k=0}^{\infty} (1/(2^{4k+1}\xi^{2k+1}))(\textstyle\int V)^{2k} \int \delta V$$

Transforming the last term we have used Prop.4 d). Using the property $\int f Tg = -\int (Tf) g$ and Prop.4 c) we obtain

$$\delta H_s =$$

$$= \int \operatorname{res} L_0\, i(i\xi)^{s}\, i\,\delta V +$$

$$+ \int \operatorname{res} L_0\, i(i\xi)^{s} \sum_{d=1}^{\infty} (-1)^d \big(i((K^-)^d + (K^+)^d) - T((K^-)^d + (K^+)^d)\big)\delta V/2 +$$

$$+ \operatorname{res} L_{0,\xi}\, i(i\xi)^{s} \sum_{k=0}^{\infty} (1/(2^{4k+1}\xi^{2k+1}))(\textstyle\int V)^{2k} \int \delta V =$$

$$= \operatorname{res} L_0 i(i\xi)^s \sum_{k=0}^{\infty} (1/(2^{4k} \delta^{2k}))(\int V)^{2k} \int \delta V +$$

$$+ \operatorname{res} L_{0,\xi}\, i\,(i\xi)^s \sum_{k=0}^{\infty} (1/(2^{4k+1} \delta^{2k+1}))(\int V)^{2k} (\int \delta V)$$

Finally

$$H_s = \operatorname{res}\,(i L_0 + L_{0\xi}/(2\delta))\, i\,(i\xi)^s \sum_{k=0}^{\infty} (1/(4\delta))^{2k} (\int V)^{2k+1}/(2k+1)$$

2). Let us compute the Poisson bracket between H_s and H_p :

$$i\{H_s, H_p\} = \int \operatorname{res}\,(L\,(\nabla H_s L)_+ - (L \nabla H_s)_+ L)\, \nabla H_p =$$

$$= \int \operatorname{res}\,((1+K^-) L_0 (1+K^+)^{-1} ((1+K^+) i (i\xi)^s (1+K^-)^{-1} (1+K^-) L_0 (1+K^+)^{-1})_+ -$$

$$- ((1+K^-) L_0 (1+K^+)^{-1} (1+K^+) i (i\xi)^s (1+K^-)^{-1})_+ (1+K^-) L_0 (1+K^+)^{-1})\, \nabla H_p$$

$$= \int \operatorname{res}\,((1+K^+) i (i\xi)^s L_0 (1+K^+)^{-1})_+ (1+K^+) i (i\xi)^p L_0 (1+K^+)^{-1} -$$

$$- \int \operatorname{res}\,((1+K^-) i (i\xi)^s L_0 (1+K^-)^{-1})_+ (1+K^-) i (i\xi)^p L_0 (1+K^-)^{-1} =$$

$$= \int \operatorname{res}((1+K^+) i (i\xi)^s L_0 (1+K^+)^{-1})_+ (i(i\xi)^p L_0 + K^+ i (i\xi)^p L_0 +$$

$$+ i(i\xi)^p L_0 \sum_{\alpha=1}^{\infty} (-1)^\alpha (K^+)^\alpha + K^+ i (i\xi)^p L_0 \sum_{\alpha=1}^{\infty} (-1)^\alpha (K^+)^\alpha -$$

$$- \int \operatorname{res}((1+K^-) i (i\xi)^s L_0 (1+K^-)^{-1})_+ (i(i\xi)^p L_0 + K^- i (i\xi)^p L_0 +$$

$$+ i(i\xi)^p L_0 \sum_{\alpha=1}^{\infty} (-1)^\alpha (K^-)^\alpha + K^- i (i\xi)^p L_0 \sum_{\alpha=1}^{\infty} (-1)^\alpha (K^-)^\alpha .$$

Using the formula

$$\sigma \circ i(i\xi)^p L_0 = i(i\xi)^p L_0 \sigma - \sum_{\beta=1}^{n+p} \partial_x^\beta \sigma\, \partial_\xi^\beta (i(i\xi)^p L_0)/\beta!$$

we obtain

$$\int \operatorname{res}((1+K^+)\, i(i\xi)^s L_0 (1+K^+)^{-1})_+ \Big((i(i\xi)^p L_0 K^+ - \sum_{\beta=1}^{n+p} \partial_x^\beta K^+ \partial_\xi^\beta (i(i\xi)^p L_0)/\beta! \Big) \sum_{\alpha=0}^{\infty} (-1)^\alpha (K^+)^\alpha +$$

$$+ i(i\xi)^p L_0 \sum_{\alpha=1}^{\infty} (-1)^\alpha (K^+)^\alpha \Big) -$$

$$- \int \operatorname{res}((1+K^-)\, i(i\xi)^s L_0 (1+K^-)^{-1})_+ \Big((i(i\xi)^p L_0 K^- -$$

$$- \sum_{\beta=1}^{n+p} \partial_x^\beta K^- \partial_\xi^\beta (i(i\xi)^p L_0)/\beta! \Big) \sum_{\alpha=0}^{\infty} (-1)^\alpha (K^-)^\alpha +$$

$$+ i(i\xi)^p L_0 \sum_{\alpha=1}^{\infty} (-1)^\alpha (K^-)^\alpha \Big) =$$

$$= - \int \operatorname{res}((1+K^+)\, i(i\xi)^s L_0 (1+K^+)^{-1})_+ \sum_{\beta=1}^{n+p} \partial_x^\beta K^+ \partial_\xi^\beta (i(i\xi)^p L_0)/\beta! \cdot$$

$$\cdot \sum_{\alpha=0}^{\infty} (-1)^\alpha (K^+)^\alpha +$$

$$+ \int \operatorname{res}((1+K^-)\, i(i\xi)^s L_0 (1+K^-)^{-1})_+ \cdot \sum_{\beta=1}^{n+p} \partial_x^\beta K^- \partial_\xi^\beta (i(i\xi)^p L_0)/\beta! \cdot$$

$$\cdot \sum_{\alpha=0}^{\infty} (-1)^\alpha (K^-)^\alpha = - \sum_{\beta=1}^{n+p} \int \operatorname{res}(1/\beta!)\, \partial_\xi^\beta (i(i\xi)^p L_0)(1+K^+)^{-1} ((1+K^+)\, i(i\xi)^s L_0 (1+K^+)^{-1})_+ \partial_x^\beta K^+ +$$

$$+ \sum_{\beta=1}^{n+p} \int \operatorname{res}(1/\beta!)\, \partial_\xi^\beta (i(i\xi)^p L_0)(1+K^-)^{-1} ((1+K^-)\, i(i\xi)^s L_0 (1+K^-)^{-1})_+ \partial_x^\beta K^-.$$

Denote by $A_\beta^\pm = (1/\beta!)\, \partial_\xi^\beta (i(i\xi)^p L_0)(1+K^\pm)^{-1}((1+K^\pm)\, i(i\xi)^s L_0 (1+K^\pm)^{-1})_+$

then $i\{H_s, H_p\} =$

$$= -\sum_{\beta=1}^{n+p} \int \operatorname{res}\big((A_\beta^+ - A_\beta^-)\, \partial_x^\beta T V/2 - (A_\beta^+ + A_\beta^-)\, \partial_x^\beta i V/2 \big) =$$

$$= \sum_{\beta=1}^{n+p} - \int \operatorname{res}\big((T(A_\beta^- - A_\beta^+) - i(A_\beta^+ + A_\beta^-))\, \partial_x^\beta V/2 \big) =$$

$$= \sum_{\beta=1}^{n+p} - c_\beta \int \operatorname{res}\, \partial_x^\beta V/2 = 0.$$

this completes the proof.

R E F E R E N C E S

1. R.I.Joseph and R.Egre, J.Phys. A11, L97(1978).
2. H.H.Chen and Y.C.Lee, Phys.Rev.Lett. 43, 264(1979).
3. J.Satsuma, M.J.Ablowitz, and Y.Kodama, Phys.Lett. 73A, 283(1979).
4. Y.Kodama, J.Satsuma, M.J.Ablowitz, Phys.Rev.Lett. 46, 687 (1981).
5. R.I.Joseph, J.Phys. A10, L225(1977).
6. T.Kubota, D.R.S.Ko, and D.Dobbs, J.Hydronaut. 12, 157(1978).
7. V.E.Zakharov and A.B.Shabat, Funct.Anal. 8 : 3, 43(1974) (Russian).
8. F.Magri, J.Math.Phys. 19, 1159(1978).
9. I.M.Gelfand and L.A.Dikii (in Russian) Preprint Inst.Appl.Math. Acad.Sci.USSR, No.136(1978).
10. I.M.Gelfand and I.Ya.Dorfman, Funct.Anal. 14: 3,71(1980) (Russian).

    B.A.Kupershmidt and G.Wilson, Invent.Math. 62, 403(1981).

    V.G.Drinfeld and V.V.Sokolov, DAN 258, No.1, 11(1981) (Russian).
11. M.Adler, Invent.Math. 50, 219(1979).
12. D.R.Lebedev and Yu.I.Manin, Funct.Anal. 13:4, 40(1979) (Russian).
13. Yu.I.Manin, Modern Problems in Math. 11(VINITI 1978) (Russian).
14. A.S.Focas and B.Fuchssteiner, Phys.Lett. 86A, 341(1981).

Д.Р.Лебедев, А.О.Радул

Обобщенные уравнения типа длинных волн в двухслойной жидкости: построение, гамильтоновость, законы сохранения

Работа поступила в ОНТИ 25.05.82

Подписано к печати 3.06.82 Т12064 Формат 60х90 I/I6
Офсетн.печ. Усл.-печ.л.I,0. Уч.-изд.л.0,7. Тираж 290 экз.
Заказ I02 Индекс 3624 Цена I0 коп.

Отпечатано в ИТЭФ, II7259, Москва, Б.Черемушкинская, 25

ИНДЕКС 3624

М.,ПРЕПРИНТ ИТЭФ, 1982, № 102, с.1-17