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SYMMETRIES AND AGGREGATES OF QUARKS AS CONSTITUENTS OF HADRONS

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PROLEGOMEN

The present material constitutes a written pending part* to a conference given at the International Symposium on Symmetries and Properties of Non-rigid Molecules held in Paris, July 1-7, 1982. The high energy physicist will learn nothing from these notes. Indeed, the main purpose of the conference was to provide the molecular physicist or chemist with an introduction concerning the use of symmetries in elementary particle physics. As a matter of fact, although today both particle and molecular physicists employ finite and Lie groups (with however a natural tendency to Lie groups for particle physicists and to finite groups for molecular physicists), the introduction of groups follow different routes in molecular and particle physics. It is out of the scope of this paper to review all the kinds of groups that occur in particle physics. Nothing will be said about external symmetry groups (relative to those symmetries that take place in the Minkowsky space) and almost nothing about gauge symmetry groups. The emphasize is placed on the (flavor) internal symmetry groups SU(n) with n = 3, 6, 4, and 8 and the approach is quantitative rather than only qualitative. There is no doubt that, in the framework of current research, the material presented here belongs to the prehistory of elementary particle physics. Nevertheless, it may be worthy to learn some aspects of prehistory before learning history.

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ABSTRACT

The interest of the Lie algebra of the group SU(n) for the classification of hadrons and the description of some of their static properties is emphasized for n = 3, 4, 6, 8. The cases n = 3 and 4 allow to introduce the quark flavors (u, d, s) and (u, d, c, s), respectively, and the consideration of the spin of hadrons leads to the chain $SU(2m) \supset SU(m) \times SU(2)$. The hadrons are described as bound states or aggregates of quarks of type quark-quark-quark for baryons and quark-antiquark for mesons. The Pauli exclusion principle applied to the three-quark baryons requires the introduction of a new quantum number, the color : each flavor of quark then comes in three colors.

INTRODUCTION

The idea of a molecular structure, although strongly criticized in the recent years, turns out to be well established. It is perhaps less known to the molecular physicist or chemist that substructures also occur in particle physics and that the strongly interacting particles or hadrons (e.g., protons and neutrons) compare in a certain sense to atoms and molecules since they would be composed of (more) elementary constituents, the quarks. Figure 1 shows where the quarks would be localized and illustrates the kinds of particles encountered in the study of fundamental (rather than elementary) particle physics. There exist two familiar classes of particles : the hadron and lepton classes. The leptons (e.g., the electron e^- and the neutrino of electron ν_e) are fermions and manifest a pointlike character. The hadrons contain the baryons (e.g., the proton p and the neutron n) that are fermions and the mesons (e.g., the pions π^-, π^0, and π^+) that are bosons. The hadrons have form factors and exhibit some evidences for a composite structure, the constituents of which (the partons) are identified to the quarks in the simplest presentations. (At this point, note that the quarks were first introduced as mathematical objects and subsequently acquired the status of physical objects, particularly with the discovery of the granular aspect of the proton.) In the simpler approach, each baryon would be an aggregate qqq of three quarks q and each meson an aggregate $q\bar{q}$ of a quark q and an antiquark $\bar{q}$ (the antiparticle of the quark). In a certain sense, we may speak of a quarkonium chemistry (compare $q\bar{q}$ with the positronium $e^+ e^-$) and of a baryonium chemistry. The quarks should be point-like particles and we may think that quarks and leptons, although not being subjected to the same interactions, could be composed of common entities (the quips of Shupe or the rishons of Harari) as suggested independently by Harari and Shupe in 1979. Two rishons (T and V with a reference to the primordial chaos Tohu-va-Vohu) could be sufficient to construct all the matter. We shall leave here the rishons since this paper mainly deal with hadrons.

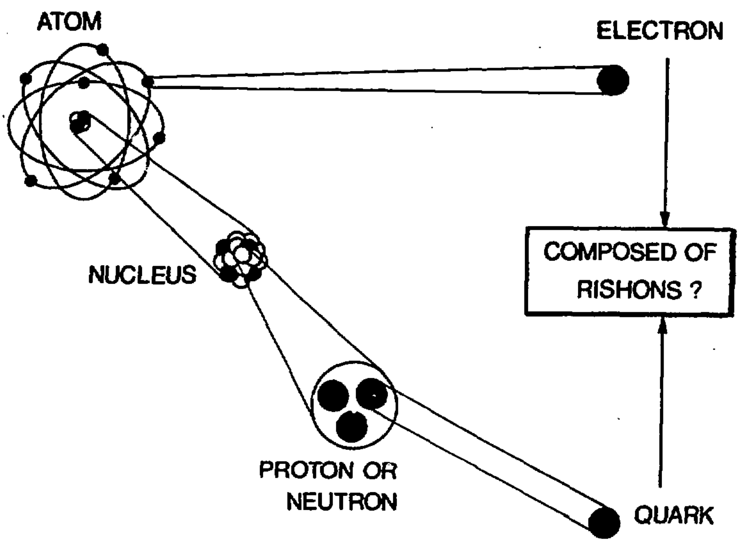

Fig. 1. Where are the quarks ?

There is however an essential difference between quarks and leptons. It is possible with a few eV to ionize an atom to obtain an electron but it seems impossible to ionize an hadron to obtain a quark. The reason for this originates in the profound difference between the forces that bound an electron to a nucleus and two quarks in a baryon. Figure 2 illustrates the various interactions encountered in Nature. The familiar long range gravitation force concerns almost all particles. It corresponds to the weaker interaction since its constant coupling is $G \sim 10^{-39}$ and would be mediated by the hypothetical massless graviton of spin J = 2. The weak force has $G \sim 10^{-10}$, is of short range ($< 10^{-13}$ cm), concerns hadrons and leptons, and is well illustrated by the neutron beta decay. The mediators (or quanta) of the weak force are predicted to be three charged bosons of spin J = 1, viz, W^+, W^-, and W^0 of mass 80, 80, and 90 times the mass of the proton, respectively. The electromagnetic force with $G \sim 1/137$ is of infinite range, concerns charged particles, and is of major relevance in chemistry and biology. The electromagnetic energies are of a few eV and are responsible for binding together the electrons and the nucleus via the exchange of a well known quantum, the massless (virtual) photon γ of spin J = 1. The strong force has $G \sim 1$ (taken as the reference), is of short range (10^{-13}cm), and concerns the domain of nuclear physics where the cohesion energies are of a few MeV. The quanta of the strong nuclear forces are

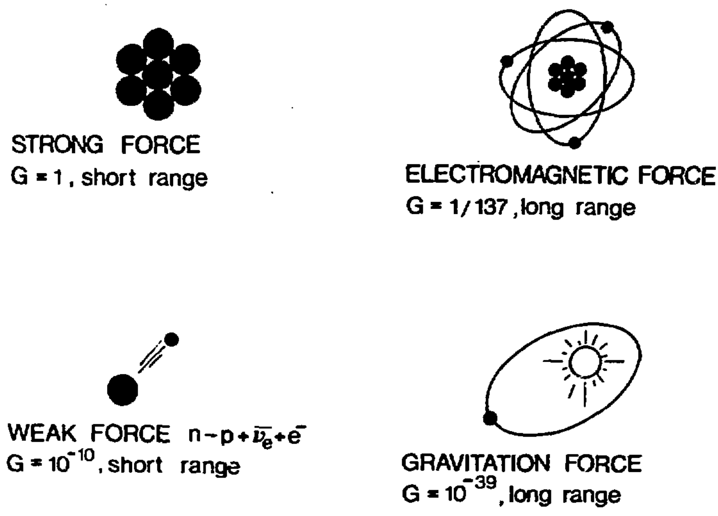

Fig. 2. The four interactions.

the meson π at longer distances and the mesons ρ and ω at shorter distances. At the subnuclear level ($< 10^{-13}$ cm), the (super)strong force ensures the cohesion of aggregates of type qqq (baryons), $\bar{q}\bar{q}\bar{q}$ (antibaryons), and $q\bar{q}$ (mesons) through the exchange of the so-called gluons. There are eight massless gluons of spin J = 1 which constitute the quanta of an unsual field, the color field. The quantum treatment of the color field (viz, the quantum chromodynamics) yields a peculiar behaviour of the coupling constant. The constant vanishes at vanishing distances, a situation known as the asymptotic or ultra-violet freedom that corresponds to a nonrigid structure, while the constant tends to infinity when the distance approaches a value corresponding to the size of the hadron, a situation known as the infra-red esclavery that corresponds to a rigid structure.

Let us now briefly sketch how the situation regarding the classification of particles has evolved during the last fifty years. In 1932, the situation is especially clear. We have, at least from a theoretical viewpoint, two baryons (p and n), two leptons (e^- and ν_e), and two mediators (the photon γ for the electromagnetic force and the pion π for the strong nuclear force). Some years later, the situation became more and more intricated because of the discovery of a great number of hadrons and hadronic resonances with the consequence of establishing an asymmetry between hadrons and leptons. Today in 1982, the symmetry (of type p, n $\leftrightarrow$ e^-, ν_e) seems to be established again since there are both experimental and theoretical evidences for six leptons (e^-, ν_e, μ^-, ν_μ, τ^-, ν_τ) and six quarks (u, d, c, s, t, b,) as well as the corresponding antiparticles. The muons μ^- and the neutrino of muon ν_μ are already familiar particles while the (heavy) lepton τ^- was discovered in 1978. The three light quarks u, d, and s, introduced as early as 1964, would be responsible for the low-lying hadrons and resonances (the baryons of type nucleon (N = n, p) and hyperon (Λ, Σ, Ξ, Ω) and the mesons of type pion (π), kaon (K), . . .). The charmed quark c received some support in 1974 with the discovery of the charmed meson $J/\psi = c\bar{c}$ soon followed by the identification of para- and ortho-charmonium states and other charmed hadrons. The bottom (or beauty) quark b was supported by the discovery of the resonance $\Upsilon = b\bar{b}$ in 1977 and more recently in 1981 of Λ_b = udb while the existence of the top (or truth) quark t was first mainly inferred by the one of the quark b, for both quarks and leptons seem to appear in pairs referred to as generations. The various presently accepted generations are the following. (The masses are indicated in MeV/c^2 that is the mass-energy equivalent unit used in this paper.)

		Leptons	Quarks
First generation	e^-	(0.511)	u (350)
	ν_e	($<60\ 10^{-6}$)	d (350)
Second generation	μ^-	(105.7)	c (1500)
	ν_μ	(<0.57)	s (500)
Third generation	τ^-	(1782)	t (16500)
	ν_τ	(<250)	b (4750)

We may think there is indeed no reason to limit to three the number of generations. The quantum chromodynamics gives 8 as the maximum number of generations while cosmological considerations lead to 4 generations.

The generation structure of quarks and leptons is explained in the framework of grand unification theories through the use of groups like SU(5), SO(10), and E(6). In these theories, only one coupling constant is necessary to describe strong, electromagnetic, and weak interactions. The three interactions become of similar intensity at ultra-high energies (10^{15} GeV). Note that the quantification of the charge Q, the equality $Q(p) = |Q(e^-)|$, and the existence of fractional charges for quarks arise naturally in the grand unification theories. Note also that from grand unification theories it is possible to predict the non-conservation of the baryon number or, in other words, the decay of the proton.

In the present paper, we shall deal with hadrons made up of quarks $q = (u, d, c, s,)$ and of their antiquarks $\bar{q} = (\bar{u}, \bar{d}, \bar{c}, \bar{s},)$, i.e., of building blocks of the first two generations. (Most of the observed particles are topless and bottomless particles.) It is the main purpose of this work to show how to use the chain of groups $SU(6) \supset SU(3) \times SU(2)$ and $SU(8) \supset SU(4) \times SU(2)$ for describing static properties of particles involving only the three flavors (u, d, s) or the four flavors (u, d, c, s), respectively. The paper ends up with a short introduction to the concept of color, the starting point towards quantum chromodynamics.

THE OLD STORY OF SU(2)

The proton p and the neutron n, although having different charges and therefore presenting different behaviours under the electromagnetic interactions, exhibit similarities as far as their interactions in the nuclear matter and their masses $(M(p) = 938.3,\ M(n) = 939.6)$ are concerned. Consequently, n and p may be regarded as two states of a unique particle, the nucleon N. This basic assumption constitutes the starting point of the isospin formalism proposed by Heisenberg in 1932.

The orthonormal state vectors $|p\rangle$ and $|n\rangle$ can then be thought of being connected by the linear operators τ_+ and τ_- in such a way that

$$\tau_+ |n\rangle = |p\rangle, \quad \tau_- |p\rangle = |n\rangle \tag{1}$$

with the conditions

$$\tau_+ |p\rangle = 0, \quad \tau_- = (\tau_+)^+ \tag{2}$$

The simplest matrix realization of τ_+ and τ_- consists of 2×2 matrices. By taking

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

we have

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4}$$

Let us call τ_3 the commutator $[\tau_+, \tau_-]$. We get in the 2×2 realization

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and

$$\tau_3 |p\rangle = |p\rangle, \quad \tau_3 |n\rangle = -|n\rangle \tag{6}$$

so that $|p\rangle$ and $|n\rangle$ are the eigenstates of (the observable) τ_3, a conclusion in accord with the basic assumption above.

In the 2×2 realization, $\tau_1 = \tau_+ + \tau_-$, $\tau_2 = (\tau_+ - \tau_-)/i$, and τ_3 identify with the Pauli matrices. The (isospin) operators $T_i = \tau_i/2$ $(i = 1, 2, 3)$ then satisfy

$$[T_k, T_\ell] = i\, \epsilon_{k\ell m} T_m \tag{7}$$

where $\epsilon_{k\ell m}$ refers to the Levi-Civita tensor.

We now leave the 2×2 realization. Equations (7) shows that the operators T_1, T_2, and T_3 may be considered as the components of a generalized angular momentum operator, the isospin operator T. From a mathematical viewpoint, they span the Lie algebra su(2). The isospin formalism thus appears to be closely connected to the Lie group SU(2). In such an approach the algebra arises first and then the group can be introduced. Indeed, to make the occurrence of SU(2) more explicit, we may postulate that the interactions between nucleons are SU(2) invariant. The two-to-one homomorphism of SU(2) onto SO(3) enables to visualize the nuclear forces Hamiltonian as being invariant under the rotations of a three-dimensional space, the isoespace.

The next step amounts to show the interest of su(2) for classification purposes. The fundamental representation 2 of su(2) comprizes two weights ($\sqrt{2}\, T_3 = -1/\sqrt{2}$ and $1/\sqrt{2}$) and in order to insure coherence with the basic assumption it is necessary to associate $T_3 = -1/2$ and $T_3 = 1/2$ with n and p , respectively. Observe that the charge Q of the two states of the nucleon N satisfies the relation

$$Q = T_3 + 1/2 \qquad (8)$$

Similarly, the three weights $\sqrt{2}\, T_3 = -\sqrt{2}, 0$, and $\sqrt{2}$ of the vector representation 3 of su(2) may be assigned to the particles π^-, π^0, and π^+, respectively. Here, the relation

$$Q = T_3 \qquad (9)$$

holds for the three states of the pion π . Equations (8) and (9) combine to give

$$Q = T_3 + B/2 \qquad (10)$$

where the baryonic number B equals 1 for (the two) baryons (N) and 0 for (the three) mesons (π).

With the discovery in 1947 of strange particles in cosmic rays, some troubles arose around Eq. (10) . For example, the kaons K^0 and K^+ are two mesons it is reasonable to put in the 2 of su(2). However, Eq. (10) does not hold any more for K^+ and K^0. To overcome this difficulty, it is appropriate to introduce a new quantum number, the strangeness S , and to concomitantly assume

$$Q = T_3 + (B+S)/2 \qquad (11)$$

as proposed by Gell-Mann and by Nishijima and Nakano in 1953. The strange mesons K^0 and K^+ then carry $S = 1$.

The sum $B + S$ can be redefined as a whole quantum number, the hypercharge Y. The ordinary (including strange) hadrons may thus be organized into groupings characterized by a value of T and one of Y. Within an su(2) multiplet associated to a given value of T, the various particles are distinguished by the corresponding values of T_3. The quantum number Y serves to (partly) separate the su(2) multiplets relative to the same value of T. The hypercharge Y of the hadrons having the isospin T is given by the average value (over T) $Y = \langle Q \rangle_T$. As an illustration, we have the baryons

$N = n, p$: $T=1/2$; $T_3=-1/2, 1/2$; $Y=1$; $B=1$; $S=0$

$\Sigma = \Sigma^-, \Sigma^0, \Sigma^+$: $T=1$; $T_3=-1, 0, 1$; $Y=0$; $B=1$; $S=-1$

$\Lambda = \Lambda^0$: $T=0$; $T_3=0$; $Y=0$; $B=1$; $S=-1$

$\Xi = \Xi^-, \Xi^0$: $T=1/2$; $T_3=-1/2, 1/2$; $Y=-1$; $B=1$; $S=-2$

and the mesons

$K = K^0, K^+$: $T=1/2$; $T_3=-1/2, 1/2$; $Y=1$; $B=0$; $S=1$

$\pi = \pi^-, \pi^0, \pi^+$: $T=1$; $T_3=-1, 0, 1$; $Y=0$; $B=0$; $S=0$

$\eta = \eta^0$: $T=0$; $T_3=0$; $Y=0$; $B=0$; $S=0$

$\bar{K} = \bar{K}^-, \bar{K}^0$: $T=1/2$; $T_3=-1/2, 1/2$; $Y=-1$; $B=0$; $S=-1$

FROM SU(2) TO SU(3) AND SU(4)

Preliminaries

The preceding résumé clearly shows the relevance of the 4-dimensional Lie algebra $su(2) + u(1)$ (more precisely the Lie algebra of $SU(2) \times U(1)/Z(2) = U(2)$) for classifying hadrons. To take full advantage of the fact there exist two (commuting) observables T_3 and Y, the best is to enlarge $su(2) + u(1)$ to a second-rank semi simple Lie algebra. In this respect, all the second-rank Lie algebras (A_2, B_2, D_2, and G_2) are a priori convenient since they all have $A_1 = su(2)$ as a subalgebra.

In fact we may start directly from a Lie group G having SU(2) as a subgroup. The group G has to be chosen in order to fit (and predict) the maximum of results. It may be expected that G be useful for describing (static) properties as for instance : (i) the grouping of particles presenting similarities in multiplets associated to (irreducible) representations of G, (ii) the masses and the magnetic moments of the various particles in a multiplet, and (iii) the branching ratios for the scattering amplitudes.

The point (i) is an affair of combining mathematical devices with physical intuition and guess based on the exploitation of experimental data. The points (ii) and (iii) more specifically remain on the use of the Wigner-Eckart theorem for G. The state vectors associated to the known (and postulated) particles and the operators corresponding to the mass, the magnetic moment, and the scattering of the hadrons are given a tensorial form with respect to the group G. Each relevant operator is expanded in terms of G irreducible tensor operators with the help of phenomenological coefficients. Repeated application of the Wigner-Eckart theorem for G produces matrix elements from which the unknown coefficients are eliminated. This yields (mass, magnetic moment, and scattering amplitude) relations which are compared to experimental results.

SU(3) and a bit of SU(4) backgrounds

Of course, various groups G have been tried. The group SU(3), introduced in the framework of the three-fold way of Sakata in 1956 and revisited in the framework of the eight-fold way of Gell-Mann and Ne'eman in 1961, rests the best candidate for describing the spectroscopy of the low-lying hadrons. To account for the nonobservation of processes with $\Delta S \neq 0$ (like $K^0 \to \mu^+ + \mu^-$ and $K^+ \to \pi^+ + \nu + \bar{\nu}$), some theoretical support have been given to the group SU(4) by Glashow, Iliopoulos, and Maiani in 1970. The consecration of SU(4) came with the November 1974 revolution after the discovery of the first charmed mesons. Here, we shall start directly with SU(4). Nevertheless, most of the developments shall be achieved for SU(3).

Each element U of SU(n) can be written as U = exp(iH), where H is a n x n traceless Hermitean matrix which thus depends on $n^2 - 1$ parameters α_k ($k=1,2,\ldots,n^2-1$). For n = 4, we may take

$$\begin{aligned} H=&\alpha_1(E_{12}+E_{21}) - i\alpha_2(E_{12}-E_{21}) + \alpha_4(E_{13}+E_{31}) - i\alpha_5(E_{13}-E_{31}) \\ &+ \alpha_6(E_{23}+E_{32}) - i\alpha_7(E_{23}-E_{32}) + \alpha_9(E_{14}+E_{41}) - i\alpha_{10}(E_{14}-E_{41}) \\ &+ \alpha_{11}(E_{24}+E_{42}) - i\alpha_{12}(E_{24}-E_{42}) + \alpha_{13}(E_{34}+E_{43}) - i\alpha_{14}(E_{34}-E_{43}) \\ &+ \alpha_3(E_{11}-E_{22}) + \alpha_8(E_{11}+E_{22}-2E_{33})/\sqrt{3} + \alpha_{15}(E_{11}+E_{22}+E_{33}-3E_{44})/\sqrt{6} \end{aligned} \qquad (12)$$

where the E_{ij} stand for the usual unit matrices satisfying the commutation relations

$$[E_{ij}, E_{k\ell}] = \delta(jk)E_{i\ell} - \delta(\ell i)E_{kj} \tag{13}$$

The infinitesimal generators λ_k of SU(4) are obtained from the self-representation of SU(4) by the well-known Taylor process :

$$\lambda_k = \frac{1}{i}\left(\frac{\partial U}{\partial \alpha_k}\right)_{\alpha=0} \qquad k = 1, 2, \ldots, 15 \tag{14}$$

As an example, we have

$$\lambda_1 = E_{12} + E_{21},\ \lambda_2 = -i(E_{12} - E_{21}),\ \lambda_3 = E_{11} - E_{22} \tag{15}$$

from which it appears that λ_1, λ_2, and λ_3 behave under commutation like the Pauli matrices. All the other commutators $[\lambda_k, \lambda_\ell]$ may be easily obtained. To parallel the passage $\sigma_k \to J_k$ or $S_k = \sigma_k/2$, let us put $F_k = \lambda_k/2$. The commutation relations

$$[F_k, F_\ell] = if_{k\ell m}F_m \tag{16}$$

(where the $f_{k\ell m}$ define the components of a third-rank antisymmetric tensor) are then readily set up and can be taken for defining the Lie algebra su(4). Clearly F_1, F_2, and F_3 span an su(2) subalgebra and $F_1, F_2, \ldots, F_8$ an su(3) subalgebra. We identify T_i with F_i for $i = 1, 2, 3$.

Let us now focus our attention on the just mentioned algebra su(3). The structure constants $f_{k\ell m}$ write

$$f_{123} = 1,\ f_{147} = f_{246} = f_{257} = f_{345} = 1/2$$
$$f_{156} = f_{367} = -1/2,\ f_{458} = f_{678} = \sqrt{3}/2 \tag{17}$$

The Cartan-Weyl basis $\{H_i, E_\alpha\}$ of su(3) is connected to the Gell-Mann basis $\{F_k\}$ through

$$H_1 = F_3/\sqrt{3},\ H_2 = F_8/\sqrt{3},\ E_{\pm 1} = (F_1 \pm iF_2)/\sqrt{6}$$
$$E_{\pm 2} = (F_4 \pm iF_5)/\sqrt{6},\ E_{\pm 3} = (F_6 \pm iF_7)/\sqrt{6} \tag{18}$$

The Cartan subalgebra spanned by H_1 and H_2 should have something to do with the commutative algebra relative to the third component T_3 of the isospin and the hypercharge Y discussed above. The link reads

$$T_3 = \sqrt{3}H_1,\quad Y = 2H_2 \tag{19}$$

A semi simple algebra of rank ℓ and order r possesses $(r+\ell)/2$ commuting operators defined in its enveloping algebra : ℓ Cartan generators, ℓ invariant operators (including the Casimir operator), and $(r-3\ell)/2$ additional operators. In the case of su(3), $\ell = 2$ and $r = 8$, so that we need only one additional operator in order to complete the set formed with T_3, Y, $F^2 = \Sigma F_i^2$ (the Casimir operator), and G^3 (the invariant of order 3). Such an additional operator may be provided with the su(2) Casimir operator T^2. The relevant eigenvectors of F^2, G^3, Y, T^2, and T_3 can be written as $|(p,q)\,Y\,T\,M_T>$, where the integers p and q (connected to the eigenvalues of F^2 and G^3) identify an irreducible representation of dimension $(p+1)(q+1)(p+q+2)/2$ of SU(3). Y and T have evident significance, and M_T denotes an eigenvalue of T_3. The basis $|(p,q)Y\,T\,M_T>$ is adapted to the chain $SU(3) \supset SU(2)_T$, where $SU(2)_T$ corresponds to those matrices of SU(3) for which the third coordinate remains unchanged. We could as well use a basis adapted to $SU(3) \supset SU(2)_U$ or $SU(3) \supset SU(2)_V$ corresponding to the first or second coordinate left unchanged, respectively. This would define the so-called U-spin and V-spin, two alternatives to the isospin T. Note that in the scheme $SU(3) \supset SU(2)_U$, the basis vectors would write $|(p,q)QUM_U>$.

It is customary to let d or $\overline{d}$ to denote the irreducible representation of dimension $d = (p+1)(q+1)(p+q+2)/2$. If d stands for the irreducible representations class (p, q), then $\overline{d}$ will stand for the complex conjugate class (q, p) of (p, q). For instance, we have the correspondence

(0, 0)	(1, 0)	(0, 1)	...	(1, 1)	(3, 0)	(0, 3)	...	(2, 2)	...
$1=\overline{1}$	3	$\overline{3}$	...	$8=\overline{8}$	10	$\overline{10}$	...	$27=\overline{27}$	...

We now list the various eigenvectors of Y, T^2, and T_3 for the representations $1, 3, \overline{3}, 8$, and 10 :

1 : A = $|1\ 0\ 0\ 0>$; 3 : B = $|3\ 1/3\ 1/2\ 1/2>$, C = $|3\ 1/3\ 1/2\ -1/2>$, D = $|3\ -2/3\ 0\ 0>$; $\overline{3}$: E = $|\overline{3}\ -1/3\ 1/2\ -1/2>$, F = $|\overline{3}\ -1/3\ 1/2\ 1/2>$, G = $|\overline{3}\ 2/3\ 0\ 0>$; 8 : H = $|8\ 1\ 1/2\ -1/2>$, I = $|8\ 1\ 1/2\ 1/2>$, J = $|8\ 0\ 1\ -1>$, K = $|8\ 0\ 1\ 0>$, L = $|8\ 0\ 1\ 1>$, M = $|8\ 0\ 0\ 0>$, N = $|8\ -1\ 1/2\ -1/2>$, O = $|8\ -1\ 1/2\ 1/2>$; 10 : P = $|10\ 1\ 3/2\ -3/2>$, Q = $|10\ 1\ 3/2\ -1/2>$, R = $|10\ 1\ 3/2\ 1/2>$, S = $|10\ 1\ 3/2\ 3/2>$, T = $|10\ 0\ 1\ -1>$, U = $|10\ 0\ 1\ 0>$, V = $|10\ 0\ 1\ 1>$, W = $|10\ -1\ 1/2\ -1/2>$, X = $|10\ -1\ 1/2\ 1/2>$, Y = $|10\ -2\ 0\ 0>$

From the preceding (weight) vectors, the reader will easily draw in the

(T_3, Y) plan the weight diagrams of the representations 1, 3, $\bar{3}$, 8, and 10. (The diagram for the identity representation 1 is of course trivial. The ones for the fundamental representations 3 and $\bar{3}$ follow from the self-representation of SU(3). The diagram for the regular representation 8 may be deduced from the knowledge of the structure constants of su(3).)

su(3) and su(4) and the classification of the hadrons

The hadrons having the same J^P (J = spin , p = parity), the same B (B = 0 for mesons and anti-mesons, B = 1 for baryons, and B = -1 for anti-baryons), similar masses, and similar strong interactions (but different electromagnetic and weak interactions) are assigned to the various weights of a (generally irreducible) representation of su(3) or su(4).

su(3). The classification can be achieved according to the eight-fold way. Following the prescription of Gell-Mann and Ne'eman, the eight $(1/2)^+$ baryons $n, p, \Sigma^-, \Sigma^0, \Sigma^+, \Lambda^0, \Xi^-, \Xi^0$ are associated to the regular representation 8. To be more explicit : $|n\rangle = H$, $|p\rangle = I$, $|\Sigma^-\rangle = J$, $|\Sigma^0\rangle = K$, $|\Sigma^+\rangle = L$, $|\Lambda^0\rangle = M$, $|\Xi^-\rangle = N$, $|\Xi^0\rangle = O$, where the notation $|n\rangle = H$ means that the SU(3) part of the state vector of n transforms as the vector H given above. The representation 10 is filled up with the ten $(3/2)^+$ baryonic resonances $\Delta^-, \Delta^0, \Delta^+, \Delta^{++}, \Sigma^{*-}, \Sigma^{*0}, \Sigma^{*+}, \Xi^{*-}, \Xi^{*0}, \Omega^-$. In the detail : $|\Delta^-\rangle = P$, $|\Delta^0\rangle = Q$, $|\Delta^+\rangle = R$, $|\Delta^{++}\rangle = S$, $|\Sigma^{*-}\rangle = T$, $|\Sigma^{*0}\rangle = U$, $|\Sigma^{*+}\rangle = V$, $|\Xi^{*-}\rangle = W$, $|\Xi^{*0}\rangle = X$, $|\Omega^-\rangle = Y$. The situation seems to be more intricate with the mesons. The pseudo-scalar mesons 0^- are arranged in the reducible representation 8 + 1. Here we have : $|K^0\rangle = H$, $|K^+\rangle = I$, $|\pi^-\rangle = J$, $|\pi^0\rangle = K$, $|\pi^+\rangle = L$, $|\bar{K}^-\rangle = N$, $|\bar{K}^0\rangle = O$, $|\eta^0\rangle = M \cos\theta_0 + A \sin\theta_0$, $|\eta'^0\rangle = -M \sin\theta_0 + A \cos\theta_0$, where the angle θ_0 accounts for the mixing of the two vectors M and A . In order to correctly describe the mesons η^0 and η'^0, it is thus necessary to consider the state vectors $|\eta^0\rangle$ and $|\eta'^0\rangle$ as being (orthornormal) superpositions of the vectors $M = |8\ 0\ 0\ 0\rangle$ and $A = |1\ 0\ 0\ 0\rangle$. We shall go back to this point below. In a similar way, the vector mesons 1^- correspond to the reducible representation 8 + 1 according to : $|K^{*0}\rangle = H$, $|K^{*+}\rangle = I$, $|\rho^-\rangle = J$, $|\rho^0\rangle = K$, $|\rho^+\rangle = L$, $|\bar{K}^{*-}\rangle = N$, $|\bar{K}^{*0}\rangle = O$, $|\omega^0\rangle = M \cos\theta_1 + A \sin\theta_1$, $|\phi^0\rangle = -M \sin\theta_1 + A \cos\theta_1$, where θ_1 denotes a mixing angle analogous to θ_0.

All the known low-lying hadrons are not exhausted by the just mentioned multiplets. Further particles may be assigned to other 8-plets, 10-plets, and 9-plets. There are also 1-plets. It is remarkable that only zero-triality representations of SU(3) occur in the classification of the baryons, mesons, and their anti-particles. In this respect, the convenient group should be SU(3)/Z(3), a subgroup of SO(8), rather than SU(3).

Finally, it is to be noticed that one of the great successes of SU(3) (or SU(3)/Z(3)) was the prediction of Ω^- in 1961 before its complete experimental identification in 1964. (Note the parallel with the prediction of Gd from the Mendeleyev periodic table.)

su(4). Although the real interest of SU(4) appeared in 1970 after the group SU(6) was introduced in 1964, we briefly discuss the rôle of su(4) in the classification of hadrons. For su(4) we have $\ell = 3$, so that the Cartan subalgebra contains in addition to T_3 and Y a third infinitesimal generator, viz, the charm

$$C = (1 - \sqrt{6}\,\lambda_{15})/4 \tag{20}$$

Consequently, the weight diagrams for the irreducible representations of su(4) must be constructed in the three-dimensional (T_3, Y, C) space and the relevant weight vectors now assume the form $|(P,Q,R)C(p,q)YTM_T\rangle$. Concomitantly, the so-called Gell-Mann-Nishijima relation writes

$$Q = T_3 + Y/2 + C/2 \tag{21}$$

The $(1/2)^+$ baryons find place in the irreducible representation $20'$ of su(4) which decomposes as $20' = (0;8) + (1;\bar{3}) + (1;6) + (2;3)$, where (0;8) stands for the 8 of su(3) with C = 0, and so on. The $(3/2)^+$ baryons inside the 10 of su(3) may be accomodated in the irreducible representation 20 of su(4) since $20 = (0;10) + (1;6) + (2;3) + (3;1)$. Finally, each su(3) 9-plet of mesons can be put in the reducible representation 15 + 1 of su(4) especially in virtue of $1 = (0;1)$ and $15 = (0;1) + (0;8) + (1;\bar{3}) + (-1;3)$.

The SU(4) symmetry furnishes a rich (not yet completely exploited) spectroscopy. The meson J/ψ, with hidden charm C = 0, discovered in 1974 belongs to a representation 15 + 1. The two mesons D^0 and D^+, with apparent charm C = 1, discovered in 1976 belong to another representation 15+1.

Mass relations

General considerations. These considerations will be devoted more specifically to SU(3). Should the SU(3) symmetry be an unbroken symmetry, the particles in an su(3) multiplet would have the same mass. To describe the experimental situation, it is necessary to suppose that the mass operator M involves, in addition to an SU(3) invariant part M_0, a part M_1 which does not behave like a scalar under SU(3). The operator M_1 allows to account for the differences observed between the average mass values of the various su(2) multiplets contained in a given su(3) (super)multiplet. Furthermore, inside a given su(2) multiplet, a second symmetry breaking operator M_2 may be introduced to explain the observed (electromagnetic) mass differences. The operators M_0, M_1, and M_2 are conveniently expanded in terms of SU(3) irreducible tensor operators $T[(p,q)Y\,T\,M_T]$. Of course M_0 involves only $T[(0,0)000]$. By taking operators of the type $T[(p,q)000]$ in the expansion of M_1, it is obvious that the particles belonging to a given su(2) multiplet will have the same mass if M reduces to $M_0 + M_1$. Finally, M_2 may be seen to comprize operators of the type $T[(p,q)010]$ in order to generate the mass splitting of a given su(2) multiplet. In the case where no mixing takes place between two su(3) multiplets, an operator equivalent defined in the enveloping algebra of su(3) may be substituted to the expansion of M in terms of $T[(p,q)Y\,T\,M_T]$: M_0 is then an arbitrary function of the invariants F^2 and G^3 while M_1 and M_2 are polynomial functions of certain infinitesimal generators of SU(3). Along this vein, if we restrict the expansion (strongly limited by selection rules à la Wigner-Eckart) of M to (p, q) = (0, 0) and (1, 1), we may take

$$M_0 = m_0, M_1 = m_1 Y + m_1' (T^2 - Y^2/4), M_2 = m_2 T_3 \qquad (22)$$

where m_0, m_1, m_1', and m_2 are phenomenological parameters which may be either determined from experiment or eliminated for producing mass relations to be compared with experiment. (We insist that the operator equivalent form of M provided by Eq. (22) turns out to be inconsistent with the mixing between state vectors belonging to two different su(3) multiplets.) To illustrate these matters, we now show how Eq. (22) may be used for (hypothetical) particles distributed in the representations 8 and 10 of su(3).

We begin with the 8. As a first approximation, we may consider $M = M_0 + M_1$, as given by Eq. (22). The eigenvalues of such an operator inside the 8 arise in a straightforward way and the parameters m_0, m_1, and m_1' can be eliminated to give

$$2M(H) + 2M(N) = 3M(M) + M(J) \quad (23)$$

where the notation $M(H)$ stands for the mass of a particle associated with the weight vector H. Further, if (for some special cases, as will be seen below) $m_1 = 0$, Eq. (23) specializes to

$$4M(H) = 3M(M) + M(J) \quad (24)$$

As a second approximation, let us consider $M = M_0 + M_1 + M_2$, as given by Eq. (22). This leads to change Eqs. (23) and (24) into respectively

$$M(H) + M(I) + M(N) + M(O) = 3M(M) + M(J) - M(K) + M(L) \quad (25)$$

and

$$2M(H) + 2M(I) = 3M(M) + M(J) - M(K) + M(L) \quad (26)$$

Furthermore, in the case where m_0, m_1, m_1', and m_2 are different from zero, we have

$$M(H) - M(I) + M(L) - M(J) + M(N) - M(O) = 0 \quad (27)$$

which for $m_1 = 0$ specializes to

$$2M(H) - 2M(I) + M(L) - M(J) = 0 \quad (28)$$

We continue with the 10. Let us first consider $M = M_0 + M_1$. In this case we obtain the same mass relation by replacing $M = m_0 + m_1 Y + m_1'(T^2 - Y^2/4)$ by the simpler operator equivalent $M = a + bY$. This follows from the fact that the equality $T^2 = (1 + Y/2)(2 + Y/2)$ holds for the representation 10 of su(3), so that $a = m_0 + 2m_1'$ and $b = m_1 + 3m_1'/2$. The operator $M = a + bY$ leads to the equal spacing rule

$$M(Y) - M(W) = M(W) - M(T) = M(T) - M(P) \quad (29)$$

Note that the operator $M = a + bY + cY^2$ would yield the relation

$$M(P) + 3M(W) = 3M(T) + M(Y) \quad (30)$$

Let us now introduce the electromagnetic perturbation M_2. By taking, $M = a + bY + cY^2 + m_2 T_3$, we get (even for $c = 0$)

$$\begin{aligned} M(P) - M(Q) &= M(Q) - M(R) = M(R) - M(S) \\ &= M(T) - M(U) = M(U) - M(V) = M(W) - M(X) \end{aligned} \quad (31)$$

<u>Application to baryons and mesons</u>. The formal mass relations (23) to (31) may be applied to ordinary hadrons. We shall examine in turn the cases

of the 8-plet of $(1/2)^+$ baryons, the 10-plet of $(3/2)^+$ baryons, and the 8-plet and 9-plet of 0^- mesons.

For the $(1/2)^+$ baryons, Eq. (25) particularizes as

$$[M(n) + M(p) + M(\Xi^-) + M(\Xi^0)]/4 = [3M(\Lambda^0) + M(\Sigma^-) - M(\Sigma^0) + M(\Sigma^+)]/4 \qquad (32)$$

By replacing the mass of each particle by the average mass of the particles in the su(2) multiplet to which the considered particle belongs, Eq. (32) becomes

$$[M(N) + M(\Xi)]/2 = [3M(\Lambda) + M(\Sigma)]/4 \qquad (33)$$

that is known as the Gell-Mann-Okubo mass formula. Equation (33) corresponds to Eq. (23) and was derived by Okubo in 1962 by requiring that the mass operator M transforms like $T[(1,1)000]$. The agreement between theory and experiment appears to be better than expected : both Eqs. (32) and (33) are satisfied to less than 1%. (Compare the values 1128 and 1135 as given by respectively the left- and right-hand side of Eqs. (32) or (33) along with the corresponding experimental uncertainties.) Finally, Eq. (27) gives the electromagnetic-like mass relation

$$M(n) - M(p) + M(\Sigma^+) - M(\Sigma^-) + M(\Xi^-) - M(\Xi^0) = 0 \qquad (34)$$

The latter relation may be obtained simply from U-spin invariance too. It was originally derived by Coleman and Glashow in 1961 and fits quite well the experimental data.

For the $(3/2)^+$ baryons, the equal spacing rule (29) writes (with evident abbreviations)

$$M(\Omega) - M(\Xi^*) = M(\Xi^*) - M(\Sigma^*) = M(\Sigma^*) - M(\Delta) \qquad (35)$$

Equation (35) is also in good accord with experiment and was used to successfully predict the mass of Ω^-. In addition, Eq. (30) yields the relation

$$[M(\Delta) + 3M(\Xi^*)]/4 = [3M(\Sigma^*) + M(\Omega)]/4 \qquad (36)$$

in remarkable agreement with experimental results. Lastly, Eq. (31) specializes to the electromagnetic-like mass relation

$$\begin{aligned} M(\Delta^-) - M(\Delta^0) &= M(\Delta^0) - M(\Delta^+) = M(\Delta^+) - M(\Delta^{++}) \\ &= M(\Sigma^{*-}) - M(\Sigma^{*0}) = M(\Sigma^{*0}) - M(\Sigma^{*+}) = M(\Xi^{*-}) - M(\Xi^{*0}) \qquad (37) \end{aligned}$$

For the 0^- mesons, we have $M(\overline{K}^-) = M(K^+)$ and $M(\overline{K}^0) = M(K^0)$ because the mass of a particle equals the one of its antiparticle, a corollary of the Lüders CPT-theorem. This may be reflected by taking $m_1 = 0$ in Eq. (22). Consequently, we use Eq. (24) to obtain

$$M(K) = [3M(\eta) + M(\pi)]/4 \tag{38}$$

a relation satisfied to only 11 %. The replacement of Eq. (38) by

$$M(K)^2 = [3M(\eta)^2 + M(\pi)^2]/4 \tag{39}$$

makes it possible to decrease the disagreement from 11 to 7 %. (There exist some questionable arguments based on quantum field theory for justifying the substitution $M \to M^2$ for mesons.) To improve the agreement, we have to go back to the abovementioned idea according to which the pseudoscalar mesons η^0 and η'^0 are described by the admixtures $|\eta^0\rangle = \cos\theta_0 \, |8000\rangle + \sin\theta_0 \, |1000\rangle$ and $|\eta'^0\rangle = -\sin\theta_0 \, |8000\rangle + \cos\theta_0 |1000\rangle$, respectively. Then, from usual two-level quantum mechanics, we get

$$\sin^2\theta_0 = \frac{M(8)^2 - M(\eta)^2}{M(\eta')^2 - M(\eta)^2} = \frac{M(1)^2 - M(\eta')^2}{M(\eta)^2 - M(\eta')^2} \tag{40}$$

and θ_0 can be computed by introducing in Eq. (40) the experimental values of $M(\eta)$ and $M(\eta')$ and the value of $M(8)$ deduced from Eq. (39) after the substitution $M(\eta) \to M(8)$. This leads to $\theta_0 \sim 10^\circ$. Similar calculations work for the vector mesons 1^- and the tensor mesons 2^+. The interest in obtaining mixing angles like θ_0, θ_1, and θ_2 lies on the fact they may be used in other calculations as, for instance , analyses of decay modes.

A word about SU(4). A mass operator M for the (broken) SU(4) symmetry is easy to derive on the same pattern as for the SU(3) symmetry. Here again, M turns out to be amenable to an operator equivalent form in the absence of interaction between multiplets of su(4). Let us mention, among others, the linear mass operator for baryons :

$$M = a + bY + c(T^2 - Y^2/4) + dC + eC^2 \tag{41}$$

and the quadratic mass operator for mesons :

$$M^2 = a + b(T^2 - Y^2/4) + cC^2 \tag{42}$$

The latter two operator equivalents have been employed in the recent years for predicting the mass of some charmed hadrons. By way of illustration, for the 0^- mesons in the representation 15(+1) of su(4), the quadratic mass operator (42) yields

$$5[M(K)^2 - M(\pi)^2] = 9[M(F)^2 - M(D)^2] \quad (43)$$

By introducing in Eq. (43) the experimental values for M(K), M(π), and M(D), we obtain M(F) = 1900 as an estimate for the mass of the mesons $\overline{F}^-$ and F^+. Observe that the use of M instead of M^2 in Eq. (43) would lead to M(F) = 2070 . (Equation (43) may be derived as follows. Find from Eqs. (12) to (20) the $f_{k\ell m}$ required to set up the matrices of T_3, Y, and C in the regular representation 15 or (1, 0, 1) of su(4). The eigenvectors of these matrices supply the weight vectors $|(1, 0, 1)C(p,q)\ Y\ T\ M_T\rangle$ which are also eigenstates of the operator (42). Put D^0 and D^+ in a 2-plet of su(2) with C = 1 and F^+ in a 1-plet of su(2) with C = 1, so that D^0, D^+, and F^+ belong to a 3-plet of su(3). Then, obtain Eq. (43).)

Mass relations à la Wigner-Eckart. The method of operator equivalents is particularly efficient for obtaining a hadron mass spectrum when there are no SU(3) or SU(4) Clebsch-Gordan coefficients at disposal. However, when nontrivial mixing effects via the mass operator M arise between several multiplets of su(3) or su(4) and/or when higher order contributions to M have to be taken into consideration, it is indispensable to go back to the expression of M in terms of irreducible tensor operators. To establish a link between the two approaches, we consider the simple cases of particles classified in the 8 and the 10 of su(3).

By taking $M = \Sigma m(p,q)\ T[(p,q)\ 0\ 0\ 0]$, the calculation of the matrix elements $\langle 8\ Y\ T\ M_T|M|8\ Y\ T\ M_T\rangle$ first requires the examination of the direct product $\overline{8} \times 8$. Since $\overline{8} \times 8 = 1 + 8_s + 8_a + 10 + \overline{10} + 27$, the sum over (p, q) in M a priori restricts to $(p,q) = \overline{1}, \overline{8}, \overline{10}, 10, \overline{27}$. (The representations 8_s and 8_a are indexed according to their symmetric and antisymmetric character with respect to the permutation group S_2, respectively.) However, the representations 10 and $\overline{10}$ do not have $Y = T = M_T = 0$ components. Consequently, we get

$$M = m(1)\ T(1\ 0\ 0\ 0) + m(8)\ T(8\ 0\ 0\ 0) + m(27)\ T(27\ 0\ 0\ 0) \quad (44)$$

as far as M acts within the 8 of su(3). Repeated application of the Wigner-Eckart theorem for SU(3) in the $SU(3) \supset SU(2)_T$ basis allows to express

$< 8\ Y\ T\ M_T | M | 8\ Y\ T\ M_T >$ as a function of SU(3) Clebsch-Gordan coefficients and reduced matrix elements. Once the numerical values of the Clebsch-Gordan coefficients have been introduced, we are left with (at most) four free parameters : $s = m(1) < 8 \| T(1) \| 8 >$, $t = m(8) < 8 \| T(8_s) \| 8 >$, $u = m(8) < 8 \| T(8_a) \| 8 >$, and $v = m(27) < 8 \| T(27) \| 8 >$. The connection between the parameters s, t, u and the parameters m_0, m_1, m_1' of $M = m_0 + m_1 Y + m_1' (T^2 - Y^2/4)$ reads $s = m_0 + m_1'$, $t = \sqrt{5}\, m_1'$, $u = 2\, m_1$.

In a similar fashion, for the particles in the 10, we have to consider $\overline{10} \times 10 = 1 + 8 + 27 + 64$. The mass operator then assumes the form

$$M = m(1)\ T\ (1\ 0\ 0\ 0) + m(8)\ T\ (8\ 0\ 0\ 0) + m(27)\ T\ (27\ 0\ 0\ 0) + m(64)\ T\ (64\ 0\ 0\ 0) \tag{45}$$

as far as M acts within the 10 of su(3). We end up with (at most) four free parameters : $w = m(1) < 10 \| T(1) \| 10 >$, $x = m(8) < 10 \| T(8) \| 10 >$, $y = m(27) < 10 \| T(27) \| 10 >$, and $z = m(64) < 10 \| T(64) \| 10 >$. Finally, w and x are connected with a and b of $M = a + bY$ through $w = a$ and $x = - 2\sqrt{2} b$.

Other static properties

There are two other kinds of quantities which can be handled in the framework of the SU(3) or SU(4) symmetry : the magnetic moments and the scattering amplitudes. We shall close this section with some illustrative examples relative to the SU(3) symmetry.

Magnetic moments. The U-spin plays a central rôle for the electromagnetic interactions. As a matter of fact, the three infinitesimal generators $U_1 = F_6$, $U_2 = F_7$, and $U_3 = Y - Q/2$ of $SU(2)_U$ commute with Q . Hence, the electromagnetic interactions are invariant under the group $SU(2)_U$ and the particles inside a multiplet of $su(2)_U$ should have the same electromagnetic properties (magnetic moment, form factor, electromagnetic mass shift).

The starting point for a quantitative analysis is to associate a given irreducible tensorial character to the magnetic moment operator μ . This may be achieved by demanding that μ transforms like Q , a quite natural hypothesis. In the case where the action of μ is restricted to a given multiplet of su(3), an operator equivalent form may be written down for μ and we shall take advantage of our experience with the mass operator to briefly discuss the cases of the 8 and 10. For the 8 , the operator equivalent μ writes

$$\mu = \mu_1 Q + \mu_1'(U^2 - Q^2/4 - F^2/3) \tag{46}$$

since both Q and $U^2 - Q^2/4 - F^2/3$ transform like $Q = T_3 + Y/2$. (Note that for the mass operator M the operator $- m_1' F^2/3$ may be considered to be included in the SU(3) invariant part m_0.) For the 10, Eq. (46) simplifies to give

$$\mu = \mu_1'' Q \quad \text{with (formally)} \quad \mu_1'' = \mu_1 - 3\,\mu_1'/2 \tag{47}$$

in view of the fact that the identity $U^2 - Q^2/4 - F^2/3 = -3\,Q/2$ works out in the representation 10 of su(3). As a conclusion, the list of the free magnetic moment parameters a priori is : three parameters for the baryons $(1/2)^+$ and $(3/2)^+$ (two for $(1/2)^+$ and one for $(3/2)^+$) and four parameters for the mesons 0^- and 1^- (two for 0^- and two for 1^-). Relations between the magnetic moments of the various particles in a multiplet (p, q) of su(3) may be generated by eliminating the unknown parameters. This is more easily achieved by rewriting the state vector of each particle in the basis $|(p,q)Q\,U\,M_U>$. As a trivial example, we would obtain

$$\mu(n) = \mu(\Xi^0)\ ,\ \mu(p) = \mu(\Sigma^+)\ ,\ \mu(\Sigma^-) = \mu(\Xi^-) = -[\mu(n) + \mu(p)] \tag{48}$$

<u>Scattering amplitudes</u>. The transition amplitudes for a reaction $AB \to CD$ are conveniently studied from the matrix elements of the scattering operator S. In the limit where the SU(3) symmetry is exact (i.e., to zeroth-order approximation) $S = s(1)\ T\ (1\ 0\ 0\ 0)$ while to first-order approximation $S = s(1)\ T\ (1\ 0\ 0\ 0) + s(8)\ T\ (8\ 0\ 0\ 0)$. By using the notation $|a\alpha>$ with $a = (p, q)$ and $\alpha = Y\ T\ M_T$ for the state vector of the particle A , we have first to consider the uncoupled state vectors $|a\alpha > x\ |b\beta> = |ab\alpha\beta>$ and then the coupled state vectors $|ab\,\rho_x\,x\,\xi>$, where the multiplicity label ρ_x must be used if the representation x occurs more than once in the direct product a x b. To obtain the transition amplitudes between the initial and final state vectors, we need to calculate $A(x \to y) = < cd\ \rho_y\ y\eta\,|S|\,ab\ \rho_x\ x\ \xi >$ and, here again, the Wigner-Racah algebra of SU(3) is of special relevance. The number of free parameters is (partially) dictated by the analysis of the relevant Clebsch-Gordan series involved in the calculation of $A(x \to y)$ through the Wigner-Eckart theorem for SU(3). As a first example, in a process of type $8 \times 8 \to 8 \times 10$, like in the production of $(3/2)^+$ meson-baryon resonances from $0^-(1/2)^+ \to (3/2)^+ 1^-$ experiments $(\pi^- p \to \Delta^- \pi^+,\ K^- p \to \Sigma^{*-} \pi^+, \ldots)$, the Clebsch-Gordan series to consi-

der are $8 \times 8 = 1 + 8_s + 8_a + 10 + \overline{10} + 27$ and $8 \times 10 = 8 + 10 + 27 + 35$. Therefore, in the zeroth-order approximation there are four channels characterized by the amplitudes $A(8_s \to 8)$, $A(8_a \to 8)$, $A(10 \to 10)$, and $A(27 \to 27)$. As a second example, a process of type $8 \times 8 \to 8 \times 8$ a priori requires the eight amplitudes $A(1 \to 1)$, $A(8_s \to 8_s)$, $A(8_s \to 8_a)$, $A(8_a \to 8_s)$, $A(8_a \to 8_a)$, $A(10 \to 10)$, $A(\overline{10} \to \overline{10})$, and $A(27 \to 27)$ in the limit where the SU(3) symmetry is exact. Time reversal invariance ensures that $A(8_s \to 8_a) = A(8_a \to 8_s)$ so that we are left with seven parameters which may be combined in order to produce relations between scattering amplitudes.

FROM SU(3) AND SU(4) TO SU(6) AND SU(8)

Generalities

There are recurrent themes in Physics as in many other scientific disciplines. The first extension from SU(n) to SU(2n) was developed by Wigner in 1937 for $n = 2$. Indeed, two groups must be considered for nucleons if we restrict our attention to their isospin and spin variables : $SU(2)_T$ for the isospin T and $SU(2)_S$ for the spin S. (Of course it is also necessary to include the group SO(3) for describing the orbital variables.) If we take for granted that the nucleon-nucleon interactions are isospin- and spin—independent, the suitable group is the $(r = 6, \ell = 2)$ group $SU(2)_T \times SU(2)_S$ whose the natural extension appears to be the $(r = 15, \ell = 2)$ group SU(4). The passage from SU(n) to SU(2n) in elementary particle physics follows the same scheme as will be sketched for $n = 3$ and 4.

The passage SU(3) → SU(6)

The idea of an SU(6) symmetry was exploited by various people including Gürsey, Radicati, Sakita, and Zweig between 1964 and 1966. The introduction of the spin group SU(2) in addition to the "unitary spin" group SU(3) leads to the group SU(6) which is broken to SU(3) x SU(2) in the limit where the strong interaction forces are unitary spin- and spin-independent. The elements of SU(6) are described in the neighbourhood of the identity by the infinitesimal generators $\lambda_k \times \sigma_\ell$, $1_{3x3} \times \sigma_\ell$, and $\lambda_k \times 1_{2x2}$ $(k = 1, 2, \ldots, 8$ and $\ell = 1, 2, 3)$ which all together span the thirty-six-dimensional Lie algebra su(6). (The subset $\{1_{3x3} \times \sigma_\ell, \lambda_k \times 1_{2x2}\}$ span only the eleven-dimensional Lie algebra su(3)+su(2).)

Two irreducible representations of su(6) are of particular importance : the 56 and the regular representation 35. When restricting su(6) to su(3)+su(2), we get the following decompositions : (56) = (8, 2) + (10, 4) and 35 = (8, 1) + (1, 3) + (8, 3), where in (a, b) the representations a and b indicate the su(3) and su(2) contents, respectively. The classification of the ordinary hadrons in (super)multiplets of su(6) may thus be done as follows. The 56 can accomodate the 16 baryons $(1/2)^+$ and the 40 baryonic resonances $(3/2)^+$ while 8 mesons 0^- and the 27 mesons 1^- fill up the 35. More precisely, we have :

8 baryons $(1/2)^+$ with $2J+1 = 2$ in a 8-plet of su(3) → (8, 2)
10 baryons $(3/2)^+$ with $2J+1 = 4$ in a 10-plet of su(3) → (10, 4)
8 mesons 0^- with $2J+1 = 1$ in a 8-plet of su(3) → (8, 1)
9 mesons 1^- with $2J+1 = 3$ in a 9-plet of su(3) → (1, 3) + (8, 3)

The algebra su(6) allows to unite in a same multiplet baryons or mesons having different spins but the same parity.

The mass operator M for the SU(6) symmetry may be also discussed in terms of irreducible tensor operators or operator equivalents. More or less ad hoc operator equivalents have been proposed, as for example

$$M = a + bY + c\,(T^2 - Y^2/4) + dJ^2 \qquad (49)$$

for baryons and

$$M^2 = a + b\,(T^2 - Y^2/4) + cJ^2 \qquad (50)$$

for mesons. Thus, in addition to the SU(3) mass relations (32) to (39), it becomes feasible to connect the masses of particles carrying different spins J. In that direction, we may mention the relation

$$M(K)^2 - M(\pi)^2 = M(K^*)^2 - M(\rho)^2 \qquad (51)$$

in excellent agreement with experiment.

Finally, an approach to the magnetic moments may be developed on the basis of the SU(6) symmetry. Such an approach presents the interesting peculiarity of involving only one parameter (one for baryons, one for mesons), to be compared with the two- or one-parameter approach for baryons and the two-parameter approach for mesons in the SU(3) symmetry. The SU(6) approach leads in particular to the theoretical ratio $\mu(n)/\mu(p) = -2/3$ (0, 68 is the experimental value !), one of the great successes of SU(6). (We shall obtain the latter theoretical result in the following from the quark model.) The SU(6) model has been successful in many other directions too, in spite of its nonrelativistic character.

The passage SU(4) → SU(8)

The basic representations of su(8) to tackle are the 120 and the regular representation 63. The classification of the ordinary and charmed hadrons lies on the following decompositions in su(4)+su(2) : 120 = (20',2) + (20,4) and 63 = (15,1) + (1,3) + (15,3), where the notation is now self-explanatory. Hence the baryons $(1/2)^+$ and $(3/2)^+$ find place in the *120* while the mesons 0^- and 1^- in the 63 according to the distribution :

20	baryons	$(1/2)^+$	with $2J + 1 = 2$	in a 20-plet of	su(4) → (20',2)
20	baryons	$(3/2)^+$	with $2J + 1 = 4$	in a 20-plet of	su(4) → (20,4)
15	mesons	0^-	with $2J + 1 = 1$	in a 15-plet of	su(4) → (15,1)
16	mesons	1^-	with $2J + 1 = 3$	in a 16-plet of	su(4) → (1,3) + (15,3)

It is inspiring to look at Eqs. (41), (42), (49), and (50) for constructing an ad hoc mass operator for the broken SU(8) symmetry. In order to obtain a mass operator, in an operator equivalent form, for baryons and mesons, it is sufficient to add the contributions $eC + fC^2$ and dC^2 to the operators (49) and (50), respectively. In full analogy with Eq. (51), the so-obtained quadratic mass operator yields

$$M(F)^2 - M(D)^2 \approx M(F^*)^2 - M(D^*)^2 \quad (52)$$

QUARKS

Four quarks for Muster Mark

From what preceeds, it is apparent that the classification afforded by the algebra su(3) makes use solely of the representations 1 and 8 for mesons and antimesons and 1, 8, 10, and $\overline{10}$ for baryons and antibaryons. In the Gell-Mann and Ne'eman eight-fold way, the fundamental representations 3 and $\overline{3}$ of su(3) do not contain any particle, in contradistinction with the earlier Sakata model also referred to as the three-fold way. In the Sakata model, the particles p, n, and λ are assigned to the 3 and the antiparticles $\overline{p}$, $\overline{n}$, and $\overline{\lambda}$ to the $\overline{3}$. The sakaton (p, n, λ) thus replaces the nucleon (p, n) and the antisakaton $(\overline{p}, \overline{n}, \overline{\lambda})$ the antinucleon $(\overline{p}, \overline{n})$. This leads to several difficulties among which we may mention : the occurrence (via the usual Gell-Mann-Nishijima relation (11)) of fractional charges for p, n, and λ ; the difficulty to distribute the particles Σ^-, Σ^0, Σ^+, Ξ^-, and Ξ^0 in a multiplet of su(3) ; and the impossibility to explain certain decay amplitudes (e.g., $(3/2)^+ \to (1/2)^+ 0^-$ is strictly forbidden to zeroth- and first-order for 3×8 is not contained in 1×10 and 8×10). In addition, the Sakata model postulates that the hadrons are formed out from sakatons and antisakatons and this results with further difficulties for explaining the baryon spectrum. Nevertheless, the Sakata model as well as the proposal made by Fermi and Yang in 1949 (according to which the pion appears as a nucleon-antinucleon bound state) constitute two important preliminary steps towards the quark model.

The idea to ascribe three new entities (viz, the aces of Zweig or the quarks of Gell-Mann) to the 3 of su(3) was advanced by Zweig and independently by Gell-Mann in 1964. (The word quark appears in the poetry "Three quarks for Muster Mark" by James Joyce. The word ace is connected with cards. In a pack of cards, it seems impossible to indefinitely increase the number of aces. On the contrary, it seems possible to increase the number of quarks for Muster Mark. We thus follow the common practice by employing the word quark instead of ace.) Three quarks, u (for up), d (for down), and s (for strange or sideway), are associated to the weight vectors B, C, and D listed above, respectively. Since the restriction of su(4) to su(3) + u(1) leads to the decomposition $4 = (0 \; ; \; 3) + (1 \; ; \; 1)$, the weight vectors B, C, and D constitute three of the four weight vectors of the representation 4 or (1, 0, 0) of su(4) and we associate a fourth quark c (for charm) to the fourth weight vector of (1, 0, 0). We then have

$$|u\rangle = |(1,0,0)\ 0\ (1,0)\ 1/3\ \ 1/2\ \ 1/2\rangle, \quad |d\rangle = |(1,0,0)\ 0\ (1,0)\ 1/3\ \ 1/2\ \ -1/2\rangle$$
$$|s\rangle = |(1,0,0)\ 0\ (1,0)\ -2/3\ \ 0\ 0\rangle, \quad |c\rangle = |(1,0,0)\ 1\ (0,0)\ 0\ 0\ 0\rangle \qquad (53)$$

We can calculate the charge of the quarks u, d, c, and s from the extended Gell-Mann-Nishijima relation (21). We further attribute the baryonic number $B = 1/3$, the spin $J = 1/2$, and the parity + to each quark. The (mathematical) features of the quarks $q = (u, d, c, s)$ may then be summarized as follows. (The quarks u(up), d(down), and s(strange) are sometimes named park, nark, and lark, respectively, with an evident reference to the Sakata model.)

	(P,Q,R)	(p,q)	Y	T	M_T	C	Q	B	S	J	p
u or park :	(1,0,0)	(1,0)	1/3	1/2	1/2	0	2/3	1/3	0	1/2	+
d or nark :	(1,0,0)	(1,0)	1/3	1/2	-1/2	0	-1/3	1/3	0	1/2	+
s or lark :	(1,0,0)	(1,0)	-2/3	0	0	0	-1/3	1/3	-1	1/2	+
c or charm:	(1,0,0)	(0,0)	1/3	0	0	1	2/3	1/3	0	1/2	+

The antiquarks $\bar{q} = (\bar{u}, \bar{d}, \bar{c}, \bar{s})$ belong to the representation 4 or (0,0,1) of su(4) and have the following (mathematical) features :

	(P,Q,R)	(p,q)	Y	T	M_T	C	Q	B	S	J	p
$\bar{u}$ or antipark :	(0,0,1)	(0,1)	-1/3	1/2	-1/2	0	-2/3	-1/3	0	1/2	-
$\bar{d}$ or antinark :	(0,0,1)	(0,1)	-1/3	1/2	1/2	0	1/3	-1/3	0	1/2	-
$\bar{s}$ or antilark :	(0,0,1)	(0,1)	2/3	0	0	0	1/3	-1/3	1	1/2	-
$\bar{c}$ or anticharm:	(0,0,1)	(0,0)	-1/3	0	0	-1	-2/3	-1/3	0	1/2	-

The passage quark → antiquark parallels the passage particle → antiparticle : (the mass M, the lifetime τ,) the spin J, and the isospin T are the same for a particle and its antiparticle while the additive quantum numbers like (the leptonic number L,) the baryonic number B, the strangeness S, the charm C, the hypercharge Y, the third component T_3 of the isospin, and the charge Q change their sign when going from a particle to its antiparticle. In addition, the parity of a boson (e.g., a meson) and of its antiparticle is the same while a fermion (e.g., a baryon) and its antiparticle have opposite parities. Finally, a particle and its antiparticle belong to complex conjugate representations.

Composition of hadrons

Composition rules. It has been said that God does not play with dice. May be He plays with mécano and/or lego. In this respect, we now examine the kind of mathematical game it is possible to play with quarks and antiquarks and the reasons to assume the following symbolic composition rules for hadrons

$$\text{meson} = q\bar{q}, \qquad \text{baryon} = qqq, \qquad \text{antibaryon} = \bar{q}\bar{q}\bar{q} \tag{54}$$

where q and $\bar{q}$ stand for any quark (u, d, c, s) and antiquark $(\bar{u}, \bar{d}, \bar{c}, \bar{s})$, respectively. (It may be anticipated from Eq. (54) that the state vector or wave function is of type $q_i(1)\ \bar{q}_j(2)$ for a meson and $q_i(1)\ q_j(2)\ q_k(3)$ or $\bar{q}_i(1)\ \bar{q}_j(2)\ \bar{q}_k(3)$ for a baryon or an antibaryon.)

The rules with respect to su(n). It is well known how to obtain (at least from a group-theoretical viewpoint) a spin $J = 1$ from two spins $j = 1/2$ and, more generally, a spin J from the composition of $2J$ spins $j = 1/2$. In the terminology of group theory, it is possible to obtain any irreducible representation of su(2) by conveniently coupling a certain number of representations identical with the 2 of su(2). In a general way, any irreducible representation of a Lie algebra (or group) of rank ℓ is obtainable from ℓ fundamental irreducible representations. The algebra su(n) has $\ell = n-1$ fundamental representations including the representations n and $\bar{n}$ that come from the self-representation of SU(n) and its complex conjugate, respectively. By using standard methods, we easily obtain $n \times \bar{n} = (n^2-1) + 1$, where (n^2-1) stands for the regular representation of su(n), and $n \times n \times n = n(n+1)(n+2)/6 + (n-1)\,n\,(n+1)/3 + (n-1)\,n\,(n+1)/3 + (n-2)(n-1)n/6$, where the two representations of dimension $(n-1)\,n\,(n+1)/3$ are in fact identical. For n = 3, 6, 4, and 8 we have in particular

$$\begin{aligned}
n = 3 &: 3 \times \bar{3} = 8 + 1, \quad 3 \times 3 \times 3 = 10 + 8 + 8 + 1 \\
n = 6 &: 6 \times \bar{6} = 35 + 1, \quad 6 \times 6 \times 6 = 56 + 70 + 70 + 20 \\
n = 4 &: 4 \times \bar{4} = 15 + 1, \quad 4 \times 4 \times 4 = 20' + 20 + 20 + 4 \\
n = 8 &: 8 \times \bar{8} = 63 + 1, \quad 8 \times 8 \times 8 = 120 + 168 + 168 + 56
\end{aligned} \tag{55}$$

When comparing Eq. (55) to the abovementioned assignments of baryons and mesons to irreducible representations of su(n) with n = 3, 6, 4, and 8, we see that the composition rules as given by Eq. (54) make sense with respect to su(n). For instance, $3 \times \bar{3} = 1 + 8$ and $3 \times 3 \times 3 = 1 + 8 + 8 + 10$ reflect the fact that the mesons are organized in 8-plets, 1-plets, and 9-plets of su(3) and the baryons in 8-plets, 10-plets, and 1-plets, respectively.

The rules with respect to spin and additive numbers. By assuming that quarks are fermions with the spin $J = 1/2$, Eq. (54) shows that mesons (and antimesons) behave like bosons with the spin $J = 0$ or 1 and baryons (and antibaryons) like fermions with the spin $J = 1/2$ or $3/2$. This result matches very well the fact that, in the absence of orbital excitation, the mesons have $J = 0$ or 1 and the baryons $J = 1/2$ or $3/2$. Furthermore, the ad hoc hypothesis $B = 1/3$ for each quark has been chosen to ensure via Eq. (54) that $B = 1/3 - 1/3 = 0$ for mesons and antimesons, $B = 1/3 + 1/3 + 1/3 = 1$ for baryons, and $B = -1/3 - 1/3 - 1/3 = -1$ for antibaryons. In the same vein, we may postulate the addition rules

$$X\ (\text{meson}) = X(q) + X(\bar{q}),$$
$$X(\text{baryon}) = X(q) + X(q) + X(q),$$
$$X(\text{antibaryon}) = X(\bar{q}) + X(\bar{q}) + X(\bar{q}) \qquad (56)$$

where X denotes some other additive quantum number like Y, S, C, and Q.

<u>Aggregates of quarks in su(4)</u>. By applying the rules (54) and (56), we can find the quark structure of the known hadrons. This may be done in a straightforward manner for the (noncharmed) mesons and baryons classified according to su(3). It is enough to carefully consider the above discussed weight vectors H, I, J, ..., Y in conjunction with Eqs. (54) to (56). In a similar way, the quark structure of the charmed particles may be obtained from the combination of Eqs. (54) to (56) and of the knowledge of the three-dimensional weight diagrams of the representations 15, 20, and 20' of su(4). As a net result, we have the following quark structure for the hadrons classified according to su(4).

The baryons $(1/2)^+$ in the $20' = (0;8) + (1;\bar{3}) + (1;6) + (2;3)$ of su(4) :

$(0;8)$: $C = 0$, $Y = 1$, $T = 1/2$: $n = udd$, $p = uud$
$C = 0$, $Y = 0$, $T = 1$: $\Sigma^- = dds$, $\Sigma^0 = uds$, $\Sigma^+ = uus$
$C = 0$, $Y = 0$, $T = 0$: $\Lambda^0 = uds$
$C = 0$, $Y = -1$, $T = 1/2$: $\Xi^- = dss$, $\Xi^0 = uss$
$(1;\bar{3})$: $C = 1$, $Y = 1$, $T = 0$: $C_0^+ = udc$
$C = 1$, $Y = 0$, $T = 1/2$: $A^0 = dsc$, $A^+ = usc$
$(1;6)$: $C = 1$, $Y = 1$, $T = 1$: $C_1^0 = ddc$, $C_1^+ = udc$, $C_1^{++} = uuc$
$C = 1$, $Y = 0$, $T = 1/2$: $S^0 = dsc$, $S^+ = usc$
$C = 1$, $Y = -1$, $T = 0$: $T^0 = ssc$
$(2;3)$: $C = 2$, $Y = 1$, $T = 1/2$: $X_d^+ = dcc$, $X_u^{++} = ucc$
$C = 2$, $Y = 0$, $T = 0$: $X_s^+ = scc$

The baryons $(3/2)^+$ in the $20 = (0;10) + (1;6) + (2;3) + (3;1)$ of su(4) :

$(0;10)$: $C = 0$, $Y = 1$, $T = 3/2$: $\Delta^- = ddd$, $\Delta^0 = udd$, $\Delta^+ = uud$, $\Delta^{++} = uuu$
$C = 0$, $Y = 0$, $T = 1$: $\Sigma^{*-} = dds$, $\Sigma^{*0} = uds$, $\Sigma^{*+} = uus$
$C = 0$, $Y = -1$, $T = 1/2$: $\Xi^{*-} = dss$, $\Xi^{*0} = uss$
$C = 0$, $Y = -2$, $T = 0$: $\Omega^- = sss$

(1;6) : C = 1, Y = 1, T = 1 : $B_1^0 = ddc$, $B_2^+ = udc$, $B_3^{++} = uuc$

C = 1, Y = 0, T = 1/2 : $B_4^0 = dsc$, $B_5^+ = usc$

C = 1, Y = -1, T = 0 : $B_6^0 = ssc$

(2;3) : C = 2, Y = 1, T = 1/2 : $B_7^+ = dcc$, $B_8^{++} = ucc$

C = 2, Y = 0, T = 0 : $B_9^+ = scc$

(3;1) : C = 3, Y = 1, T = 0 : $B_{10}^{++} = ccc$

The mesons 0^- in the 15 + 1 = (0;1) + (0;8) + $(1;\bar{3})$ + (-1;3) + (0;1) of su(4) :

(0;8) : C = 0, Y = 1, T = 1/2 : $K^0 = d\bar{s}$, $K^+ = u\bar{s}$

C = 0, Y = 0, T = 1 : $\pi^- = d\bar{u}$, $\pi^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$, $\pi^+ = u\bar{d}$

C = 0, Y = 0, T = 0 : $\eta^0 = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$

C = 0, Y = -1, T = 1/2 : $\bar{K}^- = s\bar{u}$, $\bar{K}^0 = s\bar{d}$

(0;1) : C = 0, Y = 0, T = 0 : $\eta'^0 = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$

(0;1) : C = 0, Y = 0, T = 0 : $\eta_c^0 = c\bar{c}$

$(1;\bar{3})$: C = 1, Y = 1, T = 0 : $F^+ = c\bar{s}$

C = 1, Y = 0, T = 1/2 : $D^0 = c\bar{u}$, $D^+ = c\bar{d}$

(-1;3) : C = -1, Y = 0, T = 1/2 : $\bar{D}^- = d\bar{c}$, $\bar{D}^0 = u\bar{c}$

C = -1, Y = -1, T = 0 : $\bar{F}^- = s\bar{c}$

The mesons 1^- in the 15 + 1 = (0;1) + (0;8) + $(1;\bar{3})$ + (-1;3) + (0;1) of su(4) :

(0;8) : C = 0, Y = 1, T = 1/2 : $K^{*0} = d\bar{s}$, $K^{*+} = u\bar{s}$

C = 0, Y = 0, T = 1 : $\rho^- = d\bar{u}$, $\rho^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$, $\rho^+ = u\bar{d}$

C = 0, Y = 0, T = 0 : $\omega^0 = (u\bar{u} + d\bar{d})/\sqrt{2}$

C = 0, Y = -1, T = 1/2 : $\bar{K}^{*-} = s\bar{u}$, $\bar{K}^{*0} = s\bar{d}$

(0;1) : C = 0, Y = 0, T = 0 : $\phi^0 = s\bar{s}$

(0;1) : C = 0, Y = 0, T = 0 : $J/\psi = c\bar{c}$

$(1;\bar{3})$: C = 1, Y = 1, T = 0 : $F^{*+} = c\bar{s}$

C = 1, Y = 0, T = 1/2 : $D^{*0} = c\bar{u}$, $D^{*+} = c\bar{d}$

(-1;3) : C = -1, Y = 0, T = 1/2 : $\bar{D}^{*-} = d\bar{c}$, $\bar{D}^{*0} = u\bar{c}$

C = -1, Y = -1, T = 0 : $\bar{F}^{*-} = s\bar{c}$

We have already mentioned that the mixing angle θ_0 between the vectors M, i.e., $(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$ in the $q\bar{q}$ framework, and A, i.e., $(u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$ in the $q\bar{q}$ framework, is $\theta_0 \sim 10^0$ for the 0^- pseudo-scalar mesons

η^0 and η'^0. The quark structure reported above for η^0 and η'^0 corresponds to the limiting case $\sin\theta_0 = 0$ and $\cos\theta_0 = 1$. Furthermore, the mixing angle θ_1 for the 1^- vector mesons ω^0 and ϕ^0 may be computed to be $\theta_1 \sim 50^\circ$. The quark structure reported above for ω^0 and ϕ^0 corresponds to the ideal case $\sin\theta_1 = \sqrt{2}/\sqrt{3}$ and $\cos\theta_1 = 1/\sqrt{3}$.

In the classification of mesons, it should be noted that the occurrence of 9-plets (rather than 8-plets) in su(3) and of 16-plets (rather than 15-plets) in su(4) reflects the existence of nine $q\bar{q}$ in the representation $3 \times \bar{3}$ of su(3) for $q = (u, d, s)$ and of sixteen $q\bar{q}$ in the representation $4 \times \bar{4}$ of su(4) for $q = (u, d, c, s)$, respectively.

The labeling afforded by the quark model does not seem to be faithful. For instance, both the proton p and the resonance Δ^+ have the composition uud. The distinction between the two particles fully arises when the spin is taken into account. Indeed, we have $p = u(+)u(+)d(-)$ and $\Delta^+ = u(+)u(+)d(+)$, where + and - indicate a spin component $+1/2$ and $-1/2$, respectively. In $u(+)u(+)d(+)$ the three spins are parallel so that Δ^+ has $J = 3/2$ while in $u(+)u(+)d(-)$ only two spins are parallel so that p has $J = 1/2$.

Besides the association of type $q\bar{q}$ and the *mariage à trois* of type qqq or $\bar{q}\bar{q}\bar{q}$, some speculations about other associations, referred to as exotics, have been advanced. In that direction, a diquark qq would be a bound state made up of two quarks. We shall not discuss this point any longer in view of the lack of definitely established experimental evidence for the exotics.

Static properties in the quark model

The quark model constitutes more than a simple bookkeeping to conveniently summarize the intrinsic quantum numbers of the strongly interacting particles. It may equally well be useful for (re-)deriving in a simple way many results about static properties of hadrons. As an illustration, we shall consider the famous ratio $\mu(n)/\mu(p)$.

The basic hypothesis is that the magnetic moment $\mu(q)$ of the quark q is connected to its charge $Q(q)$ by $\mu(q) = Q(q)\mu_0$, where the proportionality constant μ_0 remains the same for all the quarks. In other words, we have

$$\mu(u) = 2\,\mu_0/3, \quad \mu(d) = -\,\mu_0/3, \quad \mu(c) = 2\,\mu_0/3, \quad \mu(s) = -\,\mu_0/3 \qquad (57)$$

In order to calculate the magnetic moment of an hadron as a function of μ_0, it is sufficient to apply simple angular momentum algebra. Let us consider for example $p = uud$ and $n = udd$. To account for the fact that $J = 1/2$ for n and p, we may introduce the spin of the constituent quarks by writing $p = u(+)u(+)d(-)$ and $n = u(-)d(+)d(+)$. We first have

$$\mu(u(+)u(+)) = 2(2\mu_0/3), \ \mu(d(+)d(+)) = 2(-\mu_0/3) \tag{58}$$

because the magnetic moments add in the case of parallel spins. Then, we have to combine the pair $u(+)u(+)$ with $d(-)$ to form p and $u(-)$ with the pair $d(+)d(+)$ to form n. From elementary quantum mechanics we know that, given two systems a with $J(a) = 1$ and b with $J(b) = 1/2$, we may obtain a composite system ab with $J(ab) = 1/2$, the magnetic moment $\mu(ab)$ of which is linked to $\mu(a)$ and $\mu(b)$ by $\mu(ab) = (2/3)\mu(a) - (1/3)\mu(b)$. Therefore, we get

$$\mu(p) = (2/3)(4\mu_0/3) - (1/3)(-\mu_0/3) = \mu_0,$$
$$\mu(n) = (2/3)(-2\mu_0/3) - (1/3)(2\mu_0/3) = -2\mu_0/3 \tag{59}$$

and we end up with the desired result $\mu(n)/\mu(p) = -2/3$. (Indeed, Eqs. (58) and (59) follow from simple $SU(2) \supset U(1)$ Clebsch-Gordanery.)

We close this section with some remarks on the constituent mass of the quarks. If there is no anomalous magnetic moment for the quarks u and d so that $\mu(q) = Q(q)\, e\hbar/2M(q)c$ for $q = u$ and d, then from the relation $\mu(p) = \mu_0 = (3/2)\mu(u)$ and the experimental value $\mu(p) = 2.79$ in unit of $e\hbar/2M(p)c$, we get the estimate $M(d) = M(u) \sim 350$ for the mass of the quarks u and d. Note that from $M(\Xi^- = dss) - M(\Xi^0 = uss) = 0$, a naïve reasoning would also lead to $M(d) - M(u) = 0$. In a similar vein, the difference $M(\Omega^- = sss) - M(\Xi^{*-} = dss)$ is experimentally known to be equal to 150 and we may hence naïvely take $M(s) - M(d) = 150$. We thus get the estimate $M(s) \sim 500$. There exist many reasons to think that the binding energy of quarks in hadrons is small, as also suggested by the just obtained estimates for M(u), M(d), and M(s), in spite of the apparent contradiction according to which it seems impossible to extract a quark from an hadron. Along this line, we may guess the value $M(c) = 1500$ for the mass of the charmed quark from the experimental value $M(J/\Psi = c\bar{c}) = 3095$.

Color

A new quantum number. We now introduce the concept of color that has no counterpart in other fields of Physics. Let us consider the resonance $\Delta^{++} = uuu$.

Since the spin of Δ^{++} is $J = 3/2$, the quantum numbers of the three quarks u of Δ^{++} should be the same. If we accept that quarks are fermions, the Pauli exclusion principle turns out to be violated for Δ^{++}. We may invoke several remedies as for example the consideration of parastatistics (the quarks would be parafermions of order three) or the introduction of a further quantum number, the color (the quarks would respond to the Fermi-Dirac statistics). There are two ways to color quarks : the one of Greenberg that allows for fractionally charged particles and the one of Han and Nambu that excludes fractionally charged particles. We only discuss the Greenberg way, introduced in 1964, since it is compatible with our description of quarks. Each quark q = (u, d, c, s), or in a pictorial language each flavor u, d, c, or s of quark, may exist in three colors (say blue, yellow, and red with an evident reference to the visible spectrum). The composition rule for a baryon (cf. Eq. (54)) then writes

$$\text{baryon}_{ijk} = \epsilon_{\alpha\beta\gamma}\, q_i^{\alpha}\, q_j^{\beta}\, q_k^{\gamma} \tag{60}$$

where i, j, k refer to the flavor indices and α, β, γ to the color indices. Clearly, Eq. (60) satisfies the exclusion principle. In a similar way, the composition rule for a meson reads

$$\text{meson}_{ij} = \delta_{\alpha\beta}\, q_i^{\alpha}\, \bar{q}_j^{\beta} \tag{61}$$

although there is a priori no real need to introduce the concept of color for mesons. Equations (60) and (61) express the fact that hadrons are neutral (say white) with respect to color.

Of course, the total wave function for an hadron is classified according to SU(6) x O(3) or SU(8) x O(3), where O(3) characterizes the orbital part. If the spin and unitary spin are decoupled, the wave function appears as the product of a SU(3) or SU(4) part, a spin part, and an O(3) part and must be antisymmetric with respect to the interchange of all the coordinates (including the color) of two identical quarks.

Evidences for colored quarks. There are only indirect evidences for the existence of quarks since in spite of repeated experiments it seems impossible to isolate a quark (the quarks would be confined inside hadrons). As an evidence regarding the aggregates of type qqq and $q\bar{q}$, we may mention that the experimental value for $\sigma(\pi p)/\sigma(\pi\pi)$ compares with the value $(3/2)^{2/3}$ obtained on the basis of three constituents for p and two for π.

Another evidence for the granular aspect of hadrons arises when probing nucleons with (protons and) leptons in scattering experiments. As a matter of fact, inside nucleons there are parts, the so-called partons, that scatter as free particles possibly with fractional charges. A parton model has been developed by Feynman in 1969 from the scaling property at very high energy. The partons could contain the (valence) quarks, a sea of $q\bar{q}$ pairs, and the gluons.

Further (indirect) evidence to support the quark model rests on the importance to treat in a symmetric fashion the coupling between leptons and hadrons. In this respect, the introduction of the quark c accounts for the non-observation of strangeness changing weak neutral currents and furnishes a basic ingredient for the unification of electromagnetic and weak interactions.

The main arguments in favour of the introduction of color concern the lifetime of π^0 ($\pi^0 \rightarrow 2\gamma$); the suppression of the Adler anomalies and the possibility to renormalize the Weinberg-Salam model for the unification of electromagnetic and weak interactions; and the possibility to construct a (dynamical) gauge theory of strong interactions, the so-called quantum chromodynamics (QCD).

A further test (both for flavor and color) deal with the ratio $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ for the production rates of hadrons and muons in e^+e^- annihilations. If leptons and quarks are pointlike particles, then $R = \Sigma_i (1+\varepsilon)Q_i^2$, where the sum over i extends on the flavors and colors of the involved quarks and ε is a QCD correction. For energies between 2 and 3 GeV, we have $R_{exp} \sim 2.5$ whereas $R_{the} \sim 2/3$ (or 2) when using the three light quarks without (or with) color.

A final argument in favour of color is provided by the rule $\Sigma\, Q(\text{quark}) + \Sigma\, Q(\text{lepton}) = 0$ for each generation. In the $SU(2) \times U(1)$ model of Weinberg and Salam, such a rule comes out from a renormalizibility condition and is satisfied by the four quarks (u, d, c, s) with color and the four leptons ($e^-, \nu_e, \mu^-, \nu_\mu$).

<u>A word about quantum chromodynamics</u>. To supplement the quark model by a dynamic theory requires a field of force to bind quarks together. This may be achieved with the help of nonabelian gauge field theories developed by Yang and Mills in 1954. The quanta of the color interaction field are vector bosons ($J = 1$) with zero mass and correspond to the gluons already mentioned. They are to QCD what the photon is to QED (quantum electrodynamics). The

net difference between QED and QCD is that QED responds to an abelian gauge theory so that the QED quantum (the photon) does not carry an (electromagnetic) charge while QCD responds to a nonabelian gauge theory so that the QCD quanta (the gluons) carry (color) charges. Indeed, each gluon is of type $q^\alpha \overline{q^\beta}$ (α, β = blue, yellow, or red) and hence carries two color charges. The nonabelian character of QCD follows from the fact that the eight quanta $q^\alpha \overline{q^\beta}$ (nine objects minus the white combination $\delta_{\alpha\beta}\, q^\alpha \overline{q^\beta}$) do not commute and may be associated to the regular representation 8 of a gauge group $SU(3)_c$ which constitutes an exact symmetry (to be compared with the broken symmetry $SU(n)$, $n = 3, 6, 4$, or 8). The gluons do not carry electromagnetic charges and are immune to electromagnetic and weak interactions.

A characteristic of a nonabelian gauge theory lies on the unusual behaviour of its coupling constant. In such a theory, the coupling constant vanishes when the interaction distance decreases and tends to infinity when the distance increases. (In an abelian gauge theory, the coupling constant behaves like the one of QED : it vanishes with increasing distances and tends to infinity with decreasing distances.) A nonabelian gauge theory thus allows to reproduce, not to explain, the asymptotic or ultra-violet freedom (connected to the scaling property) at small distances and the infra-red esclavery at large distances. Another interest in using a gauge theory for describing strong interactions is that gauge theories are also employed for electromagnetic and weak interactions. Therefore, we forsee an open way to unify the strong ($SU(3)_c$ invariant), weak ($SU(2)$ invariant), and electromagnetic ($U(1)$ invariant) interactions. It is intuitive that the more economical unification consists to enlarge the local symmetry $SU(3)_c \times SU(2) \times U(1)$ to $SU(5)$ as shown by Georgi and Glashow in 1975.

To close this paper, we mention several models for confinement of quarks (and gluons). (i) The nonrelativistic confining potential models which use potentials of type $A/r + B + Cr^x$ $(x > 0)$. (ii) The bag model in which the quarks confine in a bag having the size of an hadron in equilibrium under the action of an external (from an homogeneous fluid)and internal (from quarks and gluons) pressure. (iii) The string model in which a quark resembles a magnet pole and the breaking of a string amounts to produce a quark-antiquark pair. (iv) The lattice gauge theory in which the space-time continuum is mimicked by a lattice on which the quarks are distributed at distances equal to the size of an hadron and linked along the lattice lines by the color field. Note that in none of these models the confinement results from important binding energies.

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ANNEXE : THE DEUTERIUM ATOM

As an illustration of the occurrence of substructures and to celebrate the birth of a molecular-subnuclear art, we offer the following composition.