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LAMINAR FORCED CONVECTIVE/CONDUCTIVE HEAT TRANSFER BY FINITE ELEMENT METHOD

by

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Reactor Engineering Division

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Descriptors

FORCED CONVECTION

THERMAL CONDUCTION

INCOMPRESSIBLE FLOW

LAMINAR FLOW

ISOTHERMAL PROCESSES

NAVIER-STOKES EQUATION

HEAT EXCHANGERS

REACTOR COMPONENTS

FBR TYPE REACTORS

PIPES

FINITE ELEMENT METHOD

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ABSTRACT:

The present study is directed at developing a finite element computer program for solution of decoupled convective/conductive heat transfer problems. Penalty function formulation has been used to solve momentum equations and subsequently transient energy equation is solved using modified Crank-Nicolson method. The optimal upwinding scheme has been employed in energy equation to remove oscillations at high Peclet number.

NOMENCLATURE:

U_i	Velocity in X_i co-ordinate direction
P_f	Fluid density
σ_{ij}	Stress tensor
p	pressure
ρ_f	Viscosity of fluid
δ_{ij}	Kronecker delta
f_i	body force vector per unit volume
g_i	prescribed velocity in X_i - co-ordinate direction
h_i	prescribed surface traction
n_i	Unit outward normal
λ	Penalty parameter
Γ_1, Γ_2	Surface of fluid domain
W	test function, wighting functions
N_i	Shape function
ε	error tolerance

C	constant
R_e	Reynolds number
C_{pf}	Specific heat of fluid
K_{ijf}	Thermal Conductivity tensor of fluid
T	Temperature
t	time
T_s	specified temperature
q_s	specified heat flux
Q	volumetric heat generation
ϵ	error vector
h_x, h_y	side length of element in X and y co-ordinate direction
P_x, P_y	element Peclet numbers.

2. INTRODUCTION:

Convective problems are divided into two categories depending upon the force responsible for fluid motion. The momentum and energy equations are one way coupled in forced convection problems and two way coupled in natural convection problems. This decoupling allows the solution of Navier-Stoke's equations separately than energy equation.

The combined heat transfer due to convection and conduction in a region consisting of both solid body and moving fluid is of great interest in engineering practice, a typical example in nuclear reactor are heat transfer in component of heat exchanger, primary heat transport pipe in fast breeder reactor. The current practice in transient temperature distribution is obtained on the basis of an estimated heat flux at the solid/fluid interface. An assumed fluid temperature distribution and an average convective heat transfer coefficient are normally used for heat flux calculations.

The analyst is advised to use "accepted heat transfer practices" to find the temperature distribution for the ASME Boiler and Pressure Vessel code and is instructed to use the average wall temperature in 'primary stress' calculations, the average gradient through the wall for "equivalent linear stress" calculations and to lump the rest into "local thermal stress" calculations. Whether or not the current analytical practice is conservative with regard to these quantities is uncertain.

In order to eliminate the uncertainty and possible errors introduced by the interface heat flux estimate, the coupled convective and conductive heat transfer has been studied. The transportive velocity field of the moving fluid, in decoupled problem may be assumed to be known prior to analysis.

The numerical solution of convective-diffusion equation present serious difficulties when the convective term is dominant. This difficulty stem from the combination of essentially elliptic and parabolic nature of two terms present in energy equation and yield an oscillatory solution, whenever mesh size exceed a certain critical value. Roache /1/ has described the spatial oscillations as "wiggles" and has given the condition on the cell Reynold number in case of one-dimensional vorticity equation. In finite difference method this difficulty has been removed by using an upwind differencing formulation for advection terms /2/, /3/. A possible way to overcome this difficulty in the context of finite element was suggested by Zienkiewicz et al. /4/ on the same ground as given by spadling and Runchal.

In essence the procedure applies the finite element method using weighting functions of non-symmetric forms different from those originally used in shape functions. By suitable choice of such shape functions stability can be established with accuracy loss that is inherent in finite difference upwinding.

The paper has been divided into two sections. Section-3.0 deals with the solution of momentum equations and Section-4.0 describes the solution of transient energy equation.

3.2 PENALTY FUNCTION FORMULATION:

In penalty-function formulation of Navier-Stokes equation, the constitutive eqn. is replaced by

$$\sigma_{ij}^{(\lambda)} = - p^{(\lambda)} \delta_{ij} + 2 \mu_f \left(\frac{\partial u_i^{(\lambda)}}{\partial x_j} + \frac{\partial u_j^{(\lambda)}}{\partial x_i} \right) \qquad \ldots(5)$$

in which $$p^{(\lambda)} = - \lambda \frac{\partial u_i^{(\lambda)}}{\partial x_i} \qquad \ldots(6)$$

Furthermore the incompressibility condition is dropped. The boundary value problem of the penalty function formulation is recasted as follows:

$$\rho u_j^{(\lambda)} \frac{\partial u_i^{(\lambda)}}{\partial x_j} = \rho f_i + \frac{\partial \sigma_{ij}^{(\lambda)}}{\partial x_j} \qquad \ldots(7)$$

$$u_i^{(\lambda)} = g_i \quad on\ \Gamma_1$$
$$\sigma_{ij}^{(\lambda)} n_j = h_i \quad on\ \Gamma_2 \qquad \ldots(8\ a,b)$$

Only the essential feature of penalty function formulation will be presented here. For more detail the readers are advised to refer the excellent work of Huges et al. /7/.

Let W_i demote the set of test function for U_i, multiplying eqn.(7) by W_i and integrating over domain one gets

$$\int_V W_i \left(\rho_f u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} + \mu_f \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) dV = 0 \qquad \ldots(9)$$

Applying divergence theorem for pressure and viscous terms, we can write

$$\int_V \left[\rho_f W_i u_j \frac{\partial u_i}{\partial x_j} - p \frac{\partial W_i}{\partial x_i} + \mu_f \frac{\partial W_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dV -$$

$$\int_\Gamma W_i \left\{ - p + \mu_f \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\} n_i \, d\Gamma = 0 \qquad \ldots(10)$$

Substituting eqn. (6) in eqn. (10), we get

$$\int_V \left[\rho_f W_i u_j \frac{\partial u_i}{\partial x_j} + \mu_f \frac{\partial W_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dV +$$

$$\lambda \int_V \frac{\partial W_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} dV = \int_\Gamma W_i h \, d\Gamma \qquad \ldots(11)$$

Eqn.(11) is penalty function formulation for the solution of eqns. (1-3). It is important to note that eqn.(11) will not give the solution of original problem and solution will be approximate one. Some of the essential features of this method are, bigger the value of Penalty parameter λ, the incompressibility constraints are satisfied more accurately and solution will be better one. Pressure is eliminated and hence fewer degree-of-freedom, encourage to attack real life problems. It does not require complex storage and solution technique as compared to primitive variable formulation.

3.3 FINITE ELEMENT EQUATIONS:

The finite element method consists of subdividing the domain into a number of subdomains, called finite elements. The elements are connected at a number of points, called nodes. The dependent variables are approximated by continuous functions in the element in terms of variables at nodes as:

$$u = \sum_{i=1} N_i u_i$$

$$v = \sum_{i=1} N_i v_i \qquad \ldots(12)$$

Substituting (12) in eqn. (11), a set of non-linear algebraic equation can be obtained.

Finally the contribution of each element property added through the connection of appropriate nodes which yields to matrix problem.

$$\underset{\approx}{C} \cdot \underset{\sim}{V} + \underset{\sim}{H}(V) = \underset{\sim}{F} \qquad \ldots(13)$$

where

$$\underset{\approx}{C} = \mathop{\mathrm{A}}_{e=1}^{nel} \underset{\approx}{C}^e \,,\quad \underset{\sim}{H}(v) = \mathop{\mathrm{A}}_{e=1}^{nel} \underset{\approx}{H}^e(v) \;;\quad \underset{\sim}{F} = \mathop{\mathrm{A}}_{e=1}^{nel} \underset{\sim}{F}^e$$

$$\underset{\approx}{C}^e_{ab} = \int_V \underset{\approx}{B}_a^T \underset{\approx}{D}_\lambda \underset{\approx}{B}_b \, dV + \int_V \underset{\approx}{B}_a^T \underset{\approx}{D}_\mu \underset{\approx}{B}_b \, dV$$

$$\left(\underset{\sim}{H}^e(v)\right)_{ab} = \int_V \rho_f N_a u_j \frac{\partial u_i}{\partial x_j} \, dV \qquad \ldots(14)$$

$$\underset{\sim}{F}^e = \int_\Gamma N_a h_i \, d\Gamma$$

A denotes the "assembly operator".

3.4 SOLUTION PROCEDURE:

The basic numerical problem associated with solution of Navier-Stokes problem is solution of non-linear equation. Newton-Raphson method has been extensively used for solving nonlinear equations. The major drawback in Newton-Raphson method is the convergence requirement that the initial solution should be close to the actual solution. In Navier-Stoke's equation an obvious choice for initial solution is solution of momentum equation at zero Reynolds number (creeping flow or Stokes flow problem). However, several investigators /8/, /9/, /10/ have reported that the zero Reynold number problem is not an appropriate choice for the initial solution.

In view of above difficulties, an incremental Newton-Raphson scheme in which density is used as a "load parameter" has been used, within each load level we iterate until convergence is achieved. The converge solution is used as the initial guess for the next load level.

In a typical load level, the equation solved is

$$\underset{\approx}{C}^{*}(\bar{v})\,\Delta\underset{\sim}{V} = \Delta\underset{\sim}{F} \qquad \ldots(15)$$

$$\bar{\underset{\sim}{V}} \leftarrow \bar{\underset{\sim}{V}} + \Delta\underset{\sim}{V} \qquad \ldots(16)$$

where $\bar{V}$ is the latest approximation to the solution. The unbalance force is estimated as

$$\Delta\underset{\sim}{F} = \underset{\sim}{F} - \underset{\approx}{C}\cdot\bar{\underset{\sim}{V}} - \underset{\sim}{H}(\bar{V}) \qquad \ldots(17)$$

$$\underset{\approx}{C}^{*}(\bar{\underset{\sim}{V}}) = \underset{\approx}{C} + \underset{\approx}{J} \qquad \ldots(18)$$

where $\underset{\approx}{J} = \mathop{A}_{e=1}^{nel} \underset{\approx}{J}^{e}$ (Tangent convection or Jacobian matrix).

$$J^{e}_{ab} = \rho\int_{v} N_a \left[\frac{\partial u_i}{\partial x_j} N_b + \delta_{ij}\left(u_k \frac{\partial N_b}{\partial x_k}\right)\right] dv \qquad \ldots(19)$$

The criteria for iterative convergence is expressed as

$$\max_{i} \frac{|u_i^{n+1} - u_i^{n}|}{|u_i^{n}|} \leq \varepsilon \qquad \ldots(20)$$

where ε is small number

For high Reynold number the penalty parameter is selected as

$$\lambda = c\,\max(\mu, \mu Re) \qquad \ldots(21)$$

where C is constant normally used 10^8 and Re $= \frac{UL}{\nu}$,
U and L are characteristic velocity and length respectively.

3.5 NUMERICAL EXAMPLES:

There are several simple problems for which either analytical solution or numerical solution exist. Seven problems have been solved to test the computer program developed by authors. The most important investigation is accuracy of solution using crude meshes as compared to other investigators.

3.5.1 Stokes flow in channel:

This problem was chosen to test axisymmetric form of Navier-Stokes equation. The horizontal velocity at entrance is unity ($u = 1$, $v = 0$). The geometry, mesh and boundary conditions are shown in fig.1. Velocity profile at X = 4.0 and at exit are shown in Fig.2. The results match well with analytical result.

3.5.2 Solution of Burgar's equation:

Burger's equation is similar to one dimensional Navier-Stokes equation. This problem was solved to check the computer program for nonlinear termed present in momentum eqn. The solution matches very well with closed from solution and from ref. /11/. for two different Reynold number.

3.5.3 Flow through constricted tubes:

This problem has been taken to check the computer program for axisymmetric problem. Solution of the axisymmetric flow of a viscous incompressible flow through a locally constricted circular cylindrical tube is presented for low Reynold numbers in the range of 0-25. The bell-shaped constriction is specified by a Gaussian normal distribution curve, with diameter ratio ½ (the diameter of the narrowest section equal to ½ of the original cylinder diameter). The Flow is symmetric at zero Reynold number as shown in Fig.5. Flow becomes asymmetric at Reynold number 25 as shown in Fig.6. These results match very well with ref. /12/.

3.5.4 Driven Cavity Flow:

This problem has been solved almost by everyone to test the numerical schemes and compare the solution obtained by various other schemes. The geometry and boundary conditions are shown in Fig.7. This problem has been analysed using 42 linear elements. The velocity profiles for 'u' at middle plane have been shown in Fig.9 for Re = 0, 400 and 700. Fig.10 shows the vertical velocity profile at mid plane. The solution deteriorates when Re $>$ 700. It means that flow is advection dominated. The upwinding or fine mesh may yield the accurate result at higher Re number.

3.5.5 Developing Channel Flow:

This problem has been solved by Gatling /5/. The geometry and boundary conditions are shown in Fig.10. The flow is having a uniform velocity of 1 unit at entrance. At the exit boundary condition of fully developed parallel flow at infinity is replaced by a zero normal stress condition.

Velocity profile for a Reynold number of 333 (based on half channel height) are plotted in Fig.11 at various distance measured from entry. The results match very well with Gartling /5/. The results of this example was also compared with Schlichting /13/. This problem was solved using 52 linear elements.

3.5.6 Flow through sudden expansion:

This problem has been solved by Kawahara /10/ and Huyakorn et al /14/. The geometry, mesh and boundary conditions are shown in Fig.12. The problem was analysed for low Reynold's number = 60. The horizontal velocity profile at various sections are shown in Fig.13. The problem was solved with 10 biquadratic elements. The results match very well compared to ref. 10 and /14/.

3.5.7 Flow past a cylinder:

All above examples are concerned with calculation at internal flow problems. In many situation fluid may be exterior to the body. The flow past a circular cylinder is one of the examples of external flow problem. The geometry meshes and boundary condition used for this study are shown in fig.14. In present study 49 linear elements have been used compared to 42 parabolic elements by Taylor and Hood /15/. The flow was analysed for Reynold number of 100. The plot of velocity vectors are shown in Fig.15.

4.0 BASIC EQUATION AND FINITE ELEMENT FORMULATION OF ENERGY EQUATION:

Neglecting the dissipation of energy in the system, the governing partial differential equation of the coupled convective and conductive heat transfer problem in 2-dimension is

$$\rho_f C_p \frac{\partial T}{\partial t} + \rho_f C_p u_j \frac{\partial T}{\partial x_j} + \frac{\partial q_i}{\partial x_i} - Q = 0 \qquad 4.1(a)$$

with $$q_i = - K_f \frac{\partial T}{\partial x_i} \qquad 4.1(b)$$

and in solid region

$$\rho_s C_p \frac{\partial T}{\partial t} + \frac{\partial q_i}{\partial x_i} - Q = 0 \qquad 4.1(c)$$

Substituting eqn.(4.1b) in (4.1a) we get

$$\rho_f C_p \frac{\partial T}{\partial t} + \rho_f C_p u \frac{\partial T}{\partial x} + \rho_f C_p v \frac{\partial T}{\partial y} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + Q \qquad 4.2(a)$$

subjected to the boundary conditions.

$$T = T_s \quad \text{on } \Gamma_1$$

$$-k_x \frac{\partial T}{\partial x} l_x - k_y \frac{\partial T}{\partial y} l_y = \dot{q}_s \quad \text{on } \Gamma_2 \qquad 4.2(b,c)$$

The domain V is mathematically subdivided into an arbitrary number of geometrically simple regions, the modified weight residual procedure transform the governing partial differential equation into a set of ordinary differential equations and leads to a finite element formulation of the coupled heat transfer problem.

The equation (4.1) is written in operator form as

$$\mathcal{L}(T) - Q = 0 \tag{4.3a}$$

where

$$\mathcal{L} = \rho_f c_p \frac{\partial}{\partial t} + \rho_f c_p u \frac{\partial}{\partial x} + \rho_f c_p v \frac{\partial}{\partial y} - \frac{\partial}{\partial x}\left(k_x \frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial}{\partial y}\right) \tag{4.3b}$$

and let us assume a temperature distribution within element

$$\hat{T}(x,y,t) = \sum_{i=1}^{node} N_i T_i \tag{4.4}$$

If assumed temperature is substituted in (4.3a) the equation will not be satisfied and an error vector ϵ will be generated.

$$\mathcal{L}(\hat{T}) - Q = \epsilon \tag{4.5}$$

Now construct an orthogonal projection of each of this residual spaces onto subspace spanned by appropriate weighting functions. Constructing an inner product we get

$$\langle \epsilon, W \rangle = \int_V \epsilon W \, dV = 0 \tag{4.7}$$

or

$$\int_V \left[\rho_f c_p \frac{\partial \hat{T}}{\partial t} + \rho_f c_p u \frac{\partial \hat{T}}{\partial x} + \rho_f c_p v \frac{\partial \hat{T}}{\partial y} - \frac{\partial}{\partial x}\left(k_x \frac{\partial \hat{T}}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial \hat{T}}{\partial y}\right) - Q \right] W \, dx dy = 0 \tag{4.8}$$

where W is suitable weighting function.

Through the application of divergence theorem, this can be written as

$$\int_V \left[\rho_f c_p \frac{\partial \hat{T}}{\partial t} W + \rho_f c_p u \frac{\partial \hat{T}}{\partial x} W + \rho_f c_p v \frac{\partial \hat{T}}{\partial y} W + k_x \frac{\partial \hat{T}}{\partial x} \frac{\partial W}{\partial x} + k_y \frac{\partial \hat{T}}{\partial y} \frac{\partial W}{\partial y} - Q W \right] dx dy - \int_{\Gamma_2} \left[k \frac{\partial \hat{T}}{\partial n} n + q_s \right] W \, d\Gamma = 0 \tag{4.9}$$

Substituting (4.4) in (4.9) and carryind out integration, the governing matrix equation of the coupled heat transfer problem can be written as

$$\underset{\approx}{C} \cdot \dot{\underset{\sim}{T}} + \underset{\approx}{K} \cdot \underset{\sim}{T} + \underset{\approx}{K_v} \underset{\sim}{T} = \underset{\sim}{R} \qquad (4.10)$$

where $\underset{\approx}{C} = \overset{n}{\underset{e=1}{A}} \underset{\approx}{C}^e$; $\underset{\approx}{K} = \overset{n}{\underset{e=1}{A}} \underset{\approx}{K}^e$, $\underset{\approx}{K_v} = \overset{n}{\underset{e=1}{A}} K_v^e$ and

$\underset{\sim}{R} = \overset{n}{\underset{e=1}{A}} \underset{\sim}{R}^e$

n is total number of elements.

The matrix $\underset{\approx}{C}$ and $\underset{\approx}{K}$ are heat capacity and conductivity matrix respectively, $\underset{\approx}{K_v}(v)$ is advection matrix which depends upon the fluid velocity is due to mass transfer, $\underset{\sim}{R}$ is system of flux vector. The element matrices are given by

$$\underset{\approx}{C}^e_{ab} = \int_V N_a \rho c_p W_b \, dV \qquad (4.11)$$

$$\underset{\approx}{K}^e_{ab} = \int_V \left[\frac{\partial N_a}{\partial x} k_x \frac{\partial W_b}{\partial x} + \frac{\partial N_a}{\partial y} k_y \frac{\partial W_b}{\partial y} \right] dV \qquad (4.12)$$

$$(\underset{\approx}{K}^e_v)_{ab} = \int_V \left[u \frac{\partial N_a}{\partial x} W_b + v \frac{\partial N_a}{\partial y} W_b \right] dV \qquad (4.13)$$

$$\underset{\sim}{R}^e_a = \int_V Q W_a \, dV \qquad (4.14)$$

4.1 Weighting Function for isoparametric linear elements:

The weighting function for 2-dimensional, isoparametric element can be constructed in terms of product of one dimensional function /16/.

The shape function for node Ni for node $\xi = \eta = -1$ is

$$N_i = \left(\frac{1-\xi}{2}\right)\left(\frac{1-\eta}{2}\right) = n_i(\xi) \cdot n_i(\eta)$$

where

$$n_i(\xi) = \frac{1-\xi}{2} \quad ; \quad n_i(\eta) = \frac{1-\eta}{2}$$

similarly the weighting function can be written as

$$W_i = \omega_i(\xi) \cdot \omega_i(\eta)$$

where

$$\omega_i(\xi) = n_i(\xi) + \alpha_{il}\, 3(1-\xi^2)/4$$

$$\omega_i(\eta) = n_i(\eta) + 3\alpha_{im}(1-\eta^2)/4$$

The suffix l (or m) is that of adjacentnode laying along the same element side and

$$|\alpha_{il}| = |\alpha_{li}|$$

$$|\alpha_{im}| = |\alpha_{mi}|$$

The sign of α coefficient dependent on the direction of the velocity vector. The velocity vector will be variable in general, hence the average velocity vector along the element side ij, i.e. V_{ij} and given to α , a sign identical to that of the vector.

The nodal velocity vector V_i will be available and we evaluate V_{ij} as

$$V_{ij} = (V_i + V_j)\, I_{ij}/2$$

where I_{ij} is vector specifying the direction of the side which is calculated from the nodal coordinates. The four independent values of α can be given an absolute value ranging from 0 (no upwinding) to 1 (full upwinding). However, an optional value of a α for non-oscillatory solution can be obtained based on mesh size and local velocity /16/ and expressed as

$$\alpha = \coth(P/2) - 2/P$$

where $P_x = \frac{\rho c_p u h_x}{k_x}$ and $P_y = \frac{\rho c_p v h_y}{k_y}$

u and v are average velocity along an element side h_x and h_y in x and y direction respectively. It has been observed that optimal parameter gives better results than full upwinding used in finite difference scheme.

4.2 WEIGHTING FUNCTION FOR PARABOLIC ISOPARAMETRIC ELEMENT:

The upwind nine-noded element has been proved to achieve stability in all the situation than eight noded element. Hence, in present analysis only 9 noded element has been considered to model fluid regime and eight noded elements to model solid region. The idea of previous section has been extended for quadratic elements.

The shape function for the Ni node $\xi = \eta = -1$ is

$$N_i = n_i(\xi) \cdot n_i(\eta)$$

$$n_i(\xi) = \xi(\xi-1)/2 \quad ; \quad n_i(\eta) = \eta(\eta-1)/2$$

and weighting function can be written as

$$W_i = \omega_i(\xi) \cdot \omega_i(\eta)$$

$$\omega_i(\xi) = \xi(\xi-1)/2 - 5\alpha_{ij}\,\xi(\xi^2-1)/8$$

$$\omega_i(\eta) = \eta(\eta-1)/2 - 5\alpha_{ik}\,\eta(\eta^2-1)/8$$

For mid-side node, at $\xi = 0$, $\eta = -1$, the shape function

$$N_k = n_k(\xi) \cdot n_k(\eta)$$

$$n_k(\xi) = 1-\xi^2 \quad ; \quad n_k(\eta) = (1-\eta)/2$$

and weighting function can be written as

$$W_k = \omega_k(\xi) \cdot \omega_k(\eta)$$

$$\omega_k(\xi) = (1-\xi^2) + 5\beta_{km}\,\xi(\xi^2-1)/2$$

$$\omega_k(\eta) = (1-\eta)/2 + 5\beta_{kl}\,\eta(\eta^2-1)/2$$

and optional values of parameters are

$$\beta = \coth(P/4) - 4/P$$

$$\alpha = 2\tanh(P/2)(1+3\beta/P+12/P^2) - \frac{12}{P} - \beta$$

which is derived from the exact solution of one dimensional diffusion-convection equation.

4.3 Solution Procedure:

It can be seen by inspection that the advection matrix $K_V(V)$ due to moving fluid is in general, asymmetric. Tay and Davis /17/ solved the asymmetric system directly, using a banded Gaussian elimination procedure.

Three different alternatives can be used solve (4.10):

1. A general, banded equation solver may be used to factor the asymmetric system.
2. The matrix $K_V(V)$ can be placed on the right hand side of (4.10). Multiplied by some appropriate temperature field and treated as

a 'Initial flux' vector.

3. The symmetric part of $\underset{\approx}{K}_v(V)$ may be added to $\underset{\approx}{K}$ and skew-symmetric part is treated as in 2.

Gartling has suggested to use the first method on the basis of lesser oscillations at high Peclet no. Therefore, the present development has concentrated on solving eqn. (4.10) as an unsymmetric system.

To find transient solution to (4.10), modified Crank-Nicolson method was used. The modified Crank-Nicolson algorithm /18/ is given as

$$\underset{\approx}{K}_{eff} \cdot (\underset{\sim}{T})_{t+\frac{\Delta t}{2}} = (\underset{\sim}{F}_{eff})_{t+\frac{\Delta t}{2}} \qquad (4.5)$$

where

$$\underset{\approx}{K}_{eff} = \underset{\approx}{K} + \underset{\approx}{K}_v + \frac{2}{\Delta t}\underset{\approx}{C}$$

$$\underset{\sim}{F}_{eff} = 2/\Delta t \ \underset{\approx}{C}(\underset{\sim}{T})_t + (\underset{\sim}{R})_{t+\Delta t/2}$$

4.4 SELECTION OF MESH SIZE AND TIME STEP:

In the solution of steady state vorticity transport equation with a mesh size of h, the solution will be non-oscillatory if local or element Peclet Number is less than 1. i.e.

$$P = \rho c_p u h / k < 1$$

The above expression can be violated in case of upwind scheme. Similarly for transient analysis the time step Δt should be less than $\frac{\rho c_p h^2}{k}$. The above criterion has been nicely described in reference /1/.

4.5 Examples:

4.5.1 SOLUTION OF ONE DIMENSIONAL TRANSPORT EQUATION:

The one dimensional steady state vorticity equation/diffusion-Convection eqn. is

$$a \frac{\partial^2 \xi}{\partial x^2} - u \frac{\partial \xi}{\partial x} = 0$$

Where u is advection speed and a is generalised diffusion coefficient. Although vorticity does not exist in one dimensional situations, but this equation model some aspects of multidimensional equations. The advection speed is constant in x or may be function of x. The study of one dimensional vorticity equation is convenient to study its limitations.

Let us consider in the first problem the solution of steady state equation described above. For u = 0, the finite element solution gives exact continuum solutions, as shown in Fig.16. As wind blows harder i.e. $u > 0$, the ξ profile is blown downstream, as in the continuum solution. The solutions were obtained at various values of $a/u = 0, 0.1, 0.2, \ldots \ldots 0.01$.
The solution matches very well for $a/u = 0.0, 0.1, \ldots 0.05$ /1/. At $a/u = 0.01$ severe oscillations have been observed as shown in Fig.17 and the wiggles are totally absent in case of upwind finite element solution. Of course, at higher advection velocity, the wiggles will be more pronounced and upwinding is the only remedy. The results shown in Fig.17 for $\frac{a}{u}$ = 0.001 clearly indicates that no oscillations are present.

The second problem is the solution of one dimensional concentration problem reported in /19/. The governing differential equation is

$$D' \frac{\partial^2 C}{\partial x^2} - u \frac{\partial C}{\partial x} = \frac{\partial C}{\partial t}$$

where C(x, t) is concentration of solute in the fluid, D' is a dispersion coefficient, u is a constant average bulk fluid velocity, C_1 is the input concentration, and Co is the initial concentration of solute in the fluid. Fig.18 shows the finite element solution for the data shown in fig. The finite element solution and exact solution matches very well.

4.5.2 COMBINED CONDUCTION/CONVECTION BETWEEN PARALLEL PLATE AND CIRCULAR PIPE:

In this example a fully developed parabolic laminar flow of sodium between two thick parallel steel plates/pipes have been assumed (see Fig.19). A finite element mesh has been shown in Fig.20 and linear isoparametric elements inthe fluid region and in solid region have been used. Both inlet and outlet temperatures were specified and outer surfaces of plate is assumed to be perfectly insulated. The inlet velocities was assumed to be known. Fig.20 also shows the isothermal temperature lines in fluid region as well as in solid region. At higher Peclet number, the oscillations have been observed and then upwinding with optimal parameter were used to remove the oscillations. Fig.21 shows the temperature plot along the flow centre line. The effect of upwinding at higher Peclet number is worth noticing. The same problem has been solved using mixed elements (see fig.22) and results are shown in fig.23.

5.0 CONCLUSION:

During the first part of this study we have attempted to establish that penalty function formulation as an accurate, competitive and efficient method for solution of the Navier-Stoke's equations. The accuracy of the method has been established with very crude meshes employed in our study. Although it is well known that the pressure extracted from this scheme is inferior to difference method but it hardly matters in convective heat transfer problems considered by us.

Application of standard Galerkin procedures of weighting function poses certaindifficulties while solving non-self adjont energy equation. This difficulty will not be present if Nueman type boundary conditions at exit are specified, or upwinding scheme is employed. This is essential at higher Peclet number. In this report oscillations have been overcome by using optimal upwinding scheme both in linear and biquadratic elements.

6.0 ACKNOWLEDGEMENT:

The authors wish to thank Shri R.L. Sehgal for running the computer programs.

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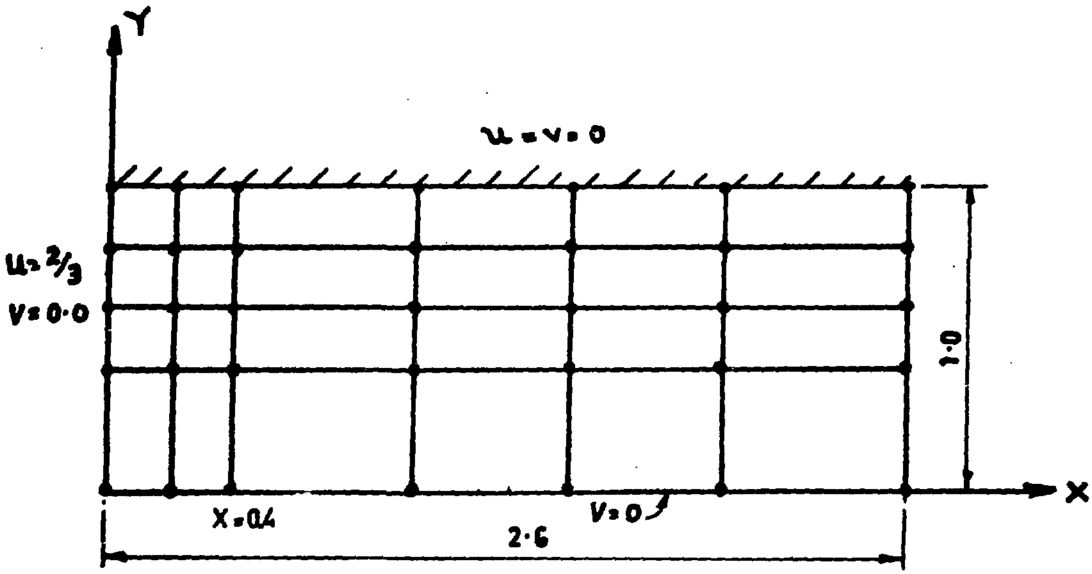

FIG. 1: MESH FOR STOKE'S FLOW THROUGH CHANNEL

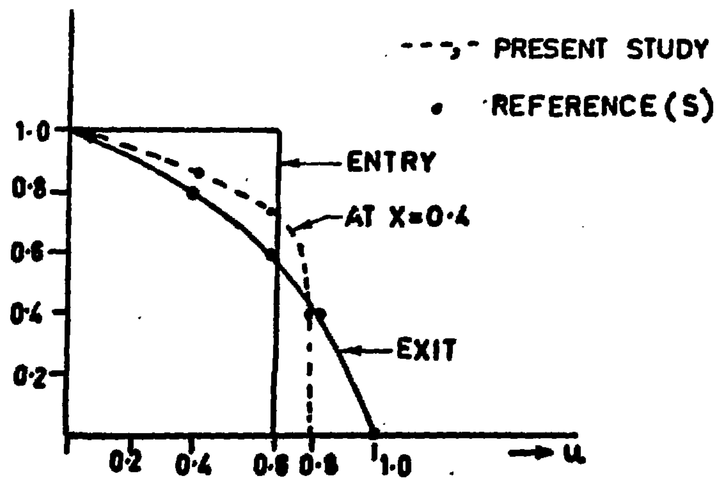

FIG. 2 : STOKE'S FLOW THROUGH A CHANNEL

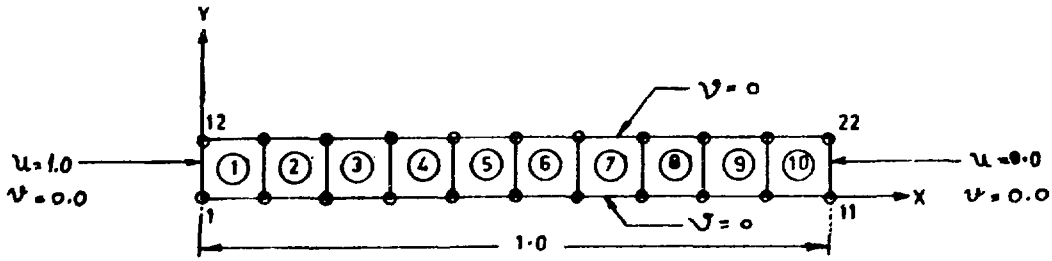

FIG.3: FINITE ELEMENT MESH AND BOUNDARY CONDITIONS FOR BURGER'S EQUATION

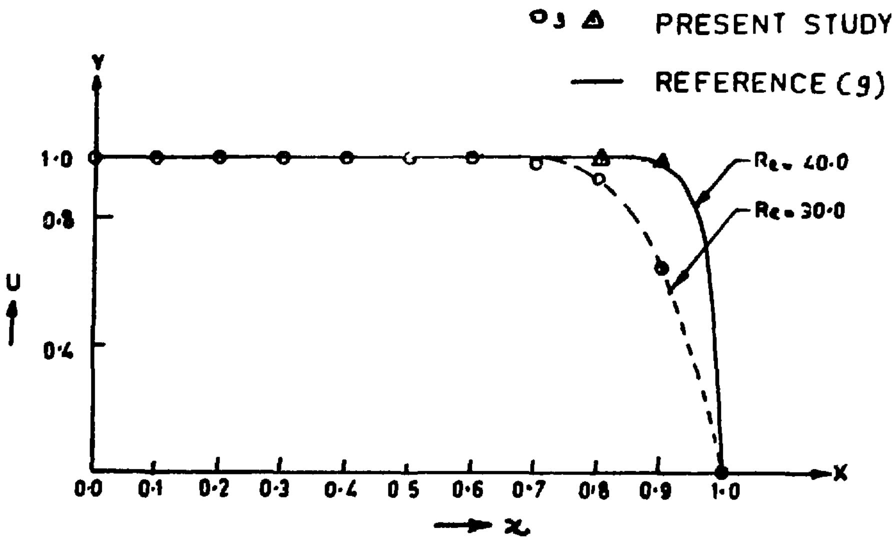

FIG.4 : FINITE ELEMENT SOLUTION OF BURGER'S EQUATION

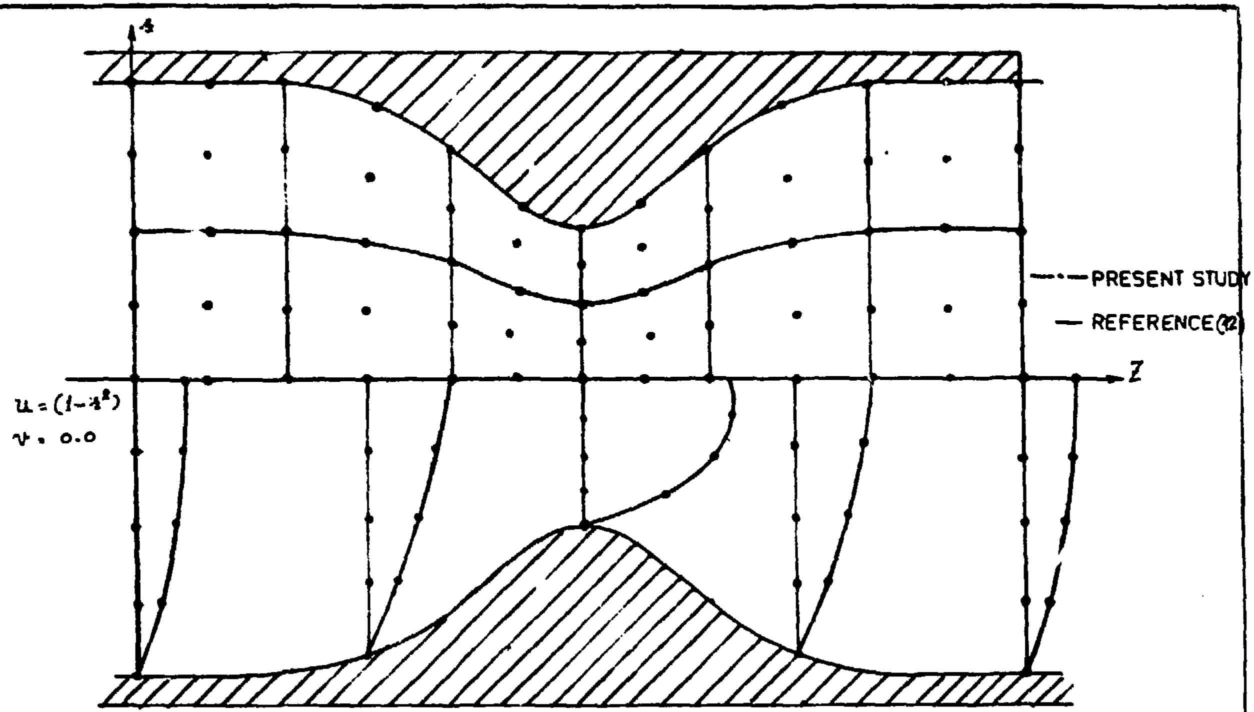

FIG. 5 : MESH AND PROFILE OF VELOCITY VECTORS Re=0

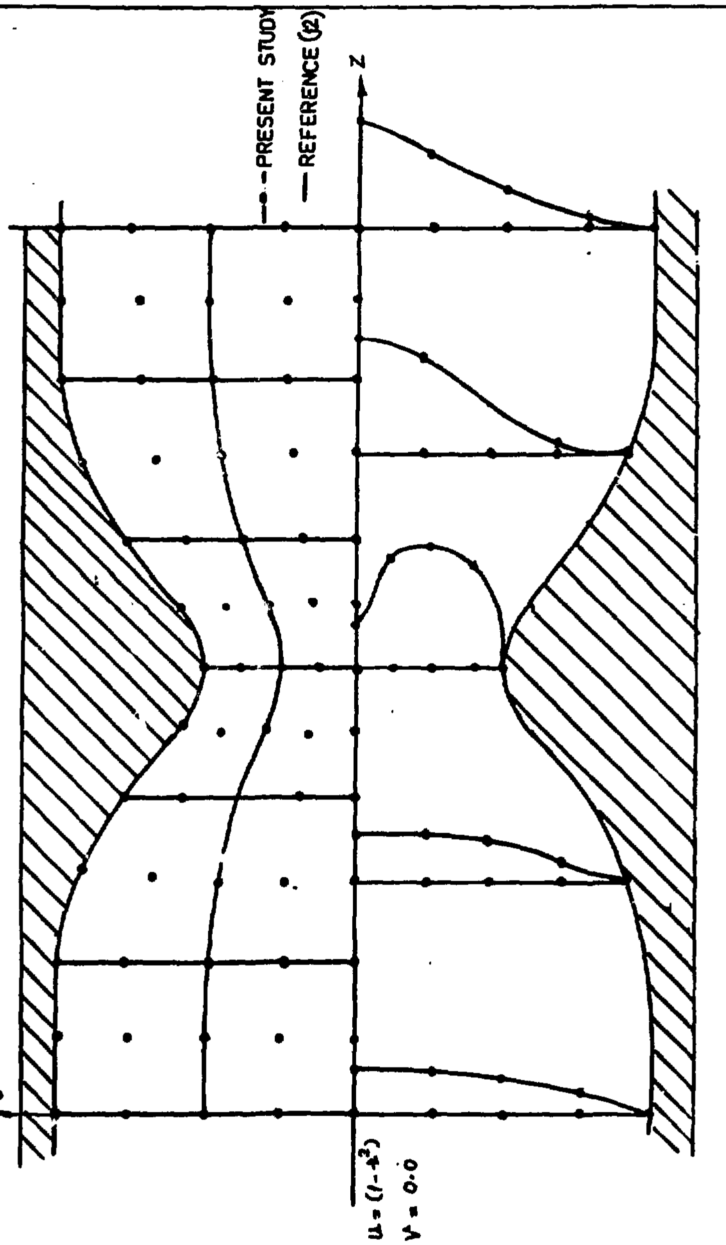

FIG. 6-1 MESH AND PROFILE OF VELOCITY VECTOR Re = 25

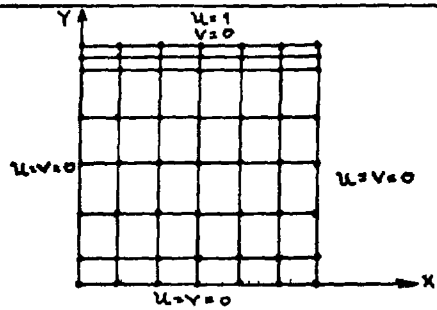

FIG. 7: FINITE ELEMENT MESH AND BOUNDARY CONDITIONS FOR DRIVEN CAVITY FLOWS

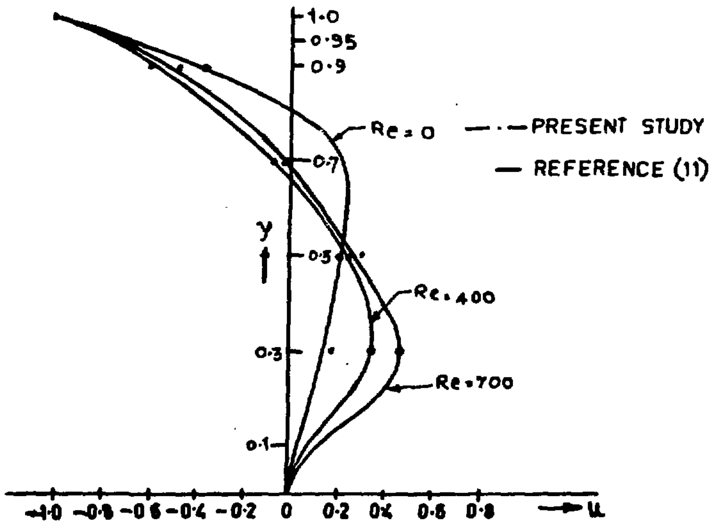

FIG. 8: DRIVEN CAVITY FLOW MIDPLANE u VELOCITY AT X= 0.5

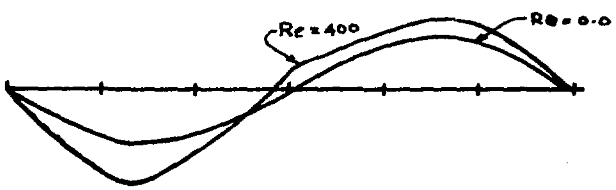

FIG. 9: DRIVEN CAVITY FLOWS MIDPLANE V VELOCITY= 0.5

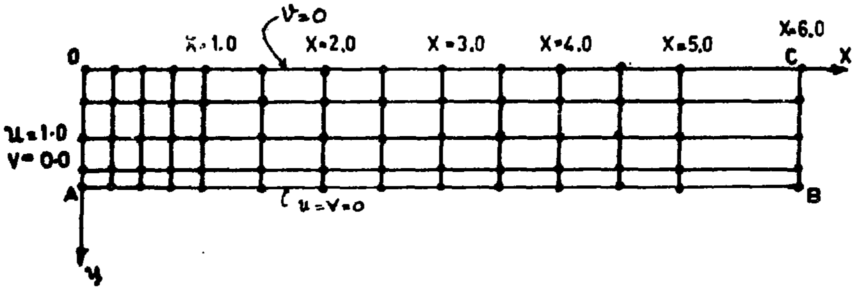

FIG. 10 : FINITE ELEMENT MESH FOR DEVELOPING FLOW IN CHANNEL

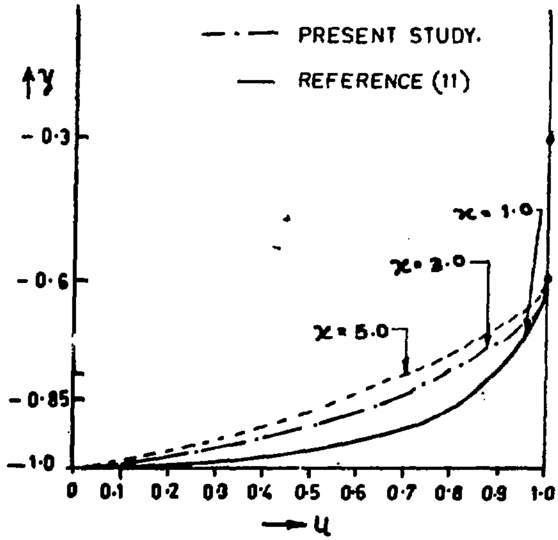

FIG. 11 : VELOCITY DISTRIBUTION FOR LAMINAR FLOW IN THE CHANNEL AT $R_E = 333$

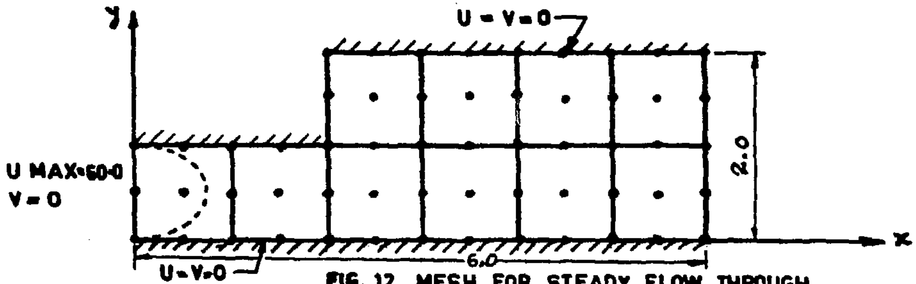

FIG. 12 MESH FOR STEADY FLOW THROUGH SUDDEN EXPANSION

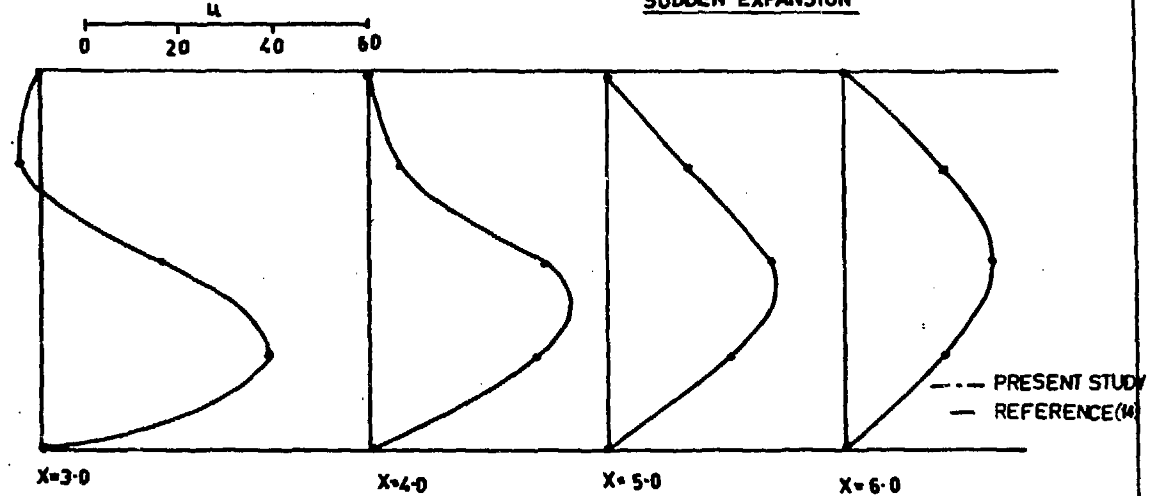

FIG. 13 COMPUTED HORIZONTAL VELOCITY PROFILE AT VARIOUS CROSS SECTION AT RE = 60

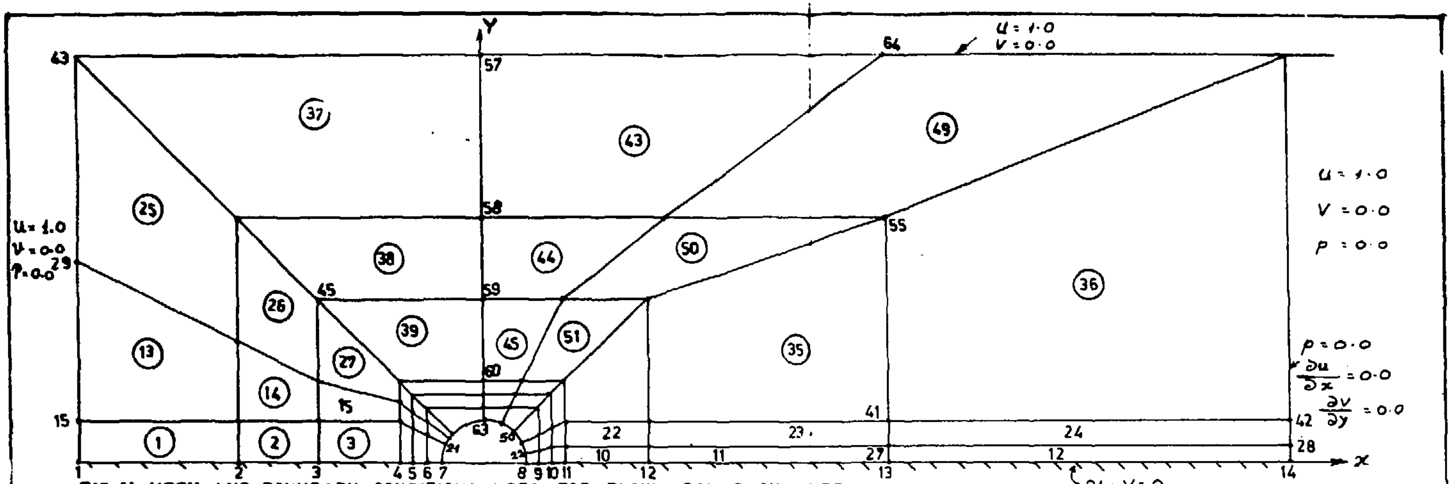

FIG. 14. MESH AND BOUNDARY CONDITIONS USED FOR FLOW AROUND CYLINDER

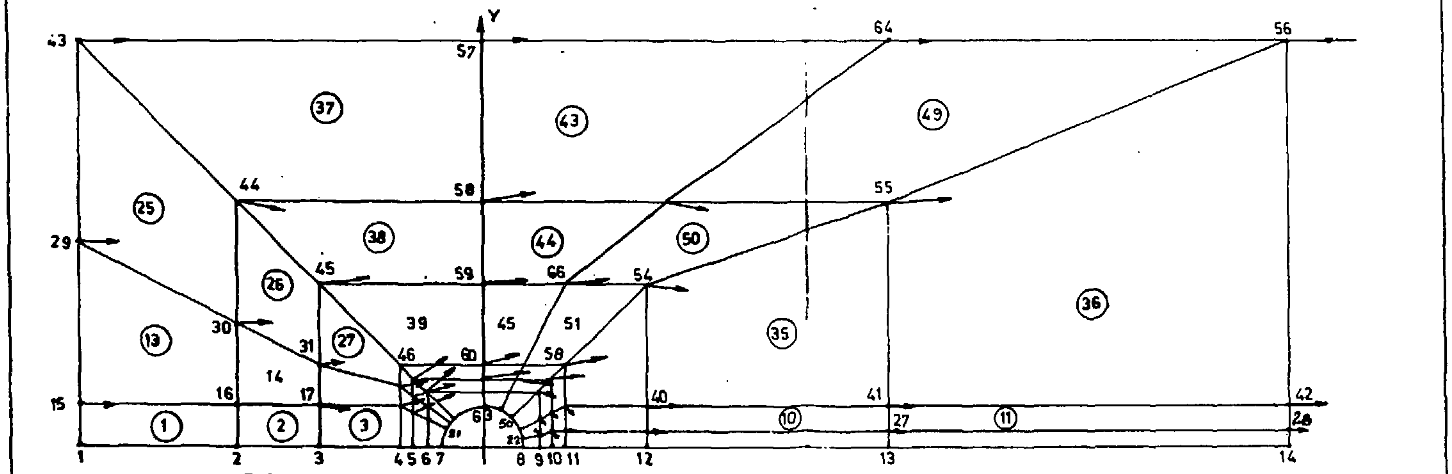

FIG. 15. VELOCITY DISTRIBUTION FOR FLOW AROUND CYLINDER

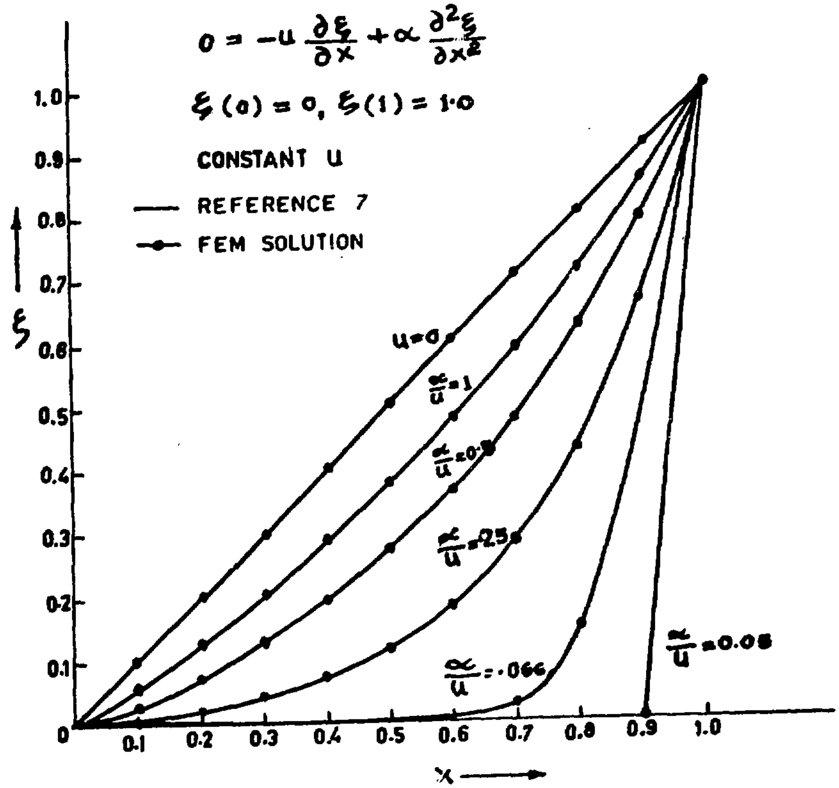

FIG.16. SOLUTION TO THE STEADY STATE LINEAR MODEL ADVECTION DIFFUSION EQUATION.

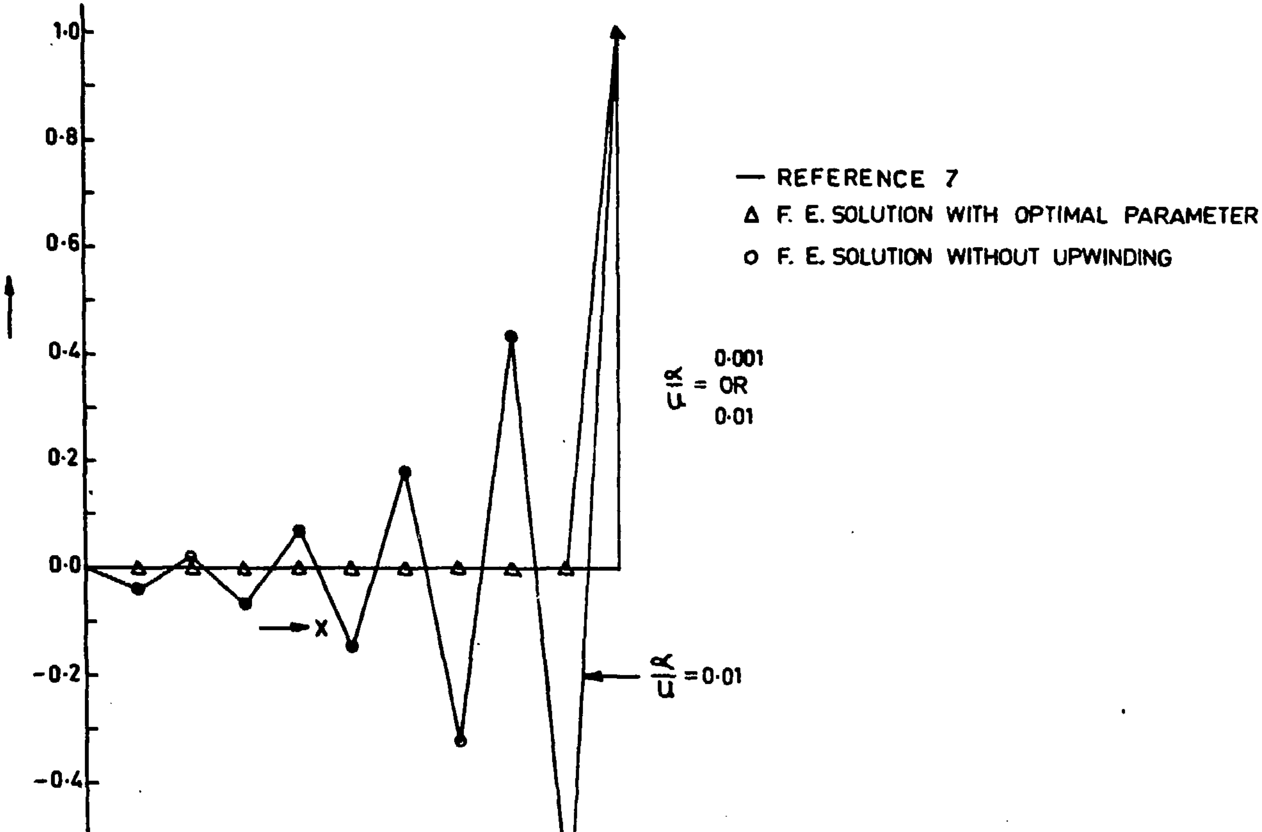

FIG.17. SOLUTION OF THE STEADY STATE LINEAR MODEL ADVECTION DIFFUSION EQUATION

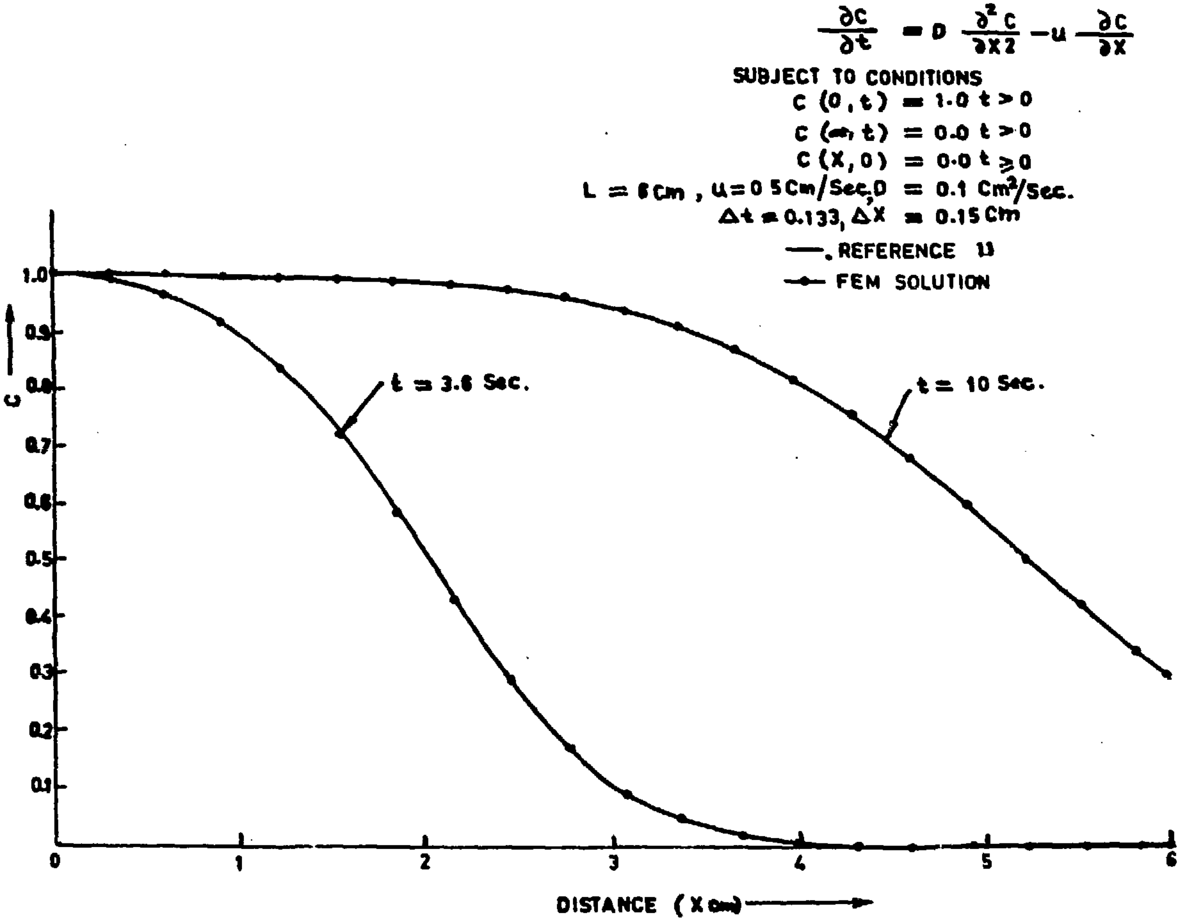

FIG.18 ONE DIMENSIONAL DIFFUSION CONVECTION PROBLEM

ρ_{SODIUM} = 58 lbm/ft^3

Cp_{SODIUM} = 33 Btu/lbm-°F

μ_{SODIUM} = 1.69 lbm/ft-hr

K_{SODIUM} = 49.8 Btu/ft-hr °F

ρ_{STEEL} = 49.0 lbm/ft^3

K_{STEEL} = 24.8 Btu/ft-hr-°F

Cp_{STEEL} = 0.14 Btu/lbm-°F

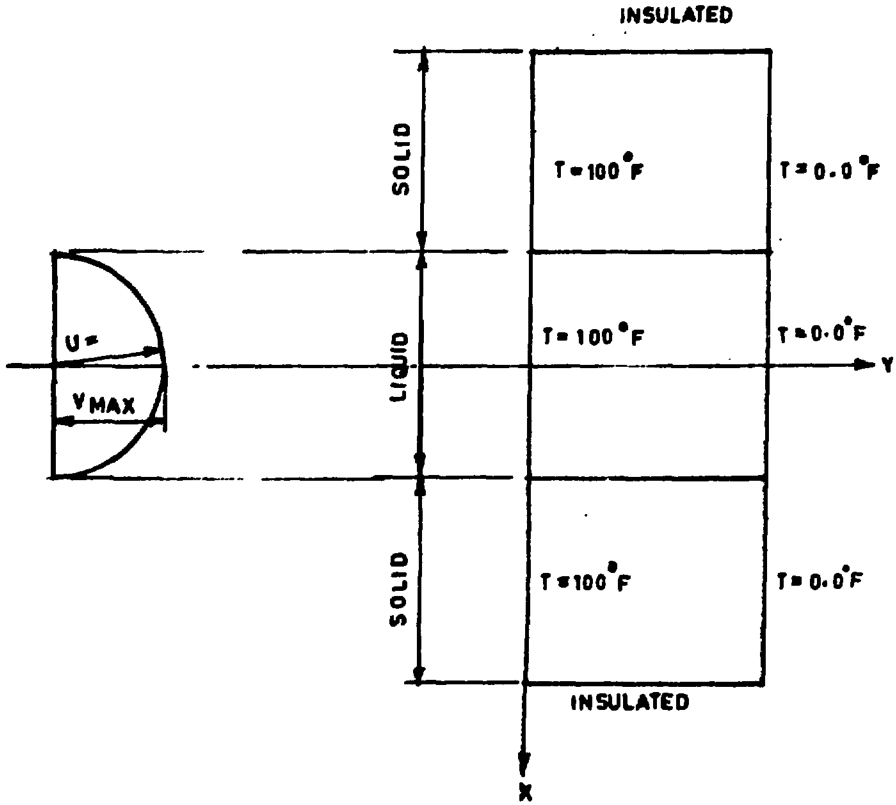

FIG. 19. THE GEOMETRY AND BOUNDARY CONDITIONS

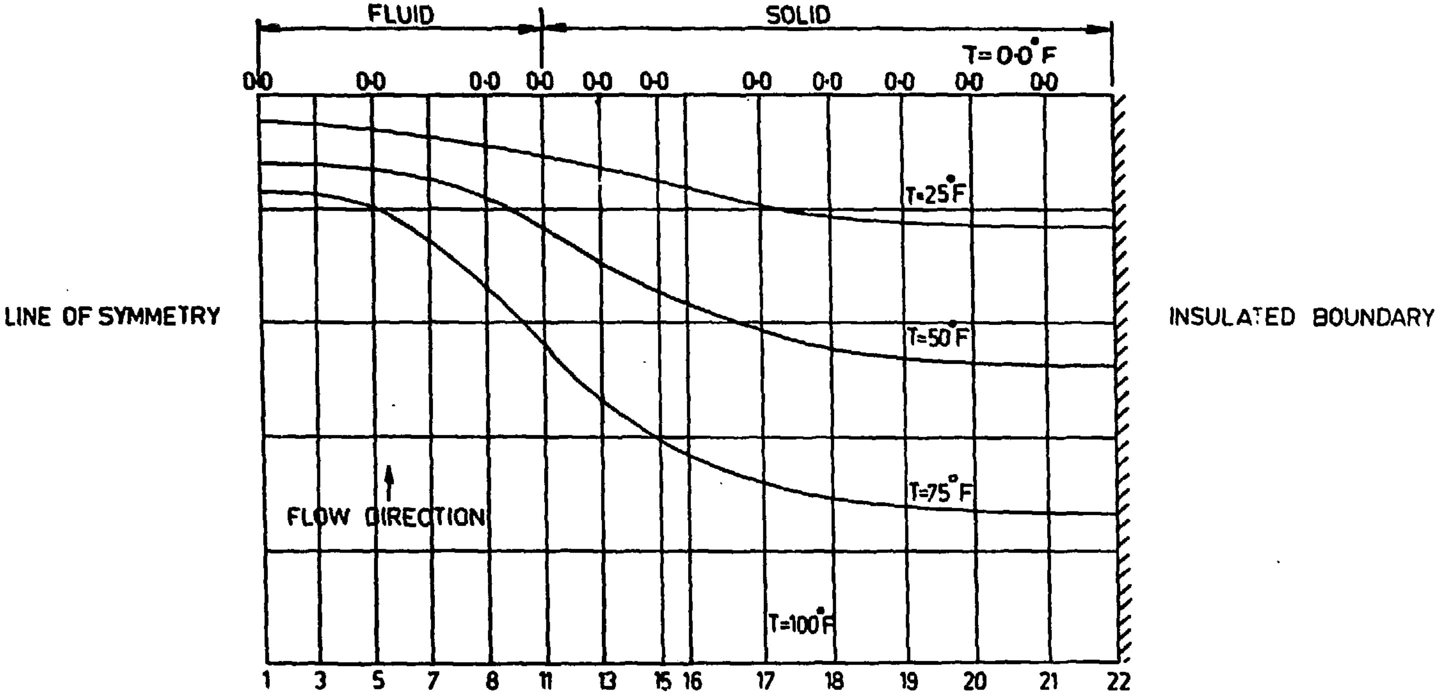

FIG.2a ISOTHERM LINES IN THE SOLID AND FLUID REGION AT PECLET NO. 15

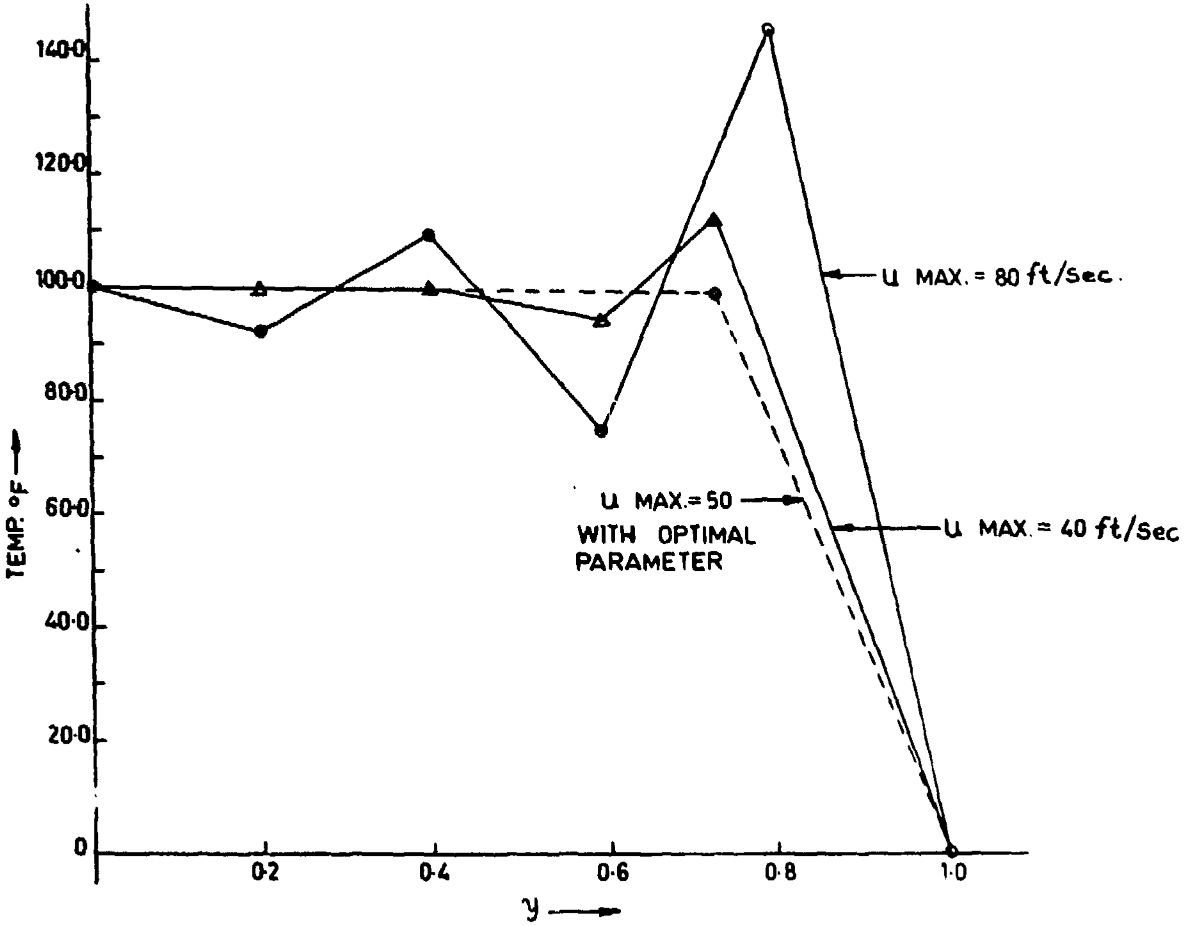

FIG.21 TEMPERATURE DISTRIBUTION IN FLUID AT X = 0·0 INCH

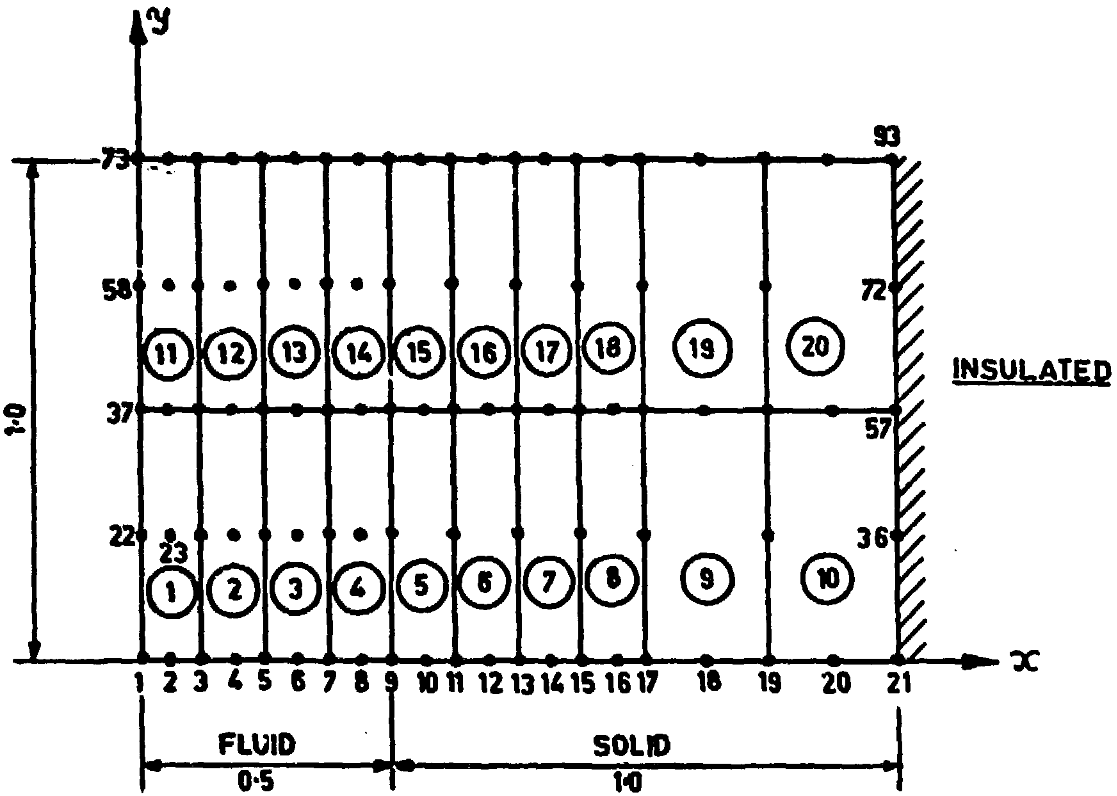

FIG.22 DISERITIZATION OF FLUID AND SOLID REGION
USING MIXED ELEMENT
[LAGRANGIAN AND SERENDIPITY]

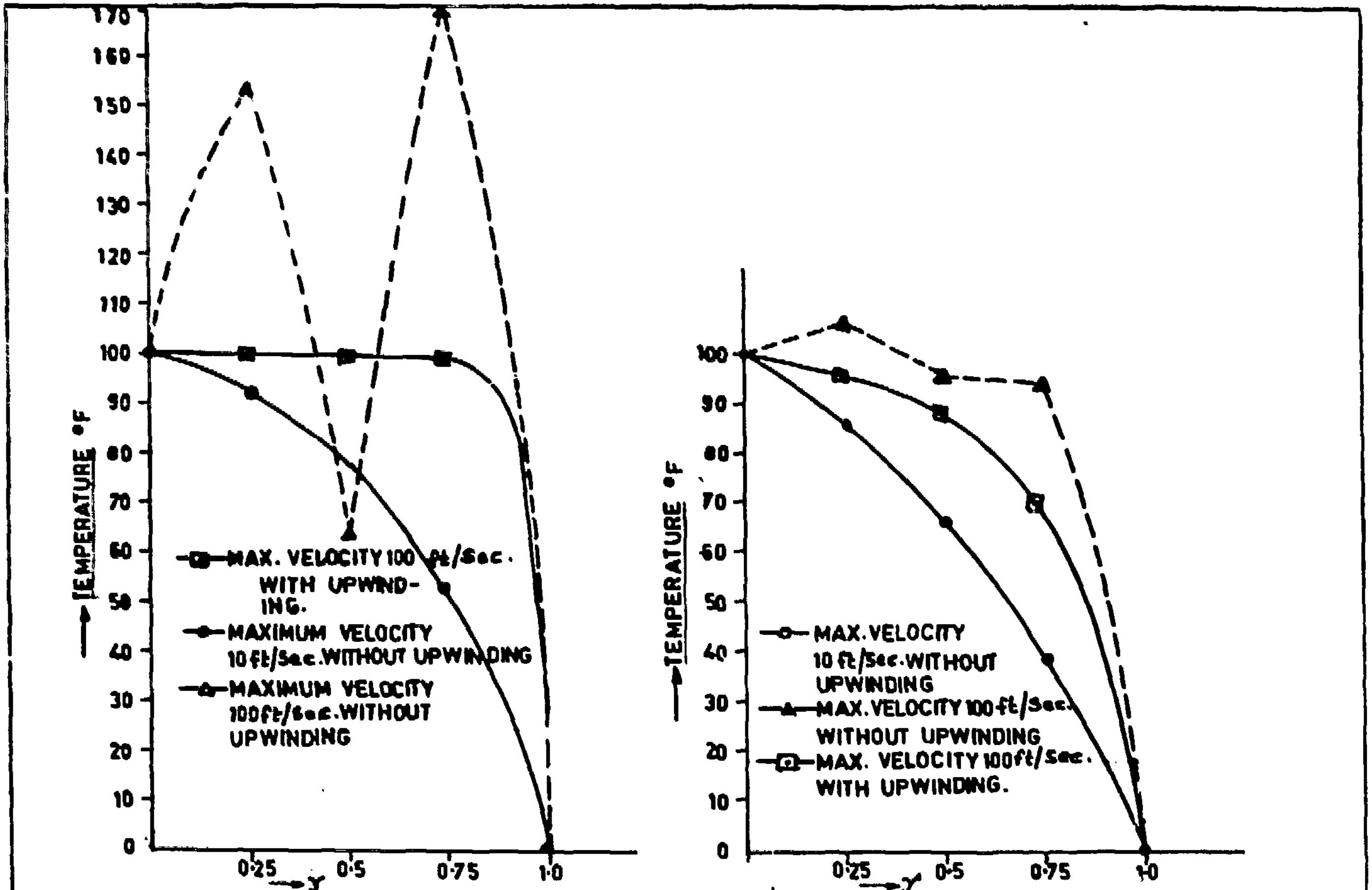

FIG.23 TEMPERATURE DISTRIBUTION ALONG FLOW CENTER LINE. AND TEMPERATURE DISTRIBUTION ALONG INTERIOR SURFACE