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CLOSED EXPRESSIONS FOR THE MAGNETIC FIELD OF TOROIDAL MULTIPOLE CONFIGURATIONS

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ABSTRACT

Closed analytic expressions for the vector potential and the magnetic field for the lower-order toroidal multipoles are presented. These expressions can be applied in the study of tokamak plasma cross section shaping. An example of such an application is included. These expressions also allow the vacuum fields required for plasma equilibrium to be specified in a general form independent of a particular coil configuration.

I. INTRODUCTION

The goal of increasing the beta capability of tokamaks has led to an interest in plasma shaping. The work to date has concentrated on choosing an expression for the plasma boundary shape and solving for the plasma parameters implied by that boundary.[1] The vacuum shaping field is determined by selecting an external coil set and solving for currents in the coils which, in a least square sense, approximate the given boundary shape when combined with the plasma fields. Another approach would be to specify the vacuum field as a sum of various order toroidal multipoles and solve for the plasma shape. This paper presents a set of simple closed expressions for the lower order toroidal multipoles which may be used for such an approach.

II. GENERAL APPROACH

The simplest field components for constructing an axisymmetric poloidal field pattern in a cylindrical coordinate system are a uniform field in the Z directions and dipoles on the Z axis, oriented along that axis. The expressions derived in the following sections are constituted from combinations of these components.

The relevant equations for a uniform field, Bo, in a cylindrical coordinate system are:

$$A\phi = \frac{B_o r}{2} \tag{1}$$

$$\psi = 2\pi rA\phi \tag{2}$$

$$B_r = 0 \tag{3}$$

$$B_z = Bo \tag{4}$$

where $A\phi$, ψ, B_r, and B_z are the vector potential, poloidal flux, radial, and axial magnetic field.

The comparable expressions for the fields of a dipole located on the Z axis at Z_o and oriented along that axis are:

$$A\phi = \frac{\mu o M}{4\pi D^3} r \tag{5}$$

$$\psi = 2\pi r A\phi \tag{6}$$

$$B_r = \frac{\mu o M}{4\pi D^5} 3r (z - Z_o) \tag{7}$$

$$B_z = \frac{\mu o M}{4\pi D^5} \left[2(z - Z_o)^2 - r^2\right] \tag{8}$$

where

$$D = \left[(z-Z_o)^2 + r^2\right]^{\frac{1}{2}} \quad . \tag{9}$$

Expressions for toroidal multipoles with a null on the midplane at R_o should have the same behavior in the immediate vicinity of this null as the two-dimensional multipole configurations described in the literature.[2-4]

This behavior at the null ($r = R_o$, $z = o$) can be summarized for the first three lower order multipoles as follows:

Quadrupole

$$B_z = o \tag{10}$$

Hexapole

$$B_z = o \tag{11}$$

$$\frac{\partial B_z}{\partial r} = o \tag{12}$$

Octupole

$$B_z = o \tag{13}$$

$$\frac{\partial B_z}{\partial r} = o \tag{14}$$

$$\frac{\partial^2 B_z}{\partial r^2} = o \quad . \tag{15}$$

The general approach is to solve for the simplest combinations of uniform field and dipoles which satisfy the relevant equations for the multipole of interest.

III. QUADRUPOLE

The quadrupole requirement, Eq. (10), is satisfied by a dipole at the origin, (r = o, z = o), and a uniform field. The resulting field expressions are:

$$A\phi = \frac{\mu o \, M_4}{4\pi D^3} + \frac{B_o r}{2} \tag{16}$$

$$\psi = 2\pi r A\phi \tag{17}$$

$$B_r = \frac{\mu o \, M_4}{4\pi D^5} \quad 3rz \tag{18}$$

$$B_z = \frac{\mu o \, M_4}{4\pi D^5} \quad (2z^2 - r^2) + B_o \tag{19}$$

where

$$B_o = \frac{\mu_o M_4}{4\pi R_o^3} \qquad (20)$$

$$D = (r^2 + z^2)^{1/2} \quad . \qquad (21)$$

The separatrix for this configuration is shown in Fig. 1.

IV. HEXAPOLE

The hexapole requirements, Eqs. (11) and (12), can be satisfied by a pair of dipoles above and below the midplane ($r = o$, $z = \pm Z_o$) added to a uniform field. The field expressions take the following form:

$$A\phi = -\frac{\mu_o M_6}{4\pi D_1^3} r + \frac{\mu_o M_6}{4\pi D_2^3} r + \frac{B_o r}{2} \qquad (22)$$

$$\psi = 2\pi r A\phi \qquad (23)$$

$$B_r = \frac{\mu_o M_6}{4\pi D_1^5} 3r(z - Z_o) + \frac{\mu_o M_6}{4\pi D_2^5} 3r(z + Z_o) \qquad (24)$$

$$B_z = \frac{\mu_o M_6}{4\pi D_1^5}\left[2(z - Z_o)^2 - r^2\right] + \frac{\mu_o M_6}{4\pi D_2^5}\left[2(z + Z_o)^2 - r^2\right] + B_o \qquad (25)$$

where

$$B_o = \frac{\mu_o M_6}{\pi R_o^3} \; \frac{8}{(5)^{5/2}} \qquad (26)$$

$$Z_o = -\frac{R_o}{2} \qquad (27)$$

$$D_1 = \left[(z - Z_o)^2 + r^2\right]^{\frac{1}{2}} \tag{28}$$

$$D_2 = \left[(z + Z_o)^2 + r^2\right]^{\frac{1}{2}} \quad . \tag{29}$$

The separatrix pattern for the hexapole is shown in Fig. 2.

V. OCTUPOLE

The octupole conditions, Eqs. (13)-(15), require one dipole at the origin of strength fM_8, a pair of dipoles above and below the midplane ($r = o$, $z = \pm Z_o$) each of strength M_8, and a uniform field. The resulting field equations are:

$$A\phi = \frac{\mu o \, fM_8}{4\pi D_o^3} \, r + \frac{\mu o \, M_8}{4\pi D_1^3} \, r + \frac{\mu o \, M_8}{4\pi D_2^3} \, r + \frac{B_o r}{2} \tag{30}$$

$$\psi = 2\pi r \, A\phi \tag{31}$$

$$\begin{aligned} B_r = {} & \frac{\mu o \, fM_8}{4\pi D_o^{\,5}} \, 3rz + \frac{\mu o \, M_8}{4\pi D_1^{\,5}} \, 3r \, (z - Z_o) \\ & + \frac{\mu o \, M_8}{4\pi D_2^{\,5}} \, 3r \, (z + Z_o) \end{aligned} \tag{32}$$

$$\begin{aligned} B_z = {} & \frac{\mu o \, fM_8}{4\pi D_o^{\,5}} \, (2z^2 - r^2) + \frac{\mu o \, M_8}{4\pi D_1^5} \left[2(z - Z_o)^2 - r^2\right] \\ & + \frac{\mu o \, M_8}{4\pi D_2^5} \left[2(z + Z_o)^2 - r^2\right] + Bo \end{aligned} \tag{33}$$

where

$$B_o = \frac{\mu o \, fM_8}{4\pi R_o^3} - \frac{\mu o \, M_8}{\pi R_o^{\,3}} \left[\frac{8}{(7)^{5/2}}\right] \tag{34}$$

$$Z_o = \sqrt{\frac{3}{4}}\ R_o \qquad (35)$$

$$D_o = \left[z^2 + r^2\right]^{1/2} \qquad (36)$$

$$D_1 = \left[(z - Z_o)^2 + r^2\right]^{1/2} \qquad (37)$$

$$D_2 = \left[(z+Z_o)^2 + r^2\right]^{1/2} \qquad (38)$$

$$f = \frac{4}{3}\left[\frac{8}{(7)^{5/2}} + \frac{160}{(7)^{7/2}}\right] . \qquad (39)$$

Figure 3 shows the separatrix pattern for the resulting octupole.

VI. APPLICATION TO PLASMA SHAPING

The application of these toroidal multipole expressions was tested by incorporating them into an existing MHD equilibrium code.[5]

First, a simple plasma equilibrium with no pressure was established in a uniform vertical field. As would be expected, the resultant equilibrium cross section is approximately circular. The nested flux surfaces for this case are shown in Fig. 4.

Next, a quadrupole and a hexapole component were added to the first equilibrium case. The quadrupole tends to make the equilibrium elliptical, vertically, while the hexapole tends to bend the cross section into a "bean" shape. The resultant flux surfaces are shown in Fig. 5.

A comparison of these flux surfaces with the flux surfaces obtained in shape optimization studies[6] points to the usefulness of these closed expressions in such studies.

VII. CONCLUSION

The expressions derived in this paper specify in a simple closed form the fields of the toroidal multipoles up to and including the octupole order. These expressions are exact over all space with singularities only on the Z axis. They appear to offer a general way of specifying the vacuum fields required for tokamak equilibria and should facilitate cross section optimization studies.

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FIGURE CAPTIONS

FIG. 1. Separatrix flux pattern for the toroidal quadrupole.

FIG. 2. Separatrix flux pattern for the toroidal hexapole.

FIG. 3. Separatrix flux pattern for the toroidal octupole.

FIG. 4. Plasma equilibrium for I_p = 6 Ma, B_t = 7T, B_v = -0.32T, and β = 0.0.

FIG. 5. Plasma equilibrium for I_p = 6 Ma, B_t = 7T, B_v = -0.32T, β = 0.0, M_4 = - 1E8, and M_6 = 2E8.

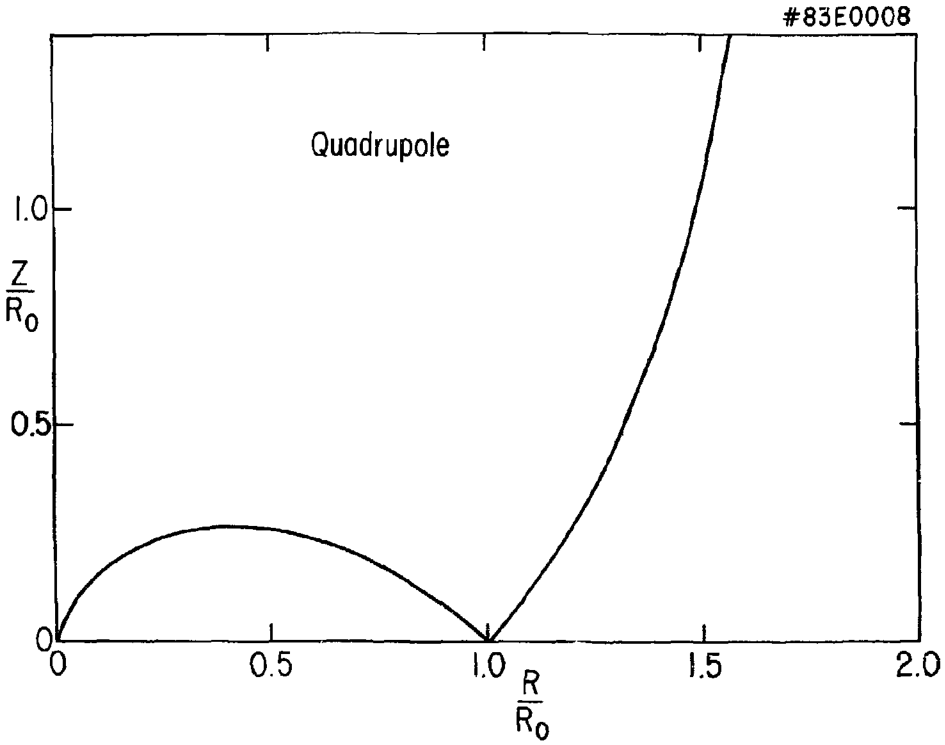


Fig. 1

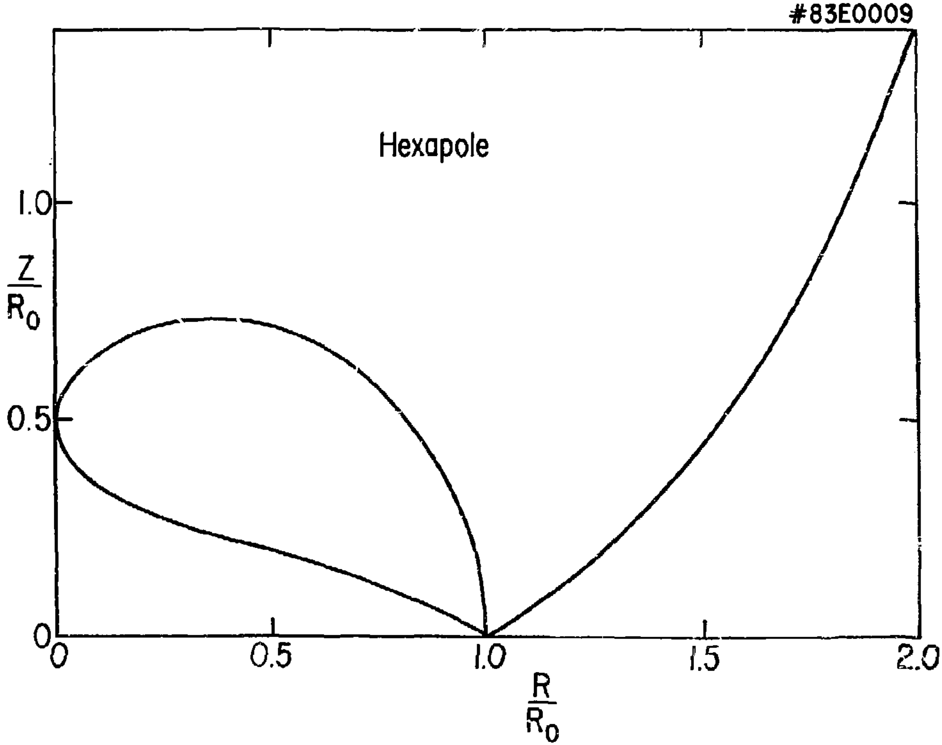
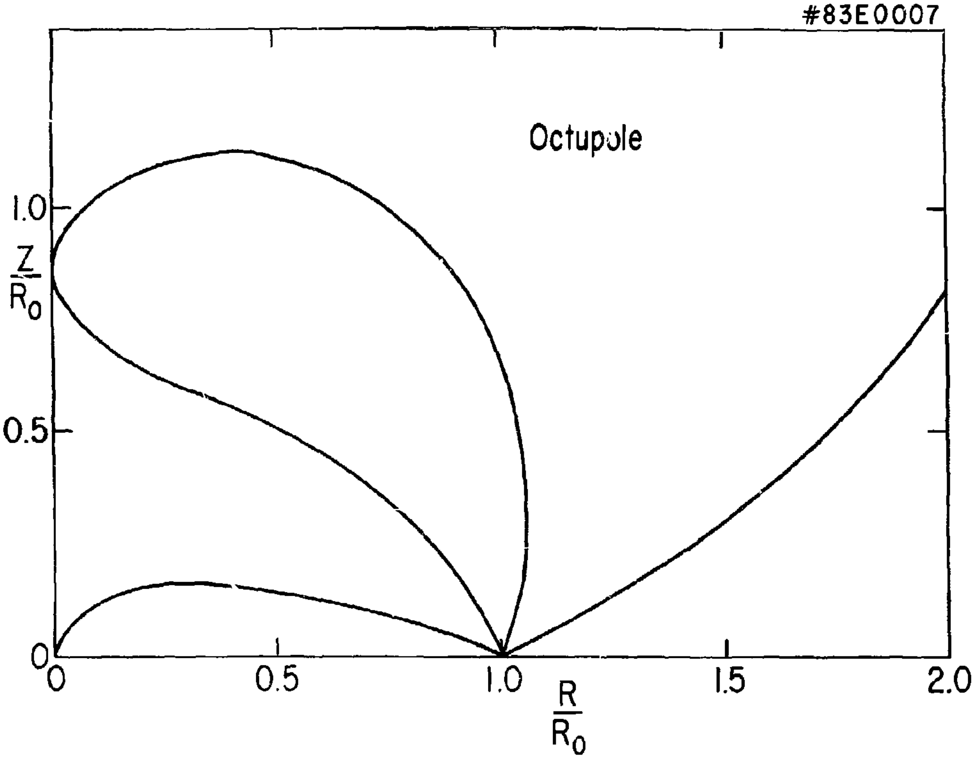


Fig. 2



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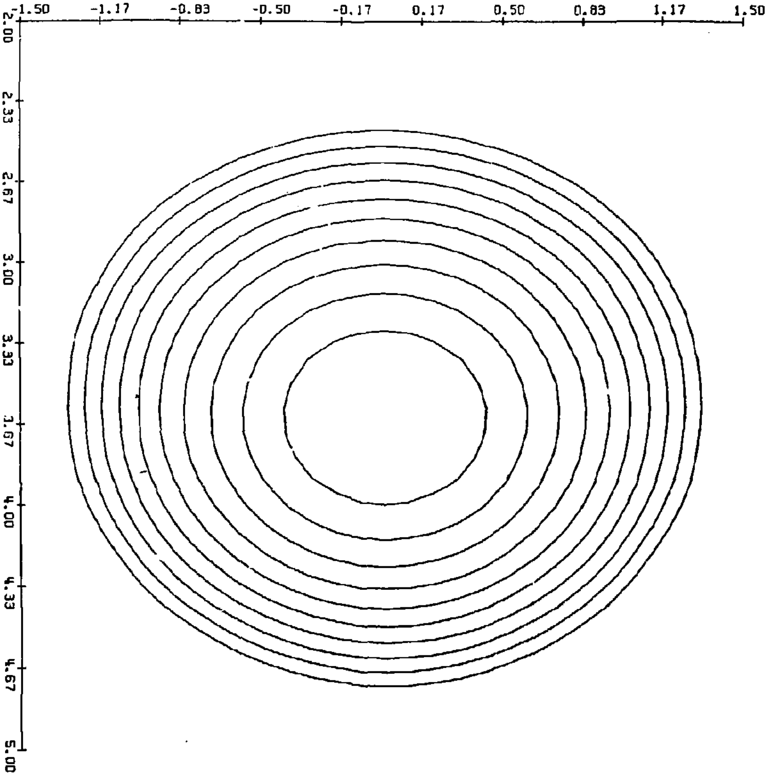
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Fig. 4

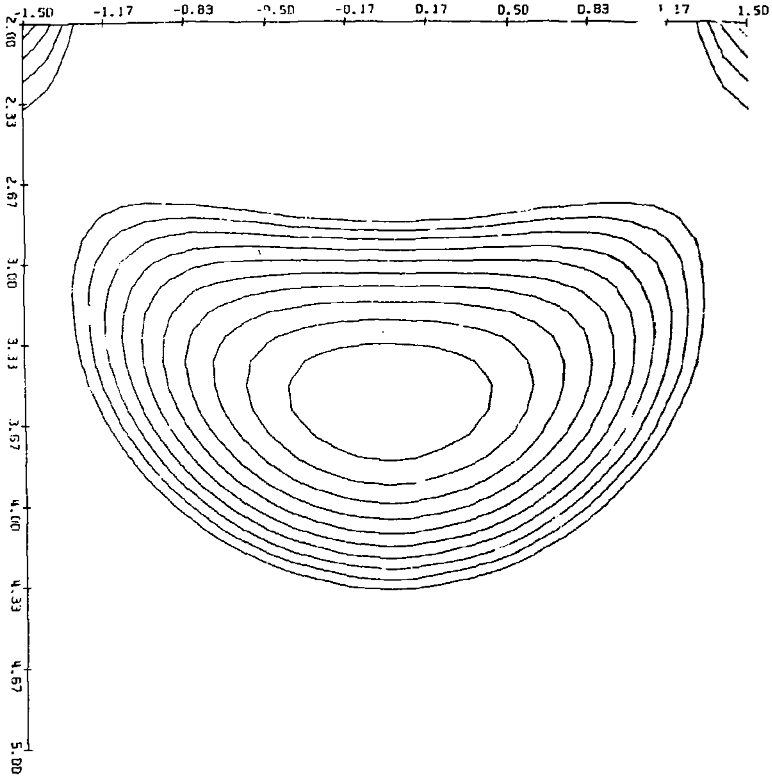
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Fig. 5

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