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VECTOR-VALUED MEASURE AND THE NECESSARY CONDITIONS FOR THE OPTIMAL CONTROL PROBLEMS OF LINEAR SYSTEMS *

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ABSTRACT

The vector-valued measure defined by the well-posed linear boundary value problems is discussed. The maximum principle of the optimal control problem with non-convex constraint is proved by using the vector-valued measure. Especially, the necessary conditions of the optimal control of elliptic systems is derived without the convexity of the control domain and the cost function.

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INTRODUCTION

LaSalle [1] and then many other authors [2,3] have used the vector-valued measure to study time optimal control of lumped parameter systems. Li and Yao [4,5] have used it to consider the time optimal control of distributed parameter systems. Li [6] and then Li and Yao [7,8] have extended this method to study general optimal control problems. In this paper, we further consider the role of the vector-valued measure in the study of the necessary conditions for general optimal control problems of linear systems. Especially, we prove the maximum principle of optimal control of elliptic systems without the convexity of the control domain U and the cost function f^0.

VECTOR-VALUED MEASURE DEFINED BY LINEAR SYSTEMS

Let Ω be an open set of space R^N with smooth boundary $\partial\Omega$. Let L be a linear differential operator in Ω and B a normal family of the boundary operators [9].

We consider the boundary value problem

$$\begin{aligned} Ly &= f, & x &\in \Omega \\ By &= g, & x &\in \partial\Omega \end{aligned} \qquad (1)$$

Assume that the boundary value problem (1) is well posed with $f \in \mathcal{F}$ and $g \in \mathcal{G}$, i.e. for $f \in \mathcal{F}$ and $g \in \mathcal{G}$, the boundary value problem (1) has a unique solution $y \in \mathcal{H}$. Where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ satisfy the following assumptions:

Assume that $\mathcal{F}$ satisfies the following conditions:

i) $0 \in \mathcal{F}$,

ii) If $f_i \in \mathcal{F}$ and $\alpha_i \in R$ $(i = 1,2)$, then

$$\alpha_1 f_1 + \alpha_2 f_2 \in \mathcal{F},$$

iii) If $f \in \mathcal{F}$, the measurable subset E of Ω is given and $f_E: \Omega \to R$ defined by

$$f_E(x) = \begin{cases} f(x) & \text{when } x \in E, \\ 0 & \text{when } x \in \Omega \setminus E, \end{cases}$$

then $f_E \in \mathcal{F}$.

For $\mathcal{G}$, we assume that $\mathcal{G}$ satisfies similar conditions as $\mathcal{F}$.

We assume that $\mathcal{H}$ is a reflexive Banach space, for example, $\mathcal{H}$ is $H^m(\Omega)$ [9].

Denote the solution y of the boundary value problem (1) by

$$y = y(f,g) \quad ,$$

and denote

$$y(f) = y(f,0) \quad ,$$

$$\mu(E) = y(f_E) \quad .$$

Obviously, if the subsets E_1 and E_2 of Ω are measurable, and $E_1 \cap E_2 = \emptyset$, then

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \quad .$$

And, if $E_j (\subset \Omega)$ is measurable, $E_i \cap E_j = \emptyset \ (i \neq j)$, then for $E = \bigcup_j E_j$

$$\mu(E) = y(f_E) = \sum_j y(f_{E_j}) = \sum_j \mu(E_j) \quad .$$

Hence, given f, $\mu(E) = y(f_E)$ defines a $\mathcal{H}$-valued measure, and it is a countable additive function on the family of all measurable subsets of Ω.

Further, we assume that the measure $\mu(E) = y(f_E)$ has the following properties:

1) It is of bounded variation, i.e. if $E = \bigcup_j E_j$, $E_i \cap E_j = \emptyset \ (i \neq j)$, E and E_j measurable, then

$$\sum_j \|\mu(E_j)\| = \sum_j \|y(f_{E_j})\| < +\infty \quad .$$

2) It is absolutely continuous, i.e. for $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{j=1}^{\ell} \|\mu(E_j)\| < \epsilon \quad \forall \sum_{j=1}^{\ell} |E_j| < \delta \quad ,$$

where $|E|$ denotes the Lebesque measure of a set E in R^N

Obviously, if the $\mathcal{H}$-valued measure is absolutely continuous, then it is of bounded variation.

We suppose that the $\mathcal{H}$-valued measure $\mu(E) = y(f_E)$ defined by the boundary value problem (1) is absolutely continuous.

Example 1. The $\mathcal{H}$-valued measure defined by the Dirichlet problem

$$\begin{aligned} \Delta y &= f, \qquad x \in \Omega \\ y &= 0, \qquad x \in \partial\Omega \end{aligned} \qquad (2)$$

is absolutely continuous.

This can be proved by the Fredholm theory [12].

Example 2. Let the state equation be an evolution equation

$$\begin{aligned} \frac{dx}{dt} &= Ax + f, \qquad 0 < t < t_1 \\ x(0) &= 0 \quad , \end{aligned} \qquad (3)$$

where A is an infinitesimal generator of C_0 semigroup $T(t) \equiv e^{At}$ and f is Bochner integrable. Then the solution of (3) with f_E is $x(\cdot,f_E)$:

$$x(t,f_E) = \int_0^t e^{A(t-s)} f_E(s)\, ds = \int_{E \cap [0,t]} e^{A(t-s)} f(s)\, ds \quad .$$

Obviously, $\mu(E) = x(\cdot,f_E)$ is a $C([0,t_1];X)$-valued measure and it is absolutely continuous.

So, we consider not only the boundary value problems but also the initial value problems.

OPTIMAL CONTROL PROBLEMS

1. The state equation is given by

$$\begin{aligned} Ly &= f + b(x,u(x)), \qquad x \in \Omega \\ By &= g, \qquad x \in \partial\Omega \end{aligned} \qquad (4)$$

Let Z be a Banach space, U an arbitrary set of Z.

Let $b : \Omega \times Z \to R$ satisfy the following assumptions:
$b(x,\cdot) : Z \to R$ is continuous for almost all $x \in \Omega$,
$b(\cdot,u) : \Omega \to R$ is Lebesque integrable for all $u \in Z$.

If $u(\cdot):\Omega \to Z$ is strongly measurable, $u(x) \in U$ for almost all $x \in \Omega$, and $b(\cdot,u(\cdot))$ Lebesque integrable in Ω, $b(\cdot,u(\cdot)) \in \mathcal{Y}$, then we call $u(\cdot)$ admissible control and denote it by

$$u(\cdot) \in U_{ad} \quad .$$

2. Let $f^{o} : \Omega \times R \times Z \to R$ be continuous and there exists $\frac{\partial f^{o}}{\partial y}$ which is continuous.

For $u(\cdot) \in U_{ad}$, the unique solution of (4) is $y = y(\cdot\,;u)$. And we have the integral

$$J(u) = \int_{\Omega} f^{o}(x,y(x;u), u(x))\, dx \quad . \tag{5}$$

We call $J(u)$ the cost function.

The optimal control problem is to find $u^*(\cdot) \in U_{ad}$ such that

$$J(u^*) \leq J(u) \quad \forall\, u(\cdot) \in U_{ad} \quad . \tag{6}$$

We want to consider the necessary conditions of the optimal control $u^*(\cdot)$ for the problem (6).

Remark 1. We do not use the convexity of the control domain U.

Remark 2. We do not assume the convexity of the function f^{o} .

MAXIMUM PRINCIPLE

Theorem: Assume that for every given $f \in \mathcal{K}$, the $\mathcal{H}$-valued measure defined by the boundary value problem (1) is absolutely continuous, $u^*(\cdot)$ is the optimal control of problem (6), $y^*(\cdot) \equiv y(\cdot\,;u^*)$, and

$$-L^*\psi = \frac{\partial f^{o}(x,y^*(x), u^*(x))}{\partial y} \qquad x \in \Omega,$$

$$B^*\psi = 0 \,, \qquad x \in \partial\Omega \tag{7}$$

where L^* (and B^*) is the dual operator of L (and B). Then the maximum principle

$$\psi(x)\, b(x,u^*(x)) - f^{o}(x,y^*(x), u^*(x))$$

$$= \max_{u \in U} \{\psi(x)\, b(x,u) - f^{o}(x,y^*(x),u)\} \tag{8}$$

holds for almost all $x \in \Omega$.

For the proof we need the following lemma.

Lemma. Under the assumptions of the theorem, let $u(\cdot) \in U_{ad}$ and

$$y(\cdot) \equiv y(\cdot\,;u) \quad .$$

Then the vector-valued measure $\hat{\mu}(E) \equiv (\varphi_E, y^{o}_E, \tilde{y}_E)$ where

$$\varphi_E = \int_E \left\{ \frac{\partial f^{o}(x,y^*(x), u(x))}{\partial y} - \frac{\partial f^{o}(x, y^*(x), u^*(x))}{\partial y} \right\} \{y(x) - y^*(x)\} dx$$

$$y^{o}_E = \int_E \{f^{o}(x,y^*(x), u(x)) - f^{o}(x,y^*(x), u^*(x))\}\, dx$$

and

$$L\,\tilde{y}_E = [b(\cdot,u(\cdot)) - b(\cdot,u^*(\cdot))]_E \,, \qquad x \in \Omega$$

$$B\,\tilde{y}_E = 0 \qquad x \in \partial\Omega$$

is absolutely continuous.

Proof is obvious.

PROOF OF THEOREM

According to the lemma, $\hat{\mu}(E)$ is absolutely continuous. And by the theorem of Uhl [10], we have that for $\alpha \in (0,1)$ there exists a measurable subset E_α of Ω such that

$$\left| \alpha \int_E \left\{ \frac{\partial f^{o}(x,y^*(x), u(x))}{\partial y} - \frac{\partial f^{o}(x,y^*(x), u^*(x))}{\partial y} \right\} \{y(x) - y^*(x)\}\, dx \right.$$
$$\left. - \int_{E_\alpha} \left\{ \frac{\partial f^{o}(x,y^*(x), u(x))}{\partial y} - \frac{\partial f^{o}(x,y^*(x), u^*(x))}{\partial y} \right\} \{y(x) - y^*(x)\}\, dx \right| \leq \alpha^2 \tag{9}$$

$$\left| \alpha \int_{\Omega} \{f^{o}(x,y^*(x), u(x)) - f^{o}(x,y^*(x), u^*(x))\}\, dx \right.$$
$$\left. - \int_{E_\alpha} \{f^{o}(x,y^*(x), u(x)) - f^{o}(x,y^*(x), u^*(x))\}\, dx \right| < \alpha^2 \tag{10}$$

$$\| \alpha\, \tilde{y}_{\Omega}(\cdot) - \tilde{y}_{E_\alpha}(\cdot) \| < \alpha^2 \quad .$$

Set

$$u_\alpha(x) = \begin{cases} u(x) & \text{when } x \in E_\alpha , \\ u^*(x) & \text{when } x \in \Omega \setminus E_\alpha \end{cases} .$$

Then $u_\alpha(\cdot) \in U_{ad}$ and

$$y(\cdot;u_\alpha) = y(\cdot;u^*) + \tilde{y}_{E_\alpha}(\cdot) ,$$

$$\tilde{y}_\Omega(\cdot) = y(\cdot;u) - y(\cdot;u^*) .$$

Therefore

$$\alpha\tilde{y}_\Omega(\cdot) - \tilde{y}_{E_\alpha}(\cdot) = \alpha y(\cdot) + (1-\alpha)\, y^*(\cdot) - y(\cdot;u_\alpha) .$$

and hence

$$\|\alpha y(\cdot) + (1-\alpha)\, y^*(\cdot) - y(\cdot;u_\alpha)\| < \alpha^2. \qquad (11)$$

By (10), we have

$$\int_\Omega f^0(x,y^*(x), u(x))\, dx = \int_\Omega f^0(x,y^*(x), u^*(x))\, dx + \alpha \int_\Omega \{f^0(x,y^*(x), u(x)) - f^0(x,y^*(x), u^*(x))\}\, dx + O(\alpha^2) . \qquad (12)$$

By (9), we have

$$\int_\Omega \frac{\partial f^0(x,y^*(x), u_\alpha(x))}{\partial y} \{y(x) - y^*(x)\}\, dx =$$

$$= \int_{\Omega\setminus E_\alpha} \frac{\partial f^0(x,y^*(x), u^*(x))}{\partial y} \{y(x) - y^*(x)\}\, dx + \int_{E_\alpha} \frac{\partial f^0(x,y^*(x), u(x))}{\partial y} \{y(x) - y^*(x)\}\, dx$$

$$= \int_\Omega \frac{\partial f^0(x,y^*(x), u^*(x))}{\partial y} \{y(x) - y^*(x)\}\, dx + \alpha \int_{E_\alpha} \left\{\frac{\partial f^0(x,y^*(x), u(x))}{\partial y} - \frac{\partial f^0(x,y^*(x), u^*(x))}{\partial y}\right\} \{y(x) - y^*(x)\}\, dx$$

$$= \int_\Omega \frac{\partial f^0(x,y^*(x), u^*(x))}{\partial y} \{y(x) - y^*(x)\}\, dx + O(\alpha) . \qquad (13)$$

By (11) and the continuity of $\frac{\partial f^0}{\partial y}$, we have

$$J(u_\alpha) = \int_\Omega f^0(x,y(x;u_\alpha), u_\alpha(x))\, dx$$

$$= \int_\Omega f^0(x,y^*(x) + \alpha(y(x) - y^*(x)), u_\alpha(x))\, dx + O(\alpha^2)$$

$$= \int_\Omega f^0(x,y^*(x), u_\alpha(x))\, dx + \alpha \int_\Omega \frac{\partial f^0(x,y^*(x), u_\alpha(x))}{\partial y} [y(x) - y^*(x)]\, dx + o(\alpha) .$$

Hence by (12) and (13), we obtain

$$J(u_\alpha) = \int_\Omega f^0(x,y^*(x), u^*(x))\, dx + \alpha \int_\Omega \{f^0(x,y^*(x), u(x)) - f^0(x,y^*(x),u^*(x))\} dx + \alpha \int_\Omega \frac{\partial f^0(x,y^*(x), u^*(x))}{\partial y} \{y(x) - y^*(x)\}\, dx + o(\alpha) .$$

But $J(u_\alpha) \geq J(u^*)$, so we have

$$\alpha \int_\Omega \{f^0(x,y^*(x), u(x)) - f^0(x,y^*(x), u^*(x))\}\, dx + \alpha \int_\Omega \frac{\partial f^0(x,y^*(x), u^*(x))}{\partial y} \{y(x) - y^*(x)\}\, dx + o(\alpha) \geq 0 .$$

Hence

$$\int_\Omega \{f^0(x,y^*(x), u(x)) - f^0(x,y^*(x), u^*(x))\}\, dx + \int_\Omega \frac{\partial f^0(x,y^*(x), u^*(x))}{y} \{y(x) - y^*(x)\}\, dx \geq 0$$

By the definition of $\psi(x)$, we have

$$\int_\Omega \{f^0(x,y^*(x), u(x)) - f^0(x,y^*(x), u^*(x))\}\, dx - \int_\Omega \{y(x)-y^*(x)\}\, L^*\psi(x)dx \geq 0.$$

Thus by the Green formula [9], we have

$$\int_\Omega \psi(x)\, \{Ly^*(x) - Ly(x)\}\, dx - \int_\Omega \{f^0(x,y^*(x), u^*(x))-f^0(x,y^*(x), u(x))\}\, dx \geq 0.$$

Hence for $u(\cdot) \in U_{ad}$, we have

$$\int_\Omega \psi(x)\{b(x,u^*(x))-b(x,u(x))\}\, dx - \int_\Omega \{f^0(x,y^*(x), u^*(x))-f^0(x,y^*(x), u(x))\}\, dx \geq 0.$$

Thus, from the above inequality and by the usual method [4,11] we can prove that the maximum principle (7) holds for almost all $x \in \Omega$. Q.E.D.

Remark 3. For the boundary control problem

$$Ly = f, \qquad x \in \Omega$$
$$By = g + \tilde{b}(x,u(x)), \qquad x \in \partial\Omega \quad ,$$

we can use the above method to derive the maximum principle.

Remark 4. For the elliptic system

$$\Delta y = f + b(x,u(x)), \qquad x \in \Omega$$
$$y = g \quad , \qquad x \in \partial\Omega$$

and the cost function (f), the maximum principle of optimal control holds without the conditions on the convexity of U and f^0 .

Remark 5. The $C([0,1];R)$-valued measure $\mu(E)$ defined by

$$\dot{x}(t) = 1_E, \quad x(0) = 0$$

is absolutely continuous. Obviously, its range is non-convex. This is an example for which the range of $\mu(E)$ is non-convex in the case of infinite dimension [13].

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