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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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FOR THE CLASSICAL LIE ALGEBRA GENERATORS

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INTERNATIONAL
ATOMIC ENERGY
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UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1982 MIRAMARE-TRIESTE



International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
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THE DERIVATION OF THE CONVENTIONAL BASIS
FOR THE CLASSICAL LIE ALGEBRA GENERATORS*

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ABSTRACT

The explicit construction of the classical Lie algebra generators in the conventional Gell-Mann basis is derived for all irreducible unitary representations of all classical groups. The main framework is based on a description of the simple roots of the classical Lie algebras such that the inter-relations implied by the Cartan matrix of the group among these simple roots are explicit within this description.

MIRAMARE - TRIESTE

January 1982

* To be submitted for publication.

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I. INTRODUCTION

In their pioneering work ¹⁾, Glashow and Gell-Mann introduced a generator basis in order to describe the Lie algebra commutations of a classical group. This basis is very convenient to apply in all unification schemes which are realized in the framework of a group structure, simple or semi-simple. Because of this fact, this basis is the conventional basis of today's gauge theories and is generally described as

$$[T^a, T^b] = if_{abc} T^c \quad (1.1)$$

for a Lie algebra having the elements T^a as the generators of the group. In this expression, the so-called structure constants f_{abc} of the group are subject to the following conditions :

i) The f_{abc} 's are completely anti-symmetric in all their three indices, and real,

$$ii) [T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0 .$$

This second condition is known as the 'Jacobi identity'. However, in the conventional basis for the Eight-fold Way ²⁾, these generators are defined as generalized Pauli matrices ^{*)}. In fact, these are not directly group generators since they have a different normalization which is fixed as being

$$\text{Tr}(T^a(R)T^b(R)) = 2\delta_{ab} \quad (1.2)$$

from those of (1.1). This normalization condition is in fact directly related to the eigenvalues of the second degree Casimir operators for a representation R. It can be shown that the normalization (1.2) can be expressed as

$$\text{Tr}(T^a(R)T^b(R)) = N(R) \delta_{ab} \quad (1.3)$$

where the eigenvalues $N(R)$ are explicitly representation dependent. The

*) The Pauli matrices define a basis for SU(2) Lie algebra as being

$$[\tau^a, \tau^b] = 2i\epsilon_{abc} \tau^c$$

and hence they are specified as

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

second and also third order ³⁾ Casimir operators are well known today's gauge theories.

Being in relation with the existence of the conventional basis, the existence of the structure constants has been shown by Racah ⁴⁾. However, the existence theorem of the conventional basis is generally due to Serre's theorem ⁵⁾. On the other hand, the explicit construction of these generators in a matrix form is less known for any irreducible representation of the classical groups excepts for some special cases including especially the fundamental representations. But, it is clear that the explicit construction of the matrix generators for any irreducible representation of any classical group is worthwhile, especially for unification schemes. In this work we explicitly carry out this programme. In the following we will give a general method to explicitly construct the fundamental representations of the chains A_N, B_N, C_N and D_N in the Cartan classification ⁶⁾. Then, with the aid of our recent work ⁷⁾ concerning the representation theory of these algebras, the explicit matrix representations can be obtained for any other representation of the groups corresponding to the above algebras. For this, we use only the inter-relations among the fundamental representation weights (FRW), which are implicit in the Cartan matrix ⁸⁾ of the group in consideration. The same programme will also soon be explained for the exceptional groups in a subsequent paper.

II. THE DERIVATION OF THE GELL-MANN BASIS

Of course, the standard formulation in the Cartan-Weyl basis of the Lie algebras is the main point in order to extract any other generator basis. This formulation will be developed afterwards in the Cartan-Dynkin (root-weight) theory ^{*}). The roots and weights are the central concepts in this theory. In fact, the groups can be determined as the root systems ϕ for each of the chains A_N, B_N, C_N and D_N . Then the corresponding Lie algebras are determined under the following commutation relations satisfied by the standard generators $H^a, E_\alpha, E_{-\alpha}$:

- 1) $[H^a, H^b] = 0$
- 2) $[H^a, E_\alpha] = \alpha^a E_\alpha, \quad \alpha \in \phi$
- 3) $[E_\alpha, E_{-\alpha}] = \alpha^a H^a, \quad \alpha \in \phi^+$

^{*}) All of the properties and concepts concerning the Cartan-Dynkin (root-weight) formulation of the standard theory is best given in ref.8. This is the main source for all related discussions within this work.

$$\begin{aligned}
 4) \quad [E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta}; \quad \alpha, \beta, \alpha+\beta \in \phi \\
 4i) \quad N_{\alpha\beta} &= N_{\beta, -\alpha-\beta} = N_{-\alpha-\beta, \alpha} = -N_{\beta\alpha} = N_{-\beta, -\alpha} \\
 4ii) \quad N^2 &= \frac{1}{2} (\alpha, \alpha) n(n+1) \quad \begin{matrix} \beta-\alpha, \beta+\alpha \in \phi \\ \beta-(n+1)\alpha, \beta+(n+1)\alpha \notin \phi \end{matrix}
 \end{aligned} \tag{2.1}$$

ϕ^+ is the positive root system of the group. In essence, a positivity notion for the roots can be introduced with the aid of a scalar product defined as ^{*})

$$(\alpha, \beta) \equiv \sum_{a=1}^N \alpha^a \beta^a \tag{2.2}$$

because the roots are generally N component vectors where N is the rank of the group. By definition, this number N is the number of the diagonal generators H^a of the group in the same time. This positivity notion is best understood in terms of the N simple roots α_a of the group. ^{**)} These simple roots can also be used to define the Cartan matrix of the group, which is essential for the definition of the groups corresponding to the root systems. The Cartan matrix of a group is defined as the matrix C having the elements

$$C_{ab} \equiv 2(\alpha_a, \alpha_b) / (\alpha_a, \alpha_b) \tag{2.3}$$

These are taken as those tabulated in Ref.8. The central idea of our work is based on a formulation of the simple roots α_a in such a way that the inter-relations supposed by the Cartan matrices are explicit in this formulation. This can be done with the aid of the parameters O_{ab} . We assume these parameters to be the elements of an orthogonal matrix. Thus, the following expressions can easily be investigated:

$$\begin{aligned}
 A) \quad A_N &\equiv SU(N+1) \\
 \alpha_m^a &\approx \frac{1}{\sqrt{m}} \{ -\sqrt{(m-1)} O_{m-1,a} + \sqrt{(m+1)} O_{ma} \}, \quad m=1,2,\dots,N.
 \end{aligned} \tag{2.4}$$

^{*}) To prevent any confusion in notation, we use the super-index a for the components of the roots and also the weight vectors while the simple roots are denoted by the sub-index a .

^{**)} This point is explicitly explained in Ref.7 in which we also define the positive root systems of the chains A_N, B_N, C_N, D_N in terms of the fundamental representation weights.

$$B) \quad B_N \equiv SO(2N+1)$$

$$\alpha_m^a = \frac{1}{\sqrt{m}} \{-\sqrt{(m-1)} O_{m-1,a} + \sqrt{(m+1)} O_{ma}\}, \quad m=1,2,\dots,N-1. \quad (2.5)$$

$$\alpha_N^a = \frac{1}{\sqrt{N}} \{-\sqrt{(N-1)} O_{N-1,a} + O_{Na}\}$$

$$C) \quad C_N \equiv SP(N)$$

$$\alpha_m^a = \frac{1}{\sqrt{m}} \{-\sqrt{(m-1)} O_{m-1,a} + \sqrt{(m+1)} O_{ma}\}, \quad m=1,2,\dots,N-1.$$

$$\alpha_N^a = \frac{2}{N} \{-\sqrt{(N-1)} O_{N-1,a} + O_{Na}\} \quad (2.6)$$

$$D) \quad D_N \equiv SO(2N)$$

$$\alpha_m^a = \frac{1}{m} \{-\sqrt{(m-1)} O_{m-1,a} + \sqrt{(m+1)} O_{ma}\}, \quad m=1,2,\dots,N-3.$$

$$\alpha_{N-2}^a = -\frac{\sqrt{(N-3)}}{\sqrt{(N-2)}} O_{N-3,a} - \frac{1}{\sqrt{(N-2)}} O_{N-2,a} - \frac{1}{\sqrt{2}} \{O_{N-1,a} + O_{Na}\} \quad (2.7)$$

$$\alpha_{N-1}^a = \sqrt{2} O_{N-1,a}$$

$$\alpha_N^a = \sqrt{2} O_{Na}$$

On the other hand, these simple roots can be used to determine the fundamental representation weights (FRW). However, the following notation will be very convenient to express the results of these relations. In conclusion, the corresponding results will be obtained for the FRW's: ^a

A)

$$(m_1, m_2, m_3, \dots, m_N) \equiv \sum_{k=1}^N \frac{m_k}{\sqrt{k(k+1)}} O_{ka} \quad (2.8)$$

$$\mu_1 = (1, 1, 1, 1, 1, \dots, 1, 1, 1)$$

$$\mu_2 = (-1, 1, 1, 1, 1, \dots, 1, 1, 1)$$

$$\mu_3 = (0, -2, 1, 1, 1, \dots, 1, 1, 1)$$

$$\mu_4 = (0, 0, -3, 1, 1, \dots, 1, 1, 1)$$

.....

.....

$$\mu_{N-1} = (0, 0, 0, 0, 0, \dots, -(N-2), 1, 1)$$

$$\mu_N = (0, 0, 0, 0, 0, \dots, 0, -(N-1), 1)$$

$$\mu_{N+1} = (0, 0, 0, 0, 0, \dots, 0, 0, -N)$$

(2.9)

B), C) *)

$$(m_1, m_2, m_3, \dots, m_N) \equiv \sum_{k=1}^{N-1} \frac{m_k}{\sqrt{k(k+1)}} O_{ka} + \frac{m_N}{\sqrt{N}} O_{Na} \quad (2.10)$$

$$\mu_1 = (1, 1, 1, 1, \dots, 1, 1, 1)$$

$$\mu_2 = (-1, 1, 1, 1, \dots, 1, 1, 1)$$

$$\mu_3 = (0, -2, 1, 1, \dots, 1, 1, 1)$$

(2.11)

$$\mu_4 = (0, 0, -3, 1, \dots, 1, 1, 1)$$

.....

$$\mu_{N-1} = (0, 0, 0, 0, \dots, -(N-2), 1, 1)$$

$$\mu_N = (0, 0, 0, 0, \dots, 0, -(N-1), 1)$$

*) The fact that B and C are duals to each other is explicitly seen here. However, the difference between the corresponding fundamental representations arises due to the difference of the multiplicity of the corresponding zero weights.

D)

$$(m_1, m_2, \dots, m_N) \equiv \sum_{k=1}^{N-3} \frac{m_k}{\sqrt{k(k+1)}} O_{ka} + \frac{m_{N-2}}{\sqrt{4(N-2)}} O_{N-2,a} + \frac{1}{\sqrt{2}} \{m_{N-1} O_{N-1} + m_N O_{Na}\} \quad (2.12)$$

$$\mu_1 = (1, 1, 1, 1, \dots, 1, \pm 2, 0, 0)$$

$$\mu_2 = (-1, 1, 1, 1, \dots, 1, \pm 2, 0, 0)$$

$$\mu_3 = (0, -2, 1, 1, \dots, 1, \pm 2, 0, 0)$$

$$\mu_4 = (0, 0, -3, 1, \dots, 1, \pm 2, 0, 0)$$

.....

$$\mu_{N-2} = (0, 0, 0, 0, \dots, -(N-3), \pm 2, 0, 0) \quad (2.13)$$

$$\mu_{N-1} = (0, 0, 0, 0, \dots, 0, 1, 1)$$

$$\mu_N = (0, 0, 0, 0, \dots, 0, -1, 1)$$

Now it is easily seen that the corresponding generators λ_a in the Gell-Mann basis will be obtained automatically by the relation

$$H^a = \frac{1}{\sqrt{2}} \lambda^b O_{ba} \quad (2.14)$$

In view of expressions (2.9), (2.11) and (2.13), the explicit constructions of the generators λ_a are clear for the fundamental representation. However, within the framework of Ref.7, these results can be directly generalized to all irreducible representations because we formulate there all irreducible representations of the chains A_N, B_N, C_N, D_N in terms of their FRW's. On the other hand, it is immediately seen that the normalization (1.3) is determined in this case as

$$\text{Tr}\{\lambda^a(R_0)\lambda^b(R_0)\} = 2 \delta_{ab} \quad (2.15)$$

which is essential for the Gell-Mann basis.

III. THE DESCRIPTION OF CHARGED GENERATORS

Although the previous section reflects our central idea for this work, it is worthwhile to study some aspects for the generators E_α which we would like to call "charged generators" of the Lie algebra. *) However, the matrix elements of these generators can be fixed up to some factor generally. The central point here is expression 2 of (2.1). This expression specifies the non-zero elements of the generators E_α . Additionally, the values of this non-zero element are determined by the relations 4), 4i) and 4ii) in (2.1). Hence in favour of the Serre theorem, the Gell-Mann basis generators will be obtained by the following correspondences for each root of ϕ :

$$\begin{aligned} \lambda_{a1} &\equiv E_{-\alpha} + E_\alpha \\ \lambda_{a2} &\equiv i(E_{-\alpha} - E_\alpha) \end{aligned} \quad (3.1)$$

Hence, all elements of the Gell-Mann basis are obtained with the aid of the fundamental representation weights because the non-zero elements of the charged generators are also given with the aid of these weights.

However, there is a point which must be emphasized here. There is no ambiguity to define the non-zero elements of the charged generators. On the other hand, expressions 4), 4i) and 4ii) cannot always specify the values and especially the signs of these non-zero elements. This point is explicit even in some simple cases such as fundamental representations of SU(3) and SU(5) and spinor representations of SO(10) and SO(14) as will be shown in the appendix. In consequence, the normalization condition (2.15) becomes insufficient to fix a normalization. This point may or may not have concrete effects on physical calculations. But, for example, the specification of the proton decay operators in the conventional SU(5) model ⁹⁾ depends on such a definition. Some of the vector operators will be pseudo-vectors if the minus sign is chosen in the definition of the charged generators.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

*) In that case, the diagonal generators H^a will be called the "neutral generators" of the Lie algebra.

APPENDIX

It will be useful to give an example on the method especially for the representations other than the fundamental representations. Consider the representation 10 of SU(5). We know now ⁷⁾ that this representation has the following 10 weights expressed in terms of FRW's:

$$\mu_1 + \mu_2 = (0, 2, 2, 2)$$

$$\mu_1 + \mu_3 = (1, -1, 2, 2)$$

$$\mu_1 + \mu_4 = (1, 1, -2, 2)$$

$$\mu_1 + \mu_5 = (1, 1, 1, -3)$$

$$\mu_2 + \mu_3 = (-1, -1, 2, 2)$$

$$\mu_2 + \mu_4 = (-1, 1, -2, 2)$$

$$\mu_2 + \mu_5 = (-1, 1, 1, -3)$$

$$\mu_3 + \mu_4 = (0, -2, -2, 2)$$

$$\mu_3 + \mu_5 = (0, -2, 1, -3)$$

$$\mu_4 + \mu_5 = (0, 0, -3, -3)$$

The right-hand sides of these expression are extracted from the forms (2.9) in the case of SU(5). And now, in view of the relation (2.14), the diagonal Gell-Mann generators will be as in what follows for the representation 10:

$$\lambda^1(\underline{10}) = \text{diag}(0, 1, 1, 1, -1, -1, -1, 0, 0, 0)$$

$$\lambda^2(\underline{10}) = \frac{1}{\sqrt{3}} \text{diag}(2, -1, 1, 1, -1, 1, 1, -2, -2, 0)$$

$$\lambda^3(\underline{10}) = \frac{1}{\sqrt{6}} \text{diag}(2, 2, -2, 1, 2, -2, 1, -2, 1, -3)$$

$$\lambda^4(\underline{10}) = \frac{1}{\sqrt{10}} \text{diag}(2, 2, 2, -3, 2, 2, -3, 2, -3, -3)$$

λ

where the conventional notation is given at the extreme right-hand sides. It is explicitly seen that the corresponding normalization for this representation is $N(\underline{10}) = 6 \neq N(5) = 2$. The method works in the same way for all irreducible representations of the groups SU(N+1), SO(2N+1), SP(2N) and SO(2N).

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