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LECTURES ON THE INVERSE SCATTERING METHOD

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ABSTRACT

In a series of six lectures an elementary introduction to the theory of inverse scattering is given. The first four lectures contain a detailed theory of solitons in the framework of the KdV equation, together with the inverse scattering theory of the one-dimensional Schrödinger equation. In the fifth lecture the dressing method is described, while the sixth lecture gives a brief review of the equations soluable by the inverse scattering method.

АННОТАЦИЯ

Дается краткое изложение серии докладов о теории обратного рассеяния, прочитанных в рамках I-ой советско-венгерской группы по теории твердых тел. В первых четырех докладах была изложена теория обратиого рассеяния в случае одномерного уравления Шредингера и теория солитонов в рамках уравнений KdV. В пятой лекции кратко излагается метод "дрессинга", а в шестой лекции обозреваются те типы уравнений, решение которых можио найти, используя метод обратного рассеяния.

KIVONAT

A hat előadásból álló sorozatban elemi bevezetést adunk az inverz szórás elméletébe. Az első négy előadásban kidolgozzuk az 1-dimenziós Schrödinger egyenlet inverz szórás elméletét és részletes elméletet épitünk fel a KdV egyenlet megoldásaiként kapható szolitonokra. Az ötödik előadásban leirjuk a "dressing"-módszert, a hatodik előadásban pedig röviden áttekintjük azokat az egyenleteket, amelyek az inverz szórás módszer segitségével megoldhatók.

INTRODUCTION

The term "soliton" is one of the most popular in the world of theoretical physics. In fact, the soliton concept is exploited in almost all parts of this vast world - from the theory of elementary particles to cosmology - and with great success. The developments of soliton theory have led to the birth of a new and highly productive branch of mathematical physics, the inverse scattering method.

These six lectures constitute an elementary introduction to the theory of inverse scattering. They were read at the Department of Solid State Theory of the Central Research Institute for Physics, Budapest, in October, 1982 during the first Hungarian-Soviet workshop on solid state theory. The first four lectures contain a detailed theory of solitons in the framework of the KdV equation, together with the inverse scattering theory of the one-dimensional Schrödinger equation.

The fifth lecture demonstra . on a simple example the more elementary "dressing method" and the sixth lecture is devoted to a general review of integrable nonlinear equations of mathematical physics.

I am most grateful to Dr. András Sütő for carrying out the exacting task of preparing these lectures for publication.

Lecture One

I should first like to write up some nonlinear differential equations:

1) $u_t - 6uu_x + u_{xxx} = 0$ (1.1)

2) $r_t + r_{xx} \pm |r|^2 r = 0$ (1.2)

3) $u_{\xi\eta} = \sin u$ (1.3)

$u_{\xi\eta} = \sinh u$ (1.4)

4) $\vec{n}_{tt} - \vec{n}_{xx} + (\vec{n}_t^2 - \vec{n}_x^2)\vec{n} = 0 \quad ; \quad \vec{n}^2 = 1$ (1.5)

Each of these equations has a universal physical application. The first of them is known as the Korteveg-de Vries (KdV) equation. Let us assume a nonlinear medium having a sound-like excitation branch with the dispersion law

$$\omega^2 = s^2k^2(1 + \varepsilon k^2) \qquad \varepsilon k^2 \ll 1$$

For small εk^2 we have

$$\omega = \pm sk(1 + \frac{\varepsilon}{2}k^2 + \dots) \tag{1.6}$$

Different signs in (1.6) correspond to waves moving in opposite directions. If we regard only waves moving to the right, we have to choose the plus sign. Let $U(x,t)$ be some variable describing the medium (maybe the density variation or macroscopic velocity of the medium). From (1.6) we find that u obeys the equation

$$\frac{\partial u}{\partial t} + s\frac{\partial u}{\partial x} - \frac{\varepsilon s}{2}\cdot\frac{\partial^3 u}{\partial x^3} = 0 \qquad (1.7)$$

Now we shall include nonlinear effects in the consideration. The most simple such effect is the dependence of the sound velocity on u. Assuming u to be small, we have

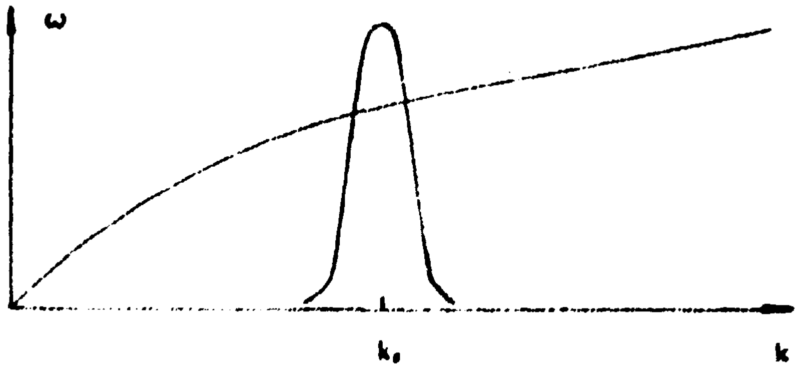
$$s = s_0 + s_1 u + \dots \qquad (1.8)$$

Substituting (1.8) in (1.7) we neglect the small term $(\varepsilon s_1/2)u\partial^3 u/\partial x^3$. After going to a coordinate system moving with velocity s_0 we obtain

$$\frac{\partial u}{\partial t} + s_1 u\frac{\partial u}{\partial x} - \frac{\varepsilon s}{2}\cdot\frac{\partial^3 u}{\partial x^3} = 0$$

An appropriate change of variables leads eventually to the KdV equation (1.1).

Equation (1.2) describes the motion of a nearly monochromatic "wave train" in a dispersive nonlinear medium. Let us suppose that the Fourier transform of the wave train, according to the spatial variable x, is a function with a sharp maximum at k_0. Then the dispersion law $\omega = \omega(k)$ can be expanded



in a Taylor series in the vicinity of k_0. With $k = k_0 + \varkappa$ we have

$$\omega(k_0+\varkappa) = \omega(k_0) + \frac{\partial\omega}{\partial k}\Big|_{k_0}\cdot\varkappa + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\Big|_{k_0}\cdot\varkappa^2 + \dots \tag{1.9}$$

Let $U(x,t)$ again be a variable describing the medium and $U(k,\omega)$ be its Foùrier transform. We define the new function $r(\varkappa,\Omega) = U(k_0+\varkappa\,,\,\omega(k_0)+\Omega)$. After inverse Fourier transform we get the function $r(x,t)$ describing the slowly varying envelope of the wave train. It obeys the equation

$$i(r_t + v_{gr} r_x) - \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\Big|_{k_0}\cdot r_{xx} = 0 \tag{1.10}$$

where $v_{gr} = \partial\omega/\partial k\,|_{k_0}$ is the group velocity. To include nonlinear effects, let us note that in a nonlinear medium the frequency of a wave at a fixed k_0 depends on the wave amplitude. For slowly varying wave trains we may consider this dependence to be local at x . If this dependence has the form

$$\omega_k \longrightarrow \omega_k + T|r|^2$$

we can change Eq. (1.10) to

$$i(r_t + v_{gr} r_x) - \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\Big|_{k_0}\cdot r_{xx} = T|r|^2 r \tag{1.11}$$

After going to a moving coordinate frame and changing the variables appropriately we get (1.2). The sign before $|r|^2 r$ coincides with the sign of the product $\omega'' T$ and cannot be changed by any transformation. Eq. (1.2) may be called the nonlinear Schrödinger equation (NSE).

Equation (1.3) is known as the "sine-Gordon" equation. It has many physical applications such as, for instance, in the theory of superconductivity for the description of Josephson oscillations. In fact, this equation has an important geometrical meaning as an equation of surfaces with constant negative curvature. In this connection it was studied by mathematicians in the last century, and some of its essential properties (e.g. Bäcklund transformation) were found more than a hundred years ago. The same is true for the related Eq. (1.4).

Eq. (1.5) is the result of minimizing the action with Lagrange density $\mathcal{L} = \vec{n}_t^2 - \vec{n}_x^2$, accounting for the additional restriction $\vec{n}^2 = 1$ (here $\vec{n}$ is a three-dimensional vector). This "$\vec{n}$ -field model" is popular as a simple model of the quark confinement theory.

We can enlarge the list of equations:

5) Boussinesque equation

$$u_{tt} - u_{xx} + (u^2)_{xx} \pm u_{xxxx} = 0$$

6) Three-wave resonance equations

$$\frac{\partial u_0}{\partial t} + v_0 \frac{\partial u_0}{\partial x} = i u_1 u_2$$

$$\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} = i u_0 u_2^*$$

$$\frac{\partial u_2}{\partial t} + v_2 \frac{\partial u_2}{\partial x} = i u_0 u_1^*$$

7) Exciton-phonon interaction equations

$$i \psi_t + \psi_{xx} = n \psi$$

$$n_t - n_x = |\psi|_x^2$$

8) Landau-Lifshitz equation for a one-dimensional ferromagnet

$$\vec{S}_t = [\vec{S}, \vec{S}_{xx}]$$

and so on. The total number of equations in this list is more than forty.

All these equations are strongly nonlinear. Nevertheless they could effectively be studied by the new mathematical method called the inverse scattering method. This method was invented by Gardner, Green, Kruskal and Miura. Their pioneering work was published in October 1967, that is, fifteen years ago.

The main idea of the method is that all these nonlinear equations are the compatibility conditions of some overdetermined systems of linear equations which are "good enough" (in a sense to be clarified later).

Let us consider the system of two linear differential equations written for a scalar complex function $\psi(x,t)$,

$$\psi_{xx} = u\psi \tag{1.12}$$

$$\psi_t = A\psi_x + B\psi \tag{1.13}$$

Differentiating (1.12) with respect to t gives

$$\psi_{xxt} = u\psi_t + u_t\psi = (u_t + uB)\psi + uA\psi_x \tag{1.14}$$

The second derivative of (1.13) with respect to x is

$$\psi_{xxt} = (A_{xx} + 2B_x + Au)\psi_x + [(A_x+B)u + B_{xx} + (Au)_x]\psi \tag{1.15}$$

Comparing (1.14) and (1.15) we find

$$(A_{xx} + 2B_x)\psi_x + (-u_t + B_{xx} + A_x u + (Au)_x)\psi = 0 \tag{1.16}$$

We must now explain how we understand the compatibility. We shall require that any solution of (1.12) given in $t=t_0$

develops into a solution of (1.13) which remains a solution of (1.12) for any $t > t_0$. Now (1.16) can be satisfied only if

$$A_{xx} + 2B_x = 0 \tag{1.17}$$

$$U_t = B_{xx} + A_x U + (AU)_x \tag{1.18}$$

or

$$U_t = -(1/2)A_{xxx} + A_x U + (AU)_x \tag{1.19}$$

Eq. (1.19) imposes one condition on two functions, U and A . Let us assume that these are also functions of the complex parameter k^2 , and Eq. (1.19) is satisfied identically in this parameter. Let

$$U = u - k^2 \qquad A = v + \alpha k^2 \tag{1.20}$$

Substituting (1.20) in (1.19), the coefficients of the zeroth and second-order terms in k must vanish. Hence we have

$$\begin{aligned} 2v_x &= \alpha u_x \\ u_t &= -(1/2)v_{xxx} + 2v_x u + u_x v \end{aligned} \tag{1.21}$$

If we choose $\alpha = 4$ then $v = 2u$ and from (1.21) we obtain

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.22}$$

Thus, the KdV Eq. (1.22) is the compatibility condition of the linear system

$$\psi_{xx} + (k^2 - u)\psi = 0 \tag{1.23}$$

$$\psi_t = (k^2 + 2u)\psi_x - u_x\psi \tag{1.24}$$

We will now exploit this remarkable fact.

Let us represent ψ in the form

$$\psi = \exp\left(i\int_x^\infty \chi\, dx\right)$$

We have for χ (using (1.23) and (1.24))

$$i\chi_x + \chi^2 + u - k^2 = 0 \tag{1.25}$$

and

$$\int_x^\infty \chi_t dy = -(4k^2+u)\chi + iu_x$$

Hence

$$\chi_t = \frac{\partial}{\partial x}\left\{(4k^2+u)\chi - iu_x\right\} \tag{1.26}$$

Let $\chi = \xi - k$, then from (1.25) we have

$$\xi = (1/2k)(u + \xi^2 + i\xi_x) \tag{1.27}$$

and from (1.26)

$$\xi_t = \frac{\partial}{\partial x}\left[(4k^2+u)\xi - ku - iu_x\right] \tag{1.28}$$

If $u \to 0$ at $x \to \pm\infty$ we can also consider $\xi_x \to 0$ at $x \to \pm\infty$. Now we have $\xi \to 0$ or $2k$ if $x \to \pm\infty$ and

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \xi\, dx = 0$$

Hence the function

$$I(k) = \int_{-\infty}^{\infty} \xi(k)\, dx$$

represents a conservation law for Eq. (1.1). It is the generating function of an infinite number of conserved quantities which do not depend on k . They are obtained as integrals of the coefficients in the asymptotic series

$$\xi = \sum_{l=0}^{\infty} \frac{1}{(2k)^{l+1}}\, \xi_l$$

For ξ_l, a recursion formula can be inferred from (1.27):

$$\xi_{n+1} = i(\xi_n)_x + \sum_{l=0}^{n-1} \xi_l \, \xi_{n-l-1} \quad ; \quad n>0 \; ; \; \xi_0 = u$$

so that

$$\xi_1 = iu_x \;, \quad \xi_2 = -u_{xx} + u^2 \;, \; \ldots$$

It is seen that

$$\int_{-\infty}^{\infty} \xi_1 dx = 0$$

Similarly, for all n

$$\int_{-\infty}^{\infty} \xi_{2n+1} \, dx = 0$$

On the other hand, the integrals

$$\int_{-\infty}^{\infty} \xi_{2n} \, dx = I_n$$

are non-vanishing quantities, independent of t. For example

$$I_0 = \int u dx \quad , \quad I_1 = \int u^2 dx \quad , \quad I_2 = \int ((u_x)^2 + 2u^3) \, dx \;,$$

$$I_3 = \int ((u_{xx})^2 - 5u^2 u_{xx} + 5u^4) \, dx \;, \; \ldots$$

All these integrals have the form

$$I_n = \int P(u, u_x, \ldots, u^{(n)}) \, dx \tag{1.29}$$

where P is a polynomial of u and its derivatives. They are called polynomial integrals of motion.

Let us make one more remark. It is, of course, not necessary to choose U and A as linear functions of k^2. In fact, they could be chosen as arbitrary rational functions of k^2.

Let us take, for example,

$$U = u - k^2 \quad , \quad A = v_1 + v_2 k^2 + 4k^4$$

Then after substitution in (1.19) we obtain an equation which is quadratic in k^2 . If we put the three coefficients equal to zero, we find

$$v_2 = 4u \qquad v_1 = 4u^2 - 2u_{xx}$$

$$u_t = 12u^2u_x - 2u_x u_{xx} - 2(u^2)_{xxx} + u_{xxxxx} \tag{1.30}$$

The last one is the so-called "second KdV equation". If A is a polynomial of arbitrary degree we get "higher" KdV equations.

It is obvious that all of these equations will have the same integrals of motion (1.29). This remains true for more general systems of equations arising if we choose A as a rational function of k^2 . But any change of U (for instance, if it is a rational function of k^2) gives rise to other equations and other conserved quantities.

Exercise 1.

Find the compatibility conditions for the system (1.12), (1.13) if

$$U = uk^2 \quad , \quad A = vk^2$$

Exercise 2.

Find the compatibility conditions if

$$U = -\frac{w^2}{k^2} + u - k^2$$

$$A = v + 4k^2$$

Lecture Two

Let us recall that the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2.1}$$

is the compatibility condition for the overdetermined linear system

$$\psi_{xx} + (k^2 - u)\psi = 0 \tag{2.2}$$

$$\psi_t = (2u + 4k^2)\psi_x - u_x\psi \tag{2.3}$$

Let ψ be a compatible solution of this system and $u \to 0$ at $x \to \pm\infty$. Let us choose the boundary conditions so that

$$\begin{aligned} &\psi \to \alpha(t,k)\, e^{-ikx} && \text{if } x \to -\infty \\ &\psi \to \beta(t,k)\, e^{-ikx} + \gamma(t,k)\, e^{ikx} && \text{if } x \to +\infty \end{aligned} \tag{2.4}$$

and substitute (2.4) in (2.3). At $x \to \pm\infty$, (2.3) reduces to

$$\psi_t = 4k^2\psi_x \tag{2.5}$$

This and (2.4) yield

$$\begin{aligned} \dot\alpha(k,t) &= -4ik^3\alpha \\ \dot\beta(k,t) &= -4ik^3\beta \\ \dot\gamma(k,t) &= +4ik^3\gamma \end{aligned} \tag{2.6}$$

A new solution of (2.2) is ψ/β . (It is, however, not the solution of (2.3)!). Asymptotically

$$\begin{aligned} &\psi/\beta \to t(k,t)\, e^{-ikx} && x \to -\infty \\ &\psi/\beta \to r(k,t)\, e^{ikx} + e^{-ikx} && x \to +\infty \end{aligned}$$

Here

$$t(k,t) = \frac{\alpha(t,k)}{\beta(t,k)} = t(k) \tag{2.7}$$

does not depend on t ($t(k)$ and t are not to be confused) and

$$r(k,t) = \frac{\gamma(t,k)}{\beta(t,k)} = r(k)\, e^{8ik^3 t} \tag{2.8}$$

The asymptotic condition (2.6) describes the scattering problem assigned to the Schrödinger equation. Here $t(k)$ is the forward scattering amplitude and $r(k)$ is the amplitude of the back-scattering.

Formulae (2.7) and (2.8) are very important and worth further discussion. Equation (2.2) is the stationary Schrödinger equation and its potential, U , depends on the parameter t . In general, the scattering data $t(k,t)$ and $r(k,t)$ also depend on t . Now (2.7) and (2.8) claim that if U obeys the KdV equation (2.1) then the time dependence of the scattering data is extremely simple: $t(k,t)$ is constant and and $r(k,t)$ is proportional to $\exp(8ik^3t)$.

This fact lies at the foundation of the inverse scattering method. Indeed, we can solve the Cauchy problem for the KdV equation using the scheme

$$U(x)\big|_{t=0} \xrightarrow{\text{I.}} r(k)\big|_{t=0} \xrightarrow{\text{II.}} r(k,t) \xrightarrow{\text{III.}} U(x,t) \tag{2.9}$$

In the first step one finds the scattering data from the initial form of the potential. Secondly, using formula (2.8) the scattering data are continued to arbitrary t . Thirdly, the potential $U(x,t)$ is restored from the scattering data. In this way, the first step is the solution of a direct scattering problem: the third one presents the inverse scattering problem; the second step is trivial. We see that for an effective exploitation of formulae (2.7) and (2.8) we need a developed theory of the direct and inverse scattering for the Schrödinger equation. The theory of direct scattering is usually described in any standard course of quantum mechanics; the inverse scattering theory is not so widely known. The latter was developed during the fifties, mainly in the Soviet Union in some works of Gel'fand, Levitan, Marchenko and Faddeev. Their main tool was the analytic continuation of the solutions of the Scrödinger equation, which we are going to discuss now.

Let us come back to formula (2.4) and define the new function

$$\varphi_1(x,k) = \psi(x,k)/\alpha(k) \qquad (2.10)$$

Asymptotically

$$\varphi_1 \longrightarrow e^{-ikx} \qquad \text{if } x \to -\infty$$
$$\varphi_1 \longrightarrow a(k)\, e^{-ikx} + b(k)\, e^{ikx} \qquad \text{if } x \to +\infty$$

where

$$a(k) = \beta/\alpha \quad , \quad b(k) = \gamma/\alpha$$

Similarly we define

$$\left.\begin{array}{ll} \varphi_2 \longrightarrow e^{ikx} & x \to -\infty \\ \psi_1 \longrightarrow e^{-ikx} & \\ \psi_2 \longrightarrow e^{ikx} & x \to +\infty \end{array}\right\} \qquad (2.11)$$

It is seen that

$$\varphi_2(-k) = \varphi_1(k) = \varphi_2^*(k)$$

$$\psi_2(-k) = \psi_1(k) = \psi_2^*(k)$$

All these functions are called Jost's functions (in fact, they were studied long before Jost, at the beginning of the century). Both of the pairs, $\varphi_{1,2}$ or $\psi_{1,2}$, form a basis in the space of the solutions of (2.2). Therefore

$$\varphi_i = T_{ik}\,\psi_k \tag{2.12}$$

The matrix T_{ik} is often called the monodromy matrix. It can be reconstructed from (2.10) as

$$T_{ik} = \begin{bmatrix} a(k) & b(k) \\ b^*(k) & a^*(k) \end{bmatrix}$$

and tnerefore

$$\begin{aligned} \varphi_1(x,k) &= a(k)\,\psi_1(x,k) + b(k)\,\psi_2(x,k) \\ \varphi_2(x,k) &= b^*(k)\,\psi_1(x,k) + a^*(k)\,\psi_2(x,k) \end{aligned} \tag{2.13}$$

Let $\{f_1, f_2\}$ denote the Wronskian of any two solutions, f_1 , f_2 , of (2.2):

$$\{f_1, f_2\} = f_1 f_{2x} - f_{1x} f_2 = -\{f_2, f_1\}$$

We recall that

$$\frac{d}{dx}\{f_1, f_2\} = 0$$

and therefore the Wronskian of Jost's functions can be constructed from their asymptotic forms, giving

$$\{\varphi_1, \varphi_2\} = \{\psi_1, \psi_2\} = 2ik \tag{2.14}$$

Substituting (2.13) into (2.14) we obtain

$$|a(k)|^2 - |b(k)|^2 = 1 \tag{2.15}$$

that is, matrix T is unimodular. The scattering data are expressed through $a(k)$ and $b(k)$ as

$$t(k) = 1/a(k) \quad , \quad r(k) = b(k)/a(k)$$

implying

$$|t(k)|^2 + |r(k)|^2 = 1$$

This is the unitarity theorem for the one-dimensional scattering. For later use we list also the Wronskians

$$\{\varphi_1, \psi_2\} = 2ik\,a(k) \qquad (2.16)$$
$$\{\varphi_1, \psi_1\} = -2ik\,b(k)$$

In what follows, we derive integral equations for Jost's functions. Let us solve (2.2) by the method of the variation of the constant. We seek Jost's function φ in the form

$$\varphi = c_1(x)e^{ikx} + c_2(x)e^{-ikx}$$

Having two new functions instead of the single old one, we may still impose the condition

$$\text{(a)} \qquad c_{1x}e^{ikx} + c_{2x}e^{-ikx} = 0$$

With this, one finds

$$\varphi_x = ik(c_1 e^{ikx} - c_2 e^{-ikx})$$

and

$$\text{(b)} \qquad c_{1x}e^{ikx} - c_{2x}e^{-ikx} = \frac{1}{ik}\,U\varphi$$

From (a) and (b) and the boundary conditions $c_1 \to 0$, $c_2 \to 1$ at $x \to -\infty$ (appropriate for φ_1) it is found that

$$c_1 = \frac{1}{2ik}\int_{-\infty}^{x} U(y)\,\varphi(y)\,e^{-iky}\,dy$$

$$c_2 = 1 - \frac{1}{2ik}\int_{-\infty}^{x} U(y)\,\varphi(y)\,e^{iky}\,dy$$

Thus we obtain

$$\varphi_1(x,k) = e^{-ikx} + \frac{1}{k}\int_{-\infty}^{x} \sin k(x-x')\,U(x')\,\varphi_1(x',k)\,dx' \qquad (2.17a)$$

or, for the function

$$\chi(x,k) = e^{ikx}\,\varphi_1(x,k)$$

the equation reads as

$$\chi(x,k) = 1 + \int_{-\infty}^{x} \frac{e^{2ik(x-x')}-1}{2ik}\,U(x')\,\chi(x',k)\,dx' \qquad (2.17b)$$

Let now $k = k_0 + i\eta$ and $\eta > 0$. Since $x - x' = z > 0$, we have

$$|e^{2ik(x-x')}| = e^{-2\eta z} \qquad (2.18)$$

Eq. (2.17) is of the Volterra type and is solvable if its kernel is bounded. According to (2.18), this holds for k in the upper half plane. Therefore, $\chi(x,k)$ can be continued analytically to the upper half plane. In the limit $|k| \to \infty$ we have

$$\chi = 1 - \frac{1}{2ik}\int_{-\infty}^{x} U(y)\,dy + O(|k|^{-2})$$
$$\varphi_1 \approx e^{-ikx}\left(1 - \frac{1}{2ik}\int_{-\infty}^{x} U(y)\,dy\right) \qquad (2.19)$$

Repeating these computations for ψ_2 , it also proves to be analytic for $\mathrm{Im}\,k > 0$, and in the limit $|k| \to \infty$,

$$\psi_2 \cong e^{ikx}\left(1 + \frac{1}{2ik}\int_x^\infty U(y)dy\right)$$

Now (2.16) tells us that $a(k)$ has an analytic continuation to the upper half plane and that

$$a(k) = 1 + O(|k|^{-1}) \qquad \text{for} \quad |k| \to \infty \tag{2.20}$$

The analytic continuation to $\operatorname{Im} k > 0$ permits us to discuss the eigenvalue problem of the operator $(\partial^2/\partial x^2 - U)$ in the space $L^2(-\infty, \infty)$. This operator is self-adjoint, its eigenstates are bound states with real negative eigenvalues

$$k^2 = -E_n = (i\kappa_n)^2 \qquad , \quad \kappa_n > 0$$

Let us consider the Jost function φ_1 , as given by the integral equation, at $k = i\kappa_n$. Let us suppose, moreover, that the support of U is finite. One can show that in this case ψ_1, ψ_2 and hence $a(k)$ and $b(k)$ are analytic functions in the whole plane. From the asymptotic formulae for φ_1 ,

$$\varphi_1 \to e^{\kappa_n x} \qquad\qquad x \to -\infty$$

$$\varphi_1 \to a(i\kappa_n)e^{\kappa_n x} + b(i\kappa_n)e^{-\kappa_n x} \qquad x \to +\infty$$

we infer that φ_1 can be a bound state only if

$$a(i\kappa_n) = 0 \tag{2.21}$$

The solutions of this equation provide the discrete spectrum of the operator. The relation

$$\varphi_1 = c_n \psi_2 \tag{2.22}$$

also holds for $k = i\kappa_n$.

In this way, we have obtained the important result that $a(k)=0$ at the points of the discrete spectrum. It is easy to see that this remains true also for potentials of non-compact support. The zeroes of $a(k)$ are simple because the eigenvalues of the one-dimensional Schrödinger operator are non-degenerate: if f_1 and f_2 are two eigenfunctions belonging to the same eigenvalue, then

$$\{f_1 ; f_2\} = 0$$

because both of these functions vanish at $x \to \pm\infty$. Therefore f_1 and f_2 are proportional to each other.

One can see from (2.6) that $a(k)=\beta/\alpha$ does not depend on t. In this way, the eigenvalues of the Schrödinger equation (2.1) do not depend on t if the time evolution of u follows the KdV equation (or, of course, if u is independent of t). On the other hand, c_n of Eq. (2.22) is time dependent. Let ψ be the compatible solution of (2.2) and (2.3) at $k=i\kappa$ with the asymptotic boundary conditions

$$\begin{aligned} &\psi \to \alpha_n(t)\, e^{\kappa_n x} && x \to -\infty \\ &\psi \to \beta_n(t)\, e^{-\kappa_n x} && x \to +\infty \end{aligned} \qquad (2.23)$$

Substituting (2.23) into Eq. (2.5) we obtain

$$\dot{\alpha}_n = 4\kappa_n^3 \alpha_n \qquad \dot{\beta}_n = -4\beta_n \kappa_n^3$$

$$\alpha_n = \alpha_{n0}\, e^{4\kappa_n^3 t} \qquad \beta_n = \beta_{n0}\, e^{-4\kappa_n^3 t}$$

According to the definition of φ_1, $\varphi_1 = \psi/\alpha_n$ and $\psi_2 = \psi/\beta_n$ and from (2.22) it is found that

$$c_n = \beta_n/\alpha_n = c_{n0}\, e^{-8\kappa_n^3 t}$$

The set of quantities $r(k)$, $\varkappa_n$, c_n is called the "scattering data"; once it is given, the inverse scattering problem can be solved in order to find the potential u .

Exercise

The potential $u(x)$ obeys the conditon

$$|u(x)| \le c\, e^{-\varkappa |x|} \qquad \text{at } |x| \to \infty \quad (\varkappa > 0)$$

What is the bound for the analytic continuation of $\varphi(x,k)$ and $a(k)$ to the lower half plane ?

Lecture Three

Let us now introduce for the Jost function the so-called triangle representation. Consider the function

$$\tilde{\psi}_2(x,k) = \psi_2(x,k) - e^{ikx}$$

We know that $\psi_2(x,k)$ is an analytic function of k in the upper half plane and that asymptotically it behaves as

$$\psi_2(x,k) \approx e^{ikx} - \frac{e^{ikx}}{2ik}\int_x^{\infty} U(x')\,dx' \tag{3.1}$$

$$(k \to \infty\ ,\quad \mathrm{Im}\, k \geq 0)$$

Hence

$$\tilde{\psi}_2(x,k) = e^{ikx}\, O(1/k) \tag{3.2}$$

and

$$\int_{-\infty}^{\infty} |\tilde{\psi}_2|^2\, dk < \infty$$

so that $\tilde{\psi}_2(x,k)$ can be Fourier-transformed in k :

$$K(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi}_2(x,k)\, e^{-iky}\, dk \tag{3.3}$$

The inverse transformation yields

$$\tilde{\psi}_2(x,k) = \int_{-\infty}^{\infty} K(x,y)\, e^{iky}\, dy$$

In (3.3) one can shift the contour of integration upwards in the upper half plane, parallel to the real axis. From (3.2) we see that the integrand of (3.3) decays exponentially with $\mathrm{Im}\, k$ going to $+\infty$, if $x > y$. Therefore $K(x,y) = 0$ if $x > y$.

Finally we have

$$\psi_2(x,k) = e^{ikx} + \int_x^{\infty} K(x,y)\, e^{iky}\, dy \qquad (3.4)$$

This equation is called the triangle representation for ψ_2 .
In (3.4) let k go to infinity. We get

$$\psi_2(x,k) \simeq e^{ikx} - \frac{K(x,x)}{ik}\, e^{ikx} + \dots \qquad (3.5)$$

Comparing (3.1) and (3.5) we obtain

$$U(x) = -2\frac{d}{dx} K(x,x) \qquad (3.6)$$

In this way, one can find the potential $U(x)$ if the kernel $K(x,y)$ of the triangle representation is known. For the Jost function $\psi_1(x,k)$ we have

$$\psi_1(x,k) = e^{-ikx} + \int_x^{\infty} K(x,y)\, e^{-iky}\, dy \qquad (3.7)$$

Let us rewrite the first equation of (2.13) in the form

$$\frac{\psi_1(x,k)}{a(k)} - e^{-ikx} = \psi_1 - e^{-ikx} + r(k)\,\psi_2 =$$

$$= \int_x^{\infty} K(x,y)\, e^{-iky}\, dy + r(k)\Big(e^{ikx} + \int_x^{\infty} K(x,y)\, e^{iky}\, dy\Big)$$

Let us then multiply this equation by $\exp(ikz)/2\pi$ and integrate over k from $-\infty$ to $+\infty$. Using the formula

$$\frac{1}{2\pi}\int e^{ik\xi}\, dk = \delta(\xi)$$

we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{\psi_1(x,k)}{a(k)} - e^{-ikx}\Big)\, e^{ikz}\, dk =$$

$$= K(x,z) + \tilde{F}(x+z) + \int_x^{\infty} K(x,y)\,\tilde{F}(y+z)\, dy \qquad (3.8)$$

Here

$$\tilde{F}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)\, e^{ik\xi}\, dk$$

Now remember that $\varphi_1(x,k)$ is an analytic function of k in the upper half plane, and

$$\frac{\varphi_1(x,k)}{a(k)} - e^{-ikx} \longrightarrow e^{-ikx}\, O\left(\frac{1}{k}\right) \qquad \text{at } k \to \infty\,,\ \operatorname{Im} k \geq 0$$

so that for $z > x$ the integral on the left-hand side of (3.8) is equal to the sum of the residues in the zeroes of $a(k)$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varphi_1(x,k)}{a(k)} - e^{-ikx}\right) e^{ikz}\, dk = \sum_n \frac{i\varphi_1(x,i\kappa_n)}{a'(i\kappa_n)}\, e^{-\kappa_n z} =$$

$$= \sum_n \frac{ic_n}{a'(i\kappa_n)}\, \psi_2(x,i\kappa_n)\, e^{-\kappa_n z} = -\sum_n M_n^2\, \psi_2(x,i\kappa_n)\, e^{-\kappa_n z} \tag{3.9}$$

Here we used the equality (2.22)

$$\varphi_1(x,i\kappa_n) = c_n \psi_2(x,i\kappa_n)$$

and introduced the notation

$$M_n^2 = \frac{c_n}{i a'(i\kappa_n)}$$

From (3.4) one finds

$$\psi_2(x,i\kappa_n) = e^{-\kappa_n x} + \int_x^{\infty} K(x,y)\, e^{-\kappa_n y}\, dy \tag{3.10}$$

Substituting (3.10) into (3.9) and (3.9) into (3.8) we get the equation (known as the Marchenko equation) for the kernel K

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y)\, F(y+z)\, dy = 0 \tag{3.11}$$

$$F(\xi) = \tilde{F}(\xi) + \sum_n M_n^2\, e^{-\kappa_n \xi} =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)\, e^{ik\xi}\, dk + \sum_n M_n^2\, e^{-\kappa_n \xi} \tag{3.12}$$

We prove that M_n is real. Consider again the Schrödinger equations

$$\psi_{xx} + (k^2 - U)\psi = 0 \tag{3.13a}$$

$$\psi^*_{xx} + (k^2 - U)\psi^* = 0 \tag{3.13b}$$

and differentiate (3.13a) with respect to k :

$$\psi_{kxx} + (k^2 - U)\psi_k = -2k\psi \tag{3.14}$$

Let us multiply (3.13b) and (3.14) respectively by ψ_k and $-\psi^*$, add them and integrate the resulting equation over x :

$$\begin{aligned} 2k\int_{-\infty}^{\infty} |\psi|^2 dx &= \int_{-\infty}^{\infty} (\psi^*_{xx}\psi_k - \psi_{kxx}\psi^*)\, dx = \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}(\psi^*_x\psi_k - \psi_{kx}\psi^*)\, dx = (\psi^*_x\psi_k - \psi_{kx}\psi^*)\Big|_{-\infty}^{\infty} \end{aligned} \tag{3.15}$$

We apply this equation to $\psi = \varphi_1$, where we had (2.13)

$$\varphi_1(k) = a(k)\psi_1(k) + b(k)\psi_2(k)$$

At $k = i\kappa_n$ $a(i\kappa_n) = 0$. Hence

$$\varphi_{1k}(i\kappa_n) \to a'(i\kappa_n)\,\psi_1(i\kappa_n) \to a'(i\kappa_n)\, e^{\kappa_n x} \quad at \quad x \to +\infty \tag{3.16}$$

and (3.15) and (3.16) yield

$$2i\int_{-\infty}^{\infty} |\varphi_1|^2 dx = -2c_n a'(i\kappa_n)$$

Therefore $a'(i\kappa_n)$ is pure imaginary and

$$M_n^2 = \int |\varphi_1|^2 dx \,/\, [i a'(i\kappa_n)]^2 > 0$$

Moreover, from (2.8)

$$M_n^2(t) = M_n^2(0)\, e^{8\kappa_n^3 t} \tag{3.17}$$

In connection with Eq. (3.16) we would mention that $\psi_1(k)$ cannot be continued analytically to the upper half plane. But formula (3.16) is nevertheless true. To prove this, it is sufficient to "cut off" the potential at $|x|>L$, for such a potential all the Jost functions are analytic in the whole complex plane. Eq. (3.16) does not depend on L , and remains true for L going to infinity.

Equations (2.8), (3.12) and (3.17) show that

$$\frac{\partial F}{\partial t} = 8\,\frac{\partial^3 F}{\partial \xi^3} \qquad (3.18)$$

Exercise 1.

Derive the Marchenko equation using functions which are analytic in the lower half plane.

Exerciese 2.

Write up $F(\xi)$ and $\tilde{F}(\xi)$ in the absence of the discrete spectrum. (Here $\tilde{F}(\xi)$ is the kernel of the Marchenko equation for the lower half plane.)

Lecture Four

Now we consider a special class of potentials in the Schrödinger equation: the so called "Bargmann's potentials". For these potentials $r(k)=0$ at all k, in other words, back-scattering is absent. So they are fully determined by the discrete spectrum of the Schrödinger operator. Let the eigenvalues be $-\kappa_n^2$ ($n=1,2,\dots,N$) . The coefficient $a(k)$ has zeroes at the points $k=i\kappa_n$.

On the other hand, from (2.15) we have $|a(k)|=1$ at real k . These facts and the analyticity in the upper half plane fully determine $a(k)$,

$$a(k)=\prod_{n=1}^{N}\frac{k-i\kappa_n}{k+i\kappa_n} \tag{4.1}$$

whence,

$$a'\big|_{k=i\kappa_n}=\frac{1}{2i\kappa_n}\prod_{m\neq n}\frac{\kappa_m-\kappa_n}{\kappa_m+\kappa_n}$$

is purely imaginary (cf. Lecture Three).

As to $F(\xi)$, it reads as

$$F(\xi)=\sum_{n=1}^{N}M_n^2\,e^{-\kappa_n\xi} \tag{4.2}$$

where

$$M_n^2=\frac{c_n}{i\,a'(i\kappa_n)}>0$$

From (4.2) we find that in the present case the kernel of the Marchenko equation

$$F(x+y)=\sum_n M_n^2\,e^{-\kappa_n(x+y)}=\sum_n M_n^2\,e^{-\kappa_n x}\cdot e^{-\kappa_n y}$$

is degenerate and the equation can be reduced to a finite system of linear algebraic equations. We put

$$K(x,y) = \sum_n \varphi_n(x)\, e^{-\varkappa_n y} \tag{4.3}$$

and for φ_n have the system of equations

$$\varphi_n(x) + \sum_{m=1}^{N} \frac{M_n^2 e^{-(\varkappa_n + \varkappa_m)x}}{\varkappa_n + \varkappa_m} \cdot \varphi_m(x) = - M_n^2 e^{-\varkappa_n x} \tag{4.4}$$

Let $A(x)$ denote the matrix of this system

$$A_{nm} = \delta_{nm} + \frac{M_n^2}{\varkappa_n + \varkappa_m} \cdot e^{-(\varkappa_n + \varkappa_m)x} \tag{4.5}$$

and let $A_n(x)$ be the matrix obtained after changing the n-th column of A_{mn} to $-M_m^2 \exp(-\varkappa_m x)$. Obviously

$$\varphi_n(x) = \frac{\det A_n(x)}{\det A(x)}$$

and

$$K(x,x) = \frac{\sum(\det A_n(x) e^{-\varkappa_n x})}{\det A(x)} \tag{4.6}$$

More explicitly

$$\det A(x) = \begin{vmatrix} 1 + \dfrac{M_1^2 e^{-2\varkappa_1 x}}{2\varkappa_1} & \dfrac{M_1^2 e^{-(\varkappa_1+\varkappa_2)x}}{\varkappa_1 + \varkappa_2} & \dots \\ \dfrac{M_2^2 e^{-(\varkappa_1+\varkappa_2)x}}{\varkappa_1 + \varkappa_2} & 1 + \dfrac{M_2^2 e^{-2\varkappa_2 x}}{2\varkappa_2} & \dots \\ \dots & \dots & \end{vmatrix}$$

Let us compute the x-derivative of $\det A$. To get it, we differentiate, consecutively, all columns and add the results. After differentiating the first column we have

$$e^{-\varkappa_1 x} \begin{bmatrix} -M_1^2 e^{-\varkappa_1 x} \\ -M_2^2 e^{-\varkappa_2 x} \\ \vdots \end{bmatrix}$$

One can see that the column in brackets is the right-hand side of (4.4). This is true for all the other columns and finally we have

$$\frac{d}{dx}\det A(x) = \sum_n (\det A_n(x))\, e^{-\kappa_n x}$$

and if this is compared with (4.6)

$$K(x,x) = \frac{d}{dx}\ln(\det A(x))$$

$$U(x) = -2\frac{d^2}{dx^2}\ln(\det A(x)) \qquad (4.7)$$

Formulae (4.7) give the description of Bargmann's potentials. From a physical point of view it is clear that all Bargmann's potentials are "potential wells": $U(x)<0$ at all x.

Let us now put $N=1$, then

$$\det A = 1 + \frac{M^2}{2\kappa} e^{-2\kappa x} = 1 + e^{-2\kappa(x-x_0)} \qquad (4.8)$$

For $U(x)$ we obtain

$$U(x) = -\frac{2\kappa^2}{ch^2\kappa(x-x_0)} \qquad (4.9)$$

If we remember that $M^2 = M_0^2 \exp(8\kappa^3 t)$, we get

$$x_0 = 4\kappa^2 t + \tilde{x}_0 \quad , \quad \tilde{x}_0 = \frac{1}{2\kappa}\ln\frac{M_0^2}{2\kappa}$$

We have obtained the remarkable solution of the KdV equation, the soliton.

The soliton is a wide-spread phenomenon in nature. In particular, they are realized as solitary waves on the surface of fluids of finite depth. These solitons were first observed by the outstanding British engineer and ship-builder J.S.Russell 150 years ago (a memorial congress in honour of this event was held in England in the summer of 1982). Theoretical studies started in the seventies of the last century and became intensive only some fifteen years ago when solitons were discovered in plasma.

During this fifteen years, computer simulations of the very old and nearly forgotten KdV equation (it was first derived in 1894) began and the first problem to be investigated was the soliton-soliton collision. Experimentalists (M. Kruskal and others) were highly astonished at seeing this collision to be totally elastic and soon the inverse scattering method was invented. Let us now consider the soliton-soliton collision, using exact solutions of the KdV equation.

Let us put $N=2$. Then

$$\det A = \begin{vmatrix} 1 + \frac{M_1^2 e^{-2\kappa_1 x}}{2\kappa_1} & \frac{M_1^2 e^{-(\kappa_1+\kappa_2)x}}{\kappa_1+\kappa_2} \\ \frac{M_2^2 e^{-(\kappa_1+\kappa_2)x}}{\kappa_1+\kappa_2} & 1 + \frac{M_2^2 e^{-2\kappa_2 x}}{2\kappa_2} \end{vmatrix} = \tag{4.10}$$

$$= 1 + M_1^2 \cdot \frac{e^{-2\kappa_1 x}}{2\kappa_1} + M_2^2 \cdot \frac{e^{-2\kappa_2 x}}{2\kappa_2} + M_1^2 M_2^2 \cdot \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} \frac{e^{-2(\kappa_1+\kappa_2)x}}{4\kappa_1\kappa_2} =$$

$$= 1 + e^{\xi_1} + e^{\xi_2} + \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} \cdot e^{\xi_1+\xi_2}$$

Here

$$\xi_1 = -2\kappa_1(x - 4\kappa_1^2 t - \tilde{x}_{10})$$

$$\xi_2 = -2\kappa_2(x - 4\kappa_2^2 t - \tilde{x}_{20}) \tag{4.11}$$

$$\tilde{x}_{1,20} = \frac{1}{2\kappa_{1,2}} \ln \frac{M_{1,20}^2}{2\kappa_{1,2}}$$

Let $\kappa_2 > \kappa_1$ and put $t \to -\infty$.

a) In the neighbourhood of $x = x(t)$ where $\xi_1 \approx 1$,

$$\xi_2 \approx -2\kappa_2(4(\kappa_1^2 - \kappa_2^2)t) \to -\infty$$

so we have

$$\det A \to 1 + e^{\xi_1}$$

and

$$u \to -\frac{2\kappa_1^2}{ch^2\kappa_1(x - 4\kappa_1^2 t - \tilde{x}_{10})}$$

b) Now we study the domain where $\xi_2 \approx 1$. In this domain

$$\xi_1 \approx -2\kappa_1 4(\kappa_2^2 - \kappa_1^2)t \to +\infty$$

and

$$\det A \to e^{\xi_1}\left[1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} e^{\xi_2}\right] \qquad (4.12)$$

After substituting (4.12) into (4.7) we have

$$u \to -\frac{2\kappa_2^2}{ch^2\kappa_2(x - 4\kappa_2^2 t - \tilde{y}_{20})} \qquad (4.13)$$

Here

$$\tilde{y}_{20} = \tilde{x}_{20} + \frac{1}{2\kappa_2} \ln\frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \qquad (4.14)$$

One can see that in any other domain $u(x,t) \to 0$ at $t \to -\infty$. So the solution (4.7) for $N = 2$ decomposes asymptotically (at $t \to -\infty$) into the system of two separate solitons with velocities $4\kappa_1^2$ and $4\kappa_2^2$.

Repeating this procedure for large positive times we find, at $t \to +\infty$,

$$u \to -\frac{2\kappa_1^2}{ch^2\kappa_1(x-4\kappa_1^2t-\tilde{y}_{10})} - \frac{2\kappa_2^2}{ch^2\kappa_2(x-4\kappa_2^2t-\tilde{x}_{20})}$$

$$\tilde{y}_{10} = \tilde{x}_{10} + \frac{1}{2\kappa_1}\ln\frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} \qquad (4.15)$$

We have found that the solution of (4.7) for $N=2$ describes the collision of two solitons; it could be called the "two-soliton solution".

The collision is totally elastic: the asymptotic velocities of the solitons do not change. However, because of the collision, the solitons get an additional shift of their centres. The faster one $(\kappa=\kappa_2)$ gets the positive shift

$$\Delta_2 = \tilde{x}_{20} - \tilde{y}_{20} = \frac{-1}{2\kappa_2}\ln\frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2}$$

the slower one is shifted by

$$\Delta_1 = \tilde{y}_{10} - \tilde{x}_{10} = \frac{1}{2\kappa_1}\ln\frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2}$$

This means that the forces between the solitons are repulsive and the solitons behave like billiard balls.

The general solution (4.7) for arbitrary N can be called the "N-soliton solution". It decomposes, at $t\to\pm\infty$, into the system of separated solitons

$$u \to -2\sum_{k=1}^{N}\frac{\kappa_k^2}{ch^2\kappa_k(x-4\kappa_k^2t-x_{k0}^{\pm})} \qquad (4.16)$$

It is possible to prove (though we will not do so here) that the shifts of the positions of the solitons due to their interactions are given by the formula

$$\Delta_k = x_k^+ - x_k^- = \frac{-1}{2\kappa_k} \sum_{l \neq k} \alpha_l \ln \frac{(\kappa_l - \kappa_k)^2}{(\kappa_l + \kappa_k)^2} \qquad (4.17)$$

where $\alpha_l = 1$ if $\kappa_l < \kappa_k$ and $\alpha_l = -1$ if $\kappa_l > \kappa_k$.

In other words, the total shift of the soliton's centre is the algebraic sum of shifts due to binary collisions with other solitons. It is curious that this shift does not depend on the details of the interaction. One can see that the N-soliton solution is the combination of elementary functions.

If the total number of solitons is large enough and the average distance between the solitons is much larger than their characteristic size, we can speak about rarified soliton gas. It can be described by a distribution function, $f = f(x,v)$. For this function, we have the kinetic equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} f(v) \cdot \tilde{v} = 0$$

$$\tilde{v} = v - \frac{1}{\sqrt{v}} \int_0^v (v - v_1) \ln \frac{(\sqrt{v} - \sqrt{v_1})^2}{(\sqrt{v} + \sqrt{v_1})^2} \cdot f(v_1)\, dv_1 + \qquad (4.18)$$

$$+ \frac{1}{\sqrt{v}} \int_v^\infty (v_1 - v) \ln \frac{(\sqrt{v} - \sqrt{v_1})^2}{(\sqrt{v} + \sqrt{v_1})^2} \cdot f(v_1)\, dv_1$$

This equation (4.18) describes the "second sound" and other effects in the soliton gas.

Exercise

Prove that the final soliton shifts do not depend on the initial positions of solitons.

Lecture Five

We have seen in the previous lectures an achievement of the inverse scattering theory, viz. the exact soliton solutions. We now show that this solution can be obtained much more simply, without developing all the machinery of the inverse scattering theory. We shall compute soliton-type solutions for one physically interesting integrable system. At first, we construct this system. Let us consider the overdetermined system of linear equations for the NxM matrix function $\Psi(x,t,\lambda)$

$$\begin{aligned} \Psi_x &= U\Psi \\ \Psi_t &= V\Psi \end{aligned} \qquad (5.1)$$

where U and V are rational matrix functions of the parameter λ . The compatibility condition for (5.1) is

$$U_t - V_x + [U,V] = 0 \qquad (5.2)$$

Let us consider the simplest case, when U and V have different and simple poles

$$\begin{aligned} U &= U_0 + \sum_{k=1}^{N_1} \frac{U_k}{\lambda-\lambda_k} \\ V &= V_0 + \sum_{k=1}^{N_2} \frac{V_k}{\lambda-\mu_k} \end{aligned} \qquad \lambda_i \neq \mu_k \qquad (5.3)$$

We assume that λ_k and μ_k are constants and $U_i(x,t)$, $V_i(x,t)$ are unknown matrix functions of x and t . Their total number is N_1+N_2+2 . After substituting (5.3) into (5.2), the vanishing of the residues and constant terms yields the N_1+N_2+4 equations

$$U_{0t} - V_{0x} + [U_0, V_0] = 0$$

$$U_{kt} + [U_k, \varphi_k] = 0 \qquad (5.4)$$

$$V_{kx} + [V_k, \psi_k] = 0$$

$$\varphi_k = V|_{\lambda=\lambda_k} = V_0 + \sum_{n=1}^{N_2} \frac{V_n}{\lambda_k - \mu_n} \quad ; \quad \psi_k = U|_{\lambda=\mu_k} = U_0 + \sum_{n=1}^{N_1} \frac{U_n}{\mu_k - \lambda_n}$$

The number of unknown functions exceeds by one the number of equations. One can see that this remains true for any distribution of the poles. This fact can be understood by noting that any solution U, V of (5.?) produces a new solution

$$\tilde{U} = gUg^{-1} + g_x g^{-1} \quad ; \qquad \tilde{V} = gVg^{-1} + g_t g^{-1} \qquad (5.5)$$

by means of an arbitrary function $g(x,t)$. This can be checked by substituting (5.5) into (5.2), but it is more instructive to mention that the function $\tilde{\Psi} = g\Psi$ satisfies the overdetermined system of linear equations

$$\tilde{\Psi}_x = \tilde{U}\Psi \quad ; \quad \tilde{\Psi}_t = \tilde{V}\Psi \qquad (5.6)$$

and that $\tilde{U}$ and $\tilde{V}$ have respectively the same poles as U and V .

The transformation (5.5) is a gauge transformation. It allows the addition of a further equation to the system (5.2) and (5.4). Systems differing only in this equation are called gauge-equivalent. For instance U_0 and V_0 can be given constant commuting matrices, $[U_0, V_0] = 0$ (the canonical gauge) or one of the matrices U_k , V_k can be a fixed diagonal matrix (the pole gauge).

Let us choose

$$U = I\lambda + u \quad , \quad V = J\lambda + v \tag{5.7}$$

where I and J are constant commuting matrices

$$I = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \qquad J = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix} \tag{5.8}$$

From (5.2) we find

$$u_t - v_x + [u,v] = 0 \quad , \quad [I,v] = [J,u] \tag{5.9}$$

The second of these equations can be solved as

$$u = u_0 + [I,Q] \qquad , \qquad v = v_0 + [J,Q] \tag{5.10}$$

where u_0 and v_0 are diagonal matrices and Q is a matrix with vanishing diagonal elements. One can see that

$$\begin{aligned} &[[J,Q],[I,Q]]_{\alpha\alpha} = \\ &= \sum_\beta \{(b_\alpha - b_\beta)(a_\beta - a_\alpha) - (a_\alpha - a_\beta)(b_\beta - b_\alpha)\} Q_{\alpha\beta} Q_{\beta\alpha} = 0 \end{aligned} \tag{5.11}$$

Hence, for the diagonal parts of u, v we have

$$u_{0t} - v_{0x} = 0$$

and there exists some diagonal matrix ϕ such that

$$u_0 = \phi_x \quad , \quad v_0 = \phi_t$$

Equations (5.1) can now be written in the form

$$\begin{aligned} \Psi_x &= (I\lambda + [I,Q])\Psi + \phi_x \Psi \\ \Psi_t &= (J\lambda + [J,Q])\Psi + \phi_t \Psi \end{aligned} \tag{5.12}$$

Let us make the transformation

$$\Psi = e^{\phi}\tilde{\Psi} \quad , \quad Q = e^{\phi}\tilde{Q}\, e^{-\phi}$$

Because the matrices I , J and ϕ are commuting, this transformation eliminates ϕ from the equations. So we may put $\phi = 0$.

Now let a_i , b_i be real, x and t pure imaginary and $Q^{+} = -Q$ an anti-hermitian matrix. Let us rewrite the linear system in the form

$$\begin{aligned} i\Psi_x &= (I\lambda + [I,Q])\Psi \\ i\Psi_t &= (J\lambda + [J,Q])\Psi \end{aligned} \qquad (5.13)$$

Its compatibility condition is

$$i[I,Q]_t - i[J,Q]_x + \big[[I,Q],[J,Q]\big] = 0 \qquad (5.14)$$

Apparently, the matrices I , J are defined by (5.14) up to an additive constant (a scalar matrix). Moreover, they should not be proportional to each other. So system (5.13) is non--trivial ((5.14) is nonlinear) if the matrix dimension $N \geq 3$. We shall study only the simplest case N=3.

Let $a_1 > a_2 > a_3$, and

$$Q_{12} = \frac{u_1}{\sqrt{a_1 - a_2}} \quad , \quad Q_{23} = \frac{u_2}{\sqrt{a_2 - a_3}} \quad , \quad Q_{13} = \frac{u_3}{\sqrt{a_1 - a_3}}$$

Let us choose, moreover, b_i so that

$$\frac{a_1 b_2 - a_2 b_1 + a_3 b_1 - a_1 b_3 + a_2 b_3 - a_3 b_2}{\sqrt{(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)}} = 1$$

Then system (5.14) reads as

$$
\begin{aligned}
&u_{1t} - v_1 u_{1x} = i u_3 u_2^* \\
&u_{2t} - v_2 u_{2x} = i u_3 u_1^* \qquad (5.15) \\
&u_{3t} - v_3 u_{3x} = i u_1 u_2
\end{aligned}
$$

where

$$v_1 = \frac{b_1 - b_2}{a_1 - a_2} \quad , \quad v_2 = \frac{b_2 - b_3}{a_2 - a_3} \quad , \quad v_3 = \frac{b_1 - b_3}{a_1 - a_3}$$

This is the so-called "three-wave interaction system", playing an important role in nonlinear optics. Equations (5.15) describe the stimulated Raman scattering in one-dimensional media without absorption of energy. In fact, u_3 is the complex envelope of the initial wave (pumping) and u_1 , u_2 are the amplitudes of secondary waves (satellites); v_i are group velocities of the waves. As we have seen, this system is the compatibility condition of the linear system (5.13) and could be solved by the inverse scattering method. If we want to follow the way described in the first four lectures we must develop the direct and inverse scattering theory for (5.13). This is possible but not particularly easy. Our aim is to show that one can compute exact solutions of (5.15), avoiding the inverse scattering theory.

Our guide is the gauge transformation (5.5), yielding a new solution of the system from a known solution.

Let Q be a solution of (5.14). If we find a matrix function $g = g(x,t,\lambda)$ (depending on λ !) and a matrix $\tilde{Q}$ not depending on

λ , such that

$$I\lambda + [I,\tilde{Q}] = g(I\lambda + [I,Q])g^{-1} + ig_x g^{-1}$$
$$J\lambda + [J,\tilde{Q}] = g(J\lambda + [J,Q])g^{-1} + ig_t g^{-1} \qquad (5.16)$$

then $\tilde{Q}$ is a new solution of (5.14).

Clearly, it is impossible that g does not depend on λ . (If $\lambda \to \infty$ then g goes to the unit matrix: if g is independent of λ , then $g = 1$.) At $\lambda \to \infty$

$$g \to 1 + \frac{G}{\lambda} \qquad (5.17)$$

Substituting (5.17) into (5.16) and comparing the asymptotics at $\lambda \to \infty$ we get

$$[I,\tilde{Q}] = [I,Q] - [I,G]$$
$$[J,\tilde{Q}] = [J,Q] - [J,G] \qquad (5.18)$$

The problem is that if we substitute in (5.16) an arbitrary matrix g , depending on λ , (5.16) and (5.18) can not be fulfilled simultaneously because of the singularities of g and g^{-1} so we must subject g to a number of constrains.

Let us consider that both g and g^{-1} are rational functions of λ , having one single pole

$$g = 1 + \frac{P_1}{\lambda - \lambda_0} \qquad g^{-1} = 1 + \frac{P_2}{\lambda - \mu_0} \qquad (5.19)'$$

From the condition $g \cdot g^{-1} = 1$ one finds

$$P_1 = (\lambda_0 - \mu_0)P \ , \quad P_2 = -(\lambda_0 - \mu_0)P$$

where $P = P^2$ is a projection matrix depending only on x and t .

To satisfy the condition $\tilde{Q}^{+} = -\tilde{Q}$ we must put

$$g^{-1}(\lambda) = g^{+}(\lambda^{*})$$

so we have

$$\mu_{0} = \lambda_{0}^{*} \quad , \quad p^{+} = p$$

and finally

$$g = 1 + \frac{\lambda_{0} - \lambda_{0}^{*}}{\lambda - \lambda_{0}} p \quad , \quad g^{-1} = 1 - \frac{\lambda_{0} - \lambda_{0}^{*}}{\lambda - \lambda_{0}^{*}} p \qquad (5.20)$$

Replacing g and g^{-1} in (5.16) by (5.20) we find that while the l.h.s. has no pole, the r.h.s. of (5.16) has singularities at the points $\lambda = \lambda_{0}$ and $\lambda = \lambda_{0}^{*}$. To avoid the singularity at $\lambda = \lambda_{0}$ we require

$$\begin{aligned} pU(1-p) + ip_{x}(1-p) &= 0 \\ pV(1-p) + ip_{t}(1-p) &= 0 \end{aligned} \qquad (5.21)$$

Here again $U = I\lambda + [I,Q]$, $V = J\lambda + [J,Q]$. Equations (5.21) seem to be nonlinear but in fact, they are linear. Let us choose the projector p in the form

$$p_{\alpha\beta} = \frac{n_{\alpha} n_{\beta}^{*}}{\sum |n_{\alpha}|^{2}} \qquad (5.22)$$

and denote $\tilde{p} = 1 - p$. Clearly $\tilde{p}^{2} = \tilde{p}$ (so $\tilde{p}$ is also a projector) and $p\tilde{p} = 0$. It means that

$$\sum_{\beta} n_{\beta}^{*} \tilde{p}_{\beta\gamma} = 0 \qquad (5.23)$$

Now we put (5.22) into (5.21). Using (5.23) we note that (5.21) is satisfied if the complex vector $n_{\alpha}^{*}(x,t)$ obeys the equations

$$i n_{\alpha x}^{*} + \sum_{\beta} n_{\beta}^{*} U_{\beta\alpha}\big|_{\lambda=\lambda_{0}} = 0 \quad , \quad i n_{\alpha t}^{*} + \sum_{\beta} n_{\beta}^{*} V_{\beta\alpha}\big|_{\lambda=\lambda_{0}} = 0$$

or (remembering that $U^{+}|_{\lambda} = U|_{\lambda^*}$, $V^{+}|_{\lambda} = V|_{\lambda^*}$)

$$i\vec{n}_x = U|_{\lambda=\lambda_0^*} \cdot \vec{n} \quad , \quad i\vec{n}_t = V|_{\lambda=\lambda_0^*} \cdot \vec{n} \tag{5.24}$$

Our next step is to choose a particular solution of (5.14) for the procedure (5.16). This may be the trivial

$$u = v = 0 \quad , \quad U = I\lambda \quad , \quad V = J\lambda$$

Equations (5.24) can be satisfied if

$$n_\alpha = c_\alpha e^{-i(a_\alpha x + b_\alpha t)\lambda_0^*} \tag{5.25}$$

Here c_α are arbitrary complex constants.

Now from (5.17), (5.18), (5.20), (5.22) we find the solution of the system (5.14) as

$$\tilde{Q}_{\alpha\beta} = -\frac{(\lambda_0 - \lambda_0^*)\, n_\alpha n_\beta^*}{\Sigma |n_\alpha|^2} \qquad (\alpha \neq \beta) \tag{5.26}$$

To analyse solution (5.26) we first put $c_3 = 0$.

In this case there is only one non-vanishing wave amplitude, $\tilde{Q}_{12} \sim u_1 \neq 0$. Substituting (5.25) into (5.26), after simple calculations we have

$$\tilde{Q}_{12} = i2\eta \frac{e^{-i[(a_1-a_2)x\xi + (b_1-b_2)t\xi - \arg(c_1/c_2)]}}{2\,\mathrm{ch}\{\eta[(a_1-a_2)x + (b_1-b_2)t + x_0]\}} \tag{5.27}$$

Here $\lambda_0^* = \xi + i\eta$ and $x_0 = (1/\eta)\ln|c_1/c_2|$. This solution may be called the soliton of the wave u_1 . In contrast to a KdV soliton this solution depends on four parameters η, ξ, x_0 and $\varphi = \arg(c_1/c_2)$. If we put $c_1 = 0$ or $c_2 = 0$ we get solitons of other waves. They are obtained from (5.27) by changing, respectively, a_1, a_2 to a_2, a_3 and b_1, b_2 to b_2, b_3 or a_1, a_2 to a_1, a_3 and b_1, b_2 to b_1, b_3 . Let us consider, finally the general solution when all $c_i \neq 0$. This is obtained by substituting (5.25) into (5.26). It is interesting to inspect this formula in the limit when $t \to \pm\infty$. An elementary analysis, similar to that performed in the previous lecture, shows that for $\eta > 0$ and

$$v_2 > v_3 > v_1$$

the matrix element $\tilde{Q}_{13}$ tends to zero at $t \to +\infty$ whereas at $t \to -\infty$ it goes to the soliton of the wave u_3 with the standard position of centre,

$$x_{30} = \frac{1}{\eta} \ln \left| \frac{c_1}{c_3} \right|$$

By contrast, $\tilde{Q}_{12}$ and $\tilde{Q}_{23}$ tend to zero at $t \to -\infty$. At $t \to +\infty$ they go to the solitons u_1 and u_2 with the centres positioned at

$$x_{10} = \frac{1}{\eta} \ln \left| \frac{c_1}{c_2} \right| \qquad , \quad x_{20} = \frac{1}{\eta} \ln \left| \frac{c_2}{c_3} \right|$$

Therefore, solution (5.26) describes the decay of the pumping soliton u_3 into two secondary wave solitons, u_1 and u_2 . Physically this corresponds to the instability of the pumping wave u_3 or in other terms, the stimulated Raman scattering of a particular form of wave-packets.

We may assume that the function $g(\lambda)$ has more than one pole. In this case the procedure for calculating the exact solutions of system (5.15) is more complicated but still elementary. The solutions describe both the decays and the collisions of different types of solitons. The method described above is called the "dressing method". We perform the "dressing" of an initial solution (a trivial one in the considered case) by means of a rational matrix $g(x,t,\lambda)$. This dressing method provides a set of exact solutions for nonlinear integrable systems, even if a complete inverse scattering theory has not been developed yet.

Exercise

Construct the soliton solution describing the process of the fusion of two secondary solitons to a pumping soliton.

Lecture Six

Our aim now is to review the systems integrable by the inverse scattering method and having physical or other applications. In fact because we mention only the most important integrable equations our list will not be complete, but even so, it will be a representative one.

At first, some words about the principles of the classification of integrable systems. Since they are compatibility conditions of some linear systems, the most natural way of classifying them is to classify the corresponding overdetermined linear systems.

1.

Scalar systems

We begin with overdetermined systems involving two scalar differential operators. The simplest form

$$\begin{aligned} \psi_{xx} &= U\psi \\ \psi_t &= A\psi_x + B\psi \end{aligned} \qquad (6.1)$$

yields the KdV equation (if $U = \lambda + u$),

(1) $\quad u_t - 6uu_x + u_{xxx} = 0$

and its higher analogues (if $A = A(\lambda)$ is a polynomial).
The case when A is rational is also treatable even though it is insufficiently investigated. We can set U to be a polynomial of λ . If U is of second order, $U = \lambda^2 + \ldots$ and $A(\lambda)$ is a first order polynomial, then (6.1) yields the so-called Kaup's

system having applications in hydrodynamics:

(2)
$$\varrho_t + \frac{\partial}{\partial x}(\varrho v) = \pm \frac{\partial^2 v}{\partial x^2}$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial \varrho}{\partial x} = 0$$

In (6.1), U can be a rational function of λ but this possibility has not yet been studied with sufficient completeness.

The next possibility is to study third-order linear systems of the form

$$\psi_{xxx} = u_1\psi_x + u_2\psi$$
$$\psi_t = A_1\psi_{xx} + A_2\psi_x + A_3\psi \qquad (6.2)$$

Only one case of such a type of systems is studied, when $u_1 = \lambda + u$. This gives the Boussinesque equation

(3) $$u_{tt} \pm u_{xx} + (u^2)_{xx} \pm u_{xxxx} = 0$$

All the scalar integrable systems have their discrete analogues. For the KdV equation this is

(4) $$\dot{N}_k = N_k(N_{k+1} - N_{k-1})$$

having applications in plasma and laser physics and in mathematical ecology. The discrete counterpart of the Boussinesque equation yields the equations of the so called Toda lattice

(5) $$\ddot{u}_n = e^{u_{n+1} - u_n} - e^{u_n - u_{n-1}}$$

An important class of integrable nonlinear equations is connected with compatibility conditions for systems of singular integral equations, associated with some boundary problems in the theory of analytic functions. The most important among them is Benjamen - Ono's equations

$$u_t - 6uu_x + u_{xx} I[u] = 0$$

(6)

$$I[u] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x')}{x - x'} dx'$$

II.

Polynomial matrix systems

We studied in Lecture Five the overdetermined linear system

$$\psi_x = U\psi \quad , \quad \psi_t = V\psi \tag{6.3}$$

where U and V are rational matrix functions of λ .

The simplest case is when U and V are first-order polynomials of λ . In this case

$$\frac{\partial u_{ij}}{\partial t} - V_{ij} \frac{\partial u_{ij}}{\partial x} = i \sum_k \varepsilon_{ijk} u_{ik} u_{kj} \ , \quad u_{ik} = u_{ki}^*$$

(7)

$$V_{ij} = \frac{b_i - b_j}{a_i - a_j} \quad , \quad \varepsilon_{ijk} = \frac{a_i b_j - a_j b_i + a_k b_i - a_i b_k + a_j b_k - a_k b_j}{\sqrt{(a_i - a_j)(a_j - a_k)(a_i - a_k)}}$$

This is the so called N-wave system, which for N=3 gives the three-wave system, studied in Lecture Five, together with some other systems such as the "explosive three-wave system"

$$(8)\qquad \begin{aligned} \frac{\partial u_1}{\partial t} + V_1 \frac{\partial u_1}{\partial x} &= i\varepsilon u_2^* u_3^* \\ \frac{\partial u_2}{\partial t} + V_2 \frac{\partial u_2}{\partial x} &= i\varepsilon u_1^* u_3^* \\ \frac{\partial u_3}{\partial t} + V_3 \frac{\partial u_3}{\partial x} &= i\varepsilon u_1^* u_2^* \end{aligned}$$

and the equations for the interaction of the first and second harmonics

$$(9)\qquad \begin{aligned} \frac{\partial u_1}{\partial t} + V_1 \frac{\partial u_1}{\partial x} &= i\varepsilon\, u_1 u_2^* \\ \frac{\partial u_2}{\partial t} + V_2 \frac{\partial u_2}{\partial x} &= i\varepsilon\, u_1^2 \end{aligned}$$

If we omit the x-dependent terms in system (7) we obtain certain integrable systems of ordinary differential equations. If $b_i = a_i^2$, $u_{ik} = iA_{ik}$ and $A_{ik} = -A_{ki} = A_{ik}^*$, this will be a system describing the free rotation of an N-dimensional solid body (Euler equations):

$$(10)\qquad \frac{\partial M}{\partial t} = [\Omega, M] \quad , \quad M = I\Omega + \Omega I$$

where I is a constant diagonal matrix.

Now we go to the case when U is a first-order and V a second order polynomial of λ . In this case it is possible to consider the matrix dimension N=2. If we do that we have two gauge-equivalent equations, viz. the nonlinear Schrödinger equation

$$(11) \qquad ir_t + r_{xx} = \pm |r|^2 r$$

and the isotropic Landau-Lifshitz equation for a one-dimensional ferromagnet:

$$(12) \qquad \vec{S}_t = [\vec{S}, \vec{S}_{xx}]$$

For N>2 we get the system of nonlinear Schrödinger equations

$$(13) \qquad ir_{nt} + r_{nxx} = \pm \sum_{k=1}^{N-1} |r_k|^2 r_n$$

and its continuous analogues

$$(14) \qquad ir(t,x,z)_t + r(t,x,z)_{xx} = \pm \int_a^b |r(t,x,z')|^2 dz' \cdot r(t,x,z)$$

If we substitute for U a first-order and for V a third-order polynomial we obtain the so-called modified KdV equation (MKdV)

$$(15) \qquad u_t \pm |u|^2 u_x + u_{xxx} = 0$$

and putting for both U and V second-order polynomials we have the "derivative nonlinear Schrödinger equation"

$$(16) \qquad ir_t + r_{xx} \pm \frac{\partial}{\partial x}(|r|^2 r) = 0$$

and the Langmuir plasmon-sonic wave intera... system

(17)
$$i r_t + r_{xx} = u\psi$$
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \pm \frac{\partial}{\partial x}|\psi|^2$$

These last three systems have applications in plasma physics.

III.

Rational matrix systems

Many important integrable systems arise as the compatibility conditions of system (6.3) where either U or V is not a polynomial. Let U be a first order polynomial and V have a single pole at the point $\lambda=0$. The following equations belong to this class:

The sine-Gordon and sinh-Gordon equations having numerous physical and mathematical applications:

(18) $$u_{\xi\eta} = \sin u$$

(19) $$u_{\xi\eta} = \sinh u$$

The Bloch-Bloembergen equations, being very popular in nonlinear optics:

(20)
$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} = i\rho$$
$$\frac{\partial \rho}{\partial t} = inE$$
$$\frac{\partial n}{\partial t} = i(\rho E^* - \rho^* E)$$

The chiral field on Lie groups:

$$g_{\xi\eta} + \frac{1}{2} g_\xi g^{-1} g_\eta + \frac{1}{2} g_\eta g^{-1} g_\xi = 0 \tag{21}$$

Here g is an element of some Lie group. This equation determines that g for which the action functional

$$S = \int Sp \frac{\partial}{\partial\xi} g \cdot \frac{\partial}{\partial\eta} g^{-1} d\xi d\eta$$

is extremal, and it has applications in quantum field theory.

The chiral field equation can be simplified. In particular we may impose $g^2 = 1$. Putting $g = 1 - 2p$ we find $p^2 = p$ so that p is a projection operator. After fixing its dimension we have a system of equations

$$-p_{\xi\eta} + 2 p_\xi (1-2p) p_\eta + 2 p_\eta (1-2p) p_\xi = 0 \tag{22}$$

This describes a chiral field on Grassmann's manyfold. In the important special case when $G = SO_3$ and $p_{\alpha\beta} = n_\alpha n_\beta / \sum n_\alpha^2$ we have

$$\vec{n}_{\xi\eta} + (\vec{n}_\xi \vec{n}_\eta)\vec{n} = 0 \qquad \vec{n}^2 = 1 \tag{23}$$

This is the so-called $\vec{n}$-field-model, which is popular for describing quark confinement.

The system under consideration includes also the so called classical spinor equations. The simplest of them being

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\varphi_n &= \frac{i}{2} \psi_n \sum_{m=1}^{N} \psi_m^* \varphi_m \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\psi_n &= \frac{i}{2} \varphi_n \sum_{m=1}^{N} \varphi_m^* \psi_m \end{aligned} \tag{24}$$

These represent the Jona-Lasinio model used in solid-state physics. Another one is the Gross-Neveu model

$$(25)\quad \begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\varphi_n &= -i\,\psi_n \sum_{m=1}^{N}(\psi_m^*\varphi_m + \varphi_m^*\psi_m) \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\psi_n &= -i\,\varphi_n \sum_{m=1}^{N}(\psi_m^*\varphi_m + \varphi_m^*\psi_m) \end{aligned}$$

which is also useful for many-body theory. Both of these systems are very closely related to chiral fields: the first one to that on Univary groups and the second one to that on symplectic groups. Spinor systems connected with orthogonal groups also exist.

Two integrable variants of the Einstein equation for a gravitational field in vacuum are very close to the chiral field equations. Let us look for the space-time metrics in the form

$$ds^2 = f(x,t)(-dt^2 + dx^2) + g_{\alpha\beta}(x,t)\,dx_\alpha dx_\beta$$
$$\alpha, \beta = 2, 3$$

The symmetric matrix $g_{\alpha\beta}$ obeys the equation

$$(26)\quad \frac{\partial}{\partial x} d(g_x g^{-1}) - \frac{\partial}{\partial t} d(g_t g^{-1}) = 0$$

Here $d = \sqrt{\det g}$ and $d_{tt} - d_{xx} = 0$

This equation describes one-dimensional nonlinear gravitational waves.

The equation

$$(27)\quad \frac{1}{r}\frac{\partial}{\partial r} r(g_r g^{-1}) + \frac{\partial}{\partial \varphi}(g_r g^{-1}) = 0$$

describes a stationary axial-symmetric gravitational field. Both (26) and (27) can be integrated by a slight modification of the techniques used for the chiral field.

I also mentioned, in connection with chiral fields, the possibility of simplifications of integrable systems. These reductions can be very sophisticated and allow one to obtain, from the chiral field equations on general complex Lie groups, equations of the type

$$(28) \qquad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi_n = \sum_{k=1}^{N_1} \alpha_{nk}\, e^{-\beta_{nk}\varphi_k} \qquad n=1,\dots,N \quad ;\ N_1 \geq N$$

Here α_{nk} and β_{nk} are some given sets of constants calculated through elements of root systems for different Lie groups.

The simplest of equations (28) (besides sine- and sinh--Gordon's equations) is

$$(29) \qquad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi = e^{\varphi} - e^{2\varphi}$$

All the previous systems are connected with system (6.3) where $U \sim \lambda$ and $V \sim 1/\lambda$. The more complicated cases have not yet been studied sufficiently rigorously but the situation when $U \sim \lambda^2$ and $V \sim 1/\lambda^2$ has been partly investigated. It consists of the massive Thirring model

$$(30) \qquad \begin{aligned} \frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_1}{\partial x} + \psi_2 &= i\,\psi_1 |\psi_2|^2 \\ \frac{\partial \psi_2}{\partial t} - \frac{\partial \psi_2}{\partial x} - \psi_1 &= i\,\psi_2 |\psi_1|^2 \end{aligned}$$

and some related equations.

IV.

The first multi-dimensional generalization

So far we have only dealt with two-dimensional integrable systems. It is very interesting to develop the inverse scattering method for systems of any equations of higher dimensions.

There are two possibilities to do this. The first is especially simple if the matrices U and V are polynomials of λ . In fact we can replace λ by $i\partial/\partial x$ in equations (6.3). If the coefficients of the polynomials U and V do not depend on λ this is just the Fourier transformation. But this substitution is possible even in the general case, when the coefficients are functions of λ . Thus it is possible for system (6.1), if U , A , B are polynomials. In such a way we get certain three (one time and two space-like) dimensional generalizations of the equations described above. Not all of these generalizations have physical meaning since most of them are too complicated. Some of the simplest generalizations are the following:

(31) $$\frac{\partial}{\partial x}(u_t - 6uu_x + u_{xxx}) = \pm\frac{\partial^2 u}{\partial y^2}$$

Equation (31) is a generalization of the KdV equation, the so called Kadomtzev-Petviashvily equation (KP equation). It describes two-dimensional waves in a weakly dispersive nonlinear medium if the y-dependence is much slower than the dependence on x . The two different signs in (31) correspond to the two different types of dispersion $\omega_k'' \gtrless 0$. Equation (31) has important applications in hydrodynamics.

Generalized three-wave system

$$
\begin{aligned}
&u_{1t} + (\vec{v}_1 \nabla)\, u_1 = i\varepsilon\, u_2^* u_3 \\
(32) \qquad &u_{2t} + (\vec{v}_2 \nabla)\, u_2 = i\varepsilon\, u_1^* u_3 \\
&u_{3t} + (\vec{v}_3 \nabla)\, u_3 = i\varepsilon\, u_1 u_2
\end{aligned}
$$

Here $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are arbitrary constant vectors in a plane (x,y) . This system can be used in nonlinear optics.

The generalization of the nonlinear Schrödinger equation

$$
\begin{aligned}
&i\psi_t + \psi_{xx} \pm \psi_{yy} + \alpha |\psi|^2 \psi + V\psi = 0 \\
(33) \qquad &\left(\frac{\partial^2}{\partial x^2} \mp \frac{\partial^2}{\partial y^2}\right) V = \frac{\alpha}{2} \frac{\partial^2}{\partial x^2} |\psi|^2
\end{aligned}
$$

This is the so called Davey-Stewartson's system. It has applications in hydrodynamics. The constant α may be equal to ± 1 , thus equation (33) has four different variants.

The last system of this type which as yet has no application but has a great deal of potential, is the system

$$
\begin{aligned}
&i\psi_t + \psi_{xx} + U\psi = 0 \\
(34) \qquad &\frac{\partial \psi}{\partial y} = \pm \frac{\partial}{\partial x} |\psi|^2
\end{aligned}
$$

We may call the procedure of replacing λ by $i\,\partial/\partial x$ the first multi-dimensional generalization of the two-dimensional integrable systems. It is not easy to perform this generalization in the general case when the dependence on λ is rational. Nevertheless, further work in this direction may lead in the future to very important results.

V.

The second multi-dimensional generalization

There is another very natural way of increasing the number of independent variables. One can see that the compatibility conditon (5.2) of the overdetermined system (5.1) involves only two types of operations, viz. differentiation and commutation. By making the substitutions

$$\begin{aligned} U &\to U_0 + \sum_{n=1}^{N_1} \frac{U_k + \partial/\partial y_k}{\lambda - \lambda_k} \\ V &\to V_0 + \sum_{n=1}^{N_2} \frac{V_k + \partial/\partial z_k}{\lambda - \mu_k} \end{aligned} \qquad (6.4)$$

the number of equations in (5.2) will not change. So we may reconsider system (6.3) but now choosing U and V in the form shown in (6.4). In this way we get systems having $N_1 + N_2 + 2$ independent variables. In the simplest case $N_1 = N_2 = 1$ and we have a system in four-dimensional space. A very simple reduction leads to

$$F_{ij} = \pm \epsilon_{ijkl} F_{kl} \qquad (35)$$

where

$$F_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} + [A_i, A_j]$$

This is the famous self-duality equation for Yang-Mills field theory. Using the dressing method we can easily find its local solutions, which depend on some arbitrary functions. But in general these solutions have singularities. The problem of finding regular solutions (so called instantons) is much more difficult and has not been solved effectively (in spite of the great progress made recently). There are two systems closely related to (35): one is the Bogomolny system for monopoles

(36) $$F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \frac{\partial \varphi}{\partial x_\gamma}$$

where φ is a new scalar field.

The general solution of this problem has recently been given a great deal of impetus by Hungarian mathematical physicists (P. Forgach et al.). The second is the relativistic - invariant system in two space variables

(37) $$(\chi_t \chi^{-1})_t = (\chi_z \chi^{-1})_z$$

which resembles but does not coincide with the chiral field equations.

Conclusions

We have listed only some of the equations integrable by the inverse scattering method and being of importance from a physical point of view. This intensively developing domain of mathematical physics is very rich in new and productive ideas. One might also mention the idea of quasi--classical transition on integrable systems, yielding an approach to Vlasov's equation for particles with δ-function type potential.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial f}{\partial v} = 0 \qquad (38)$$

$$u = \int_{-\infty}^{\infty} f\,dv$$

or to the very symmetric equation

$$\frac{\partial^2}{\partial t^2} e^{\varphi} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi \qquad (39)$$

Another idea is to study how the singularities in exact solutions of integrable equations move. In particular, it allows one to solve the systems of one-dimensional interacting particles

$$\ddot{x}_n + \frac{\partial}{\partial x_n} u = 0$$

$$u = \sum_{i<j} U(x_i - x_j)$$

for the cases when

$$U(\xi) = \frac{a}{\xi^2} \qquad (40)$$

and

$$(41) \qquad u(\xi) = \frac{a}{\sinh^2 \xi}$$

The study of self-similar solutions of integrable equations has helped to advance old mathematical problems like the Painleve' transcendents and some other mathematical problems. There is little doubt that this fine scientific discipline which rebuilds the union of fundamental (but at first glance old-fashioned) mathematics and modern physics will successfully develop in the future.

Solutions of the excercises

Lecture 1

1. $v = c/u$

$$u_t = c\frac{\partial^3}{\partial x^3}\frac{1}{\sqrt{u}}$$

This is the so called Harry Dim's equation.

2. $v = 2u$

$$u_t - 6uu_x + u_{xxx} + 4(w^2)_x = 0$$

$$w_t = 2(wu)_x$$

According to the physical interpretation, this system can be considered as a hydro-dynamic system with a special kind of dispersive sound.

Lecture 2

Analytic continuation is possible in the domain

$$\operatorname{Im} k > -\frac{\alpha}{2}$$

Lecture 3

1. $$\tilde{F}(x+y) + \tilde{K}(x,y) + \int_{-\infty}^{x} \tilde{K}(x,s)\tilde{F}(s+y)\,ds = 0$$

2. $$F(\xi) = \frac{1}{2\pi}\int \frac{b(k)}{a(k)}\, e^{ik\xi}\, dk$$

$$\tilde{F}(\xi) = \frac{1}{2\pi}\int \frac{b^*(k)}{a(k)}\, e^{ik\xi}\, dk$$

Lecture 4

Hint: Use the formula for Bargmann's potential

Lecture 5

Hint: Represent the additional projection operator in the form

$$1-p = \frac{n_\alpha n_\beta^*}{\Sigma |n_\alpha|}$$

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