INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

TIME DEPENDENT DRIFT HAMILTONIAN

Allen H. Boozer*

(Received - Apr.3, 1982)

IPPJ- 571 Apr. 1982

RESEARCH REPORT

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya 464, Japan

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*Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544, U.S.A.

I) Introduction

The evaluation of the motion of individual charged particles in a given magnetic $\vec{B}$ and electric $\vec{E}$ field is fundamental to the theory of low collisionality plasmas. Such theory is based on a kinetic equation

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\frac{\partial F}{\partial \vec{x}} + \frac{e}{m}\left(\frac{\vec{v}}{c} \times \vec{B} + \vec{E}\right)\cdot\frac{\partial F}{\partial \vec{v}} = C(F) \qquad (1)$$

with $F(\vec{x}, \vec{v}, t)$ the single particle distribution function. The relation between the kinetic equation and classical mechanics becomes more transparent if the equation is written in the equivalent form

$$\frac{\partial F}{\partial t} - \{H, F\} = C(F) \qquad (2)$$

with H the particle Hamiltonian and $\{\cdot,\cdot\}$ the Poisson bracket. By a low collisionality plasma, we mean a plasma in which the action of the collision operater C(F) is subdominant to the left hand side or Vlasov part of the kinetic equation. In other words, the distribution functions of interest must be close to solutions of the Vlasov equation

$$\frac{\partial F}{\partial t} - \{H, F\} = 0 \qquad (3)$$

The characteristics of the Vlasov equation are just the motion of individual particles in the Hamiltonian H.

Unfortunately, it is not practical to solve for the particle motion in the complex magnetic fields which penetrate both laboratory and space plasmas. A charged particle rapidly moves on a nearly circular orbit about a field line and only slowly drifts from one field line to another. The disparate time scales makes an exact integration difficult, but encourages the development of an asymptotic theory, the drift kinetic theory. The related classical mechanics problem, the guiding center drift, has only four degrees of freedom instead of six as in the exact three dimensional problem. The rapid gyration about the field lines implies the actions associated with the two components of the velocity orthogonal to a field line $\vec{v}_\perp$ are adiabaticly conserved. Both actions equal the magnetic moment μ. The lowest order evaluation of the magnetic moment gives

$$\mu \simeq \frac{1}{2}\frac{mv_\perp^2}{B} \tag{4}$$

The two conserved actions imply there are only four degrees of freedom which implies the asymptotic Hamiltonian h is only two dimensional.

Using the asymptotic Hamiltonian h, the drift kinetic equation is

$$\frac{\partial f}{\partial t} - \{h, f\} = C(f) \tag{5}$$

The guiding center distribution function f does not equal the single particle distribution F. However, for guiding center theory to be useful, the integral information of f should agree with that of F at least to lowest order in the particle gyration radius ρ to system size a. By integral information, we mean, for example, the ion neoclassical heat flux or the fraction of alpha particles lost from a fusing plasma due to trajectories which intersect the reactor walls.

It is difficult to formulate sufficient conditions for the validity of drift kinetic theory. Of course, certain necessary conditions are easier to give. It is clear that if a symmetry implies some quantity is exactly conserved by H, the asymptotic Hamiltonian h should exactly conserve an analogous quantity. In this area, traditional drift kinetic theory has problems.[1] The traditional drift kinetic equation is

$$\frac{\partial f}{\partial t} + \vec{v}_g \cdot \frac{\partial f}{\partial \vec{x}} = e(f) \tag{6}$$

with $\vec{v}_g$ Alfvén's expression[2] for the guiding center velocity. That is

$$\vec{v}_g = v_\parallel \hat{b} + \frac{c\vec{B}}{eB^2} \times (\mu \vec{\nabla} B + m v_\parallel^2 \hat{b} \cdot \vec{\nabla} \hat{b} - e\vec{E}) \tag{7}$$

with $v_\parallel$ the component of particle velocity along a field line and $\hat{b} = \vec{B}/B$. Unless $\vec{B} \cdot (\vec{\nabla} \times \vec{B}) = 0$, the traditional guiding center equation cannot be written in Hamiltonian form, Eq. (5), and it does not obey Liouville's theorem, which is particle conservation, nor is there a constant of the motion in axisymmetry.[3,4]

Efforts to rectify the problems of traditional guiding center theory[5,6,7] have placed a major emphasis on the second order terms in the gyroradius ρ to system size r. The real problem is to develop an asymptotic Hamiltonian h. Whether h is first or second order accurate is not critical, the important point is that h be asymptoticly correct. The importance of an asymptotic theory as opposed to a first order correct theory comes fundamentally from the need to follow particle for many gyroperiods. In a reactor grade plasma, even thermal ions have about 10^4 (r/ρ) gyroperiods between collisions. A first order theory need only be accurate when the number of gyroperiods is small compared to (r/ρ).

In this paper, a simple guiding center Hamiltonian h is proposed which obyes all the obvious necessary conditions for asymptotic validity. It agrees to lowest order in gyroradius to system size with Alfvén's drift velocity, Eq. (7), in both time varying and time independent fields. In axisymmetry, a canonical momentum is conserved. The Hamiltonian and the canonical variables will be given in magnetic coordinates, which not only give simpler equations but also measure the drift motion of the particles relative to the plasma structure.

II) Magnetic Coordinates

The general non-stochastic magnetic field $\vec{B}$ can be written in a contravariant and a covariant form[8,4]

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\phi \times \vec{\nabla}\psi_p(\psi) \tag{8}$$

$$\vec{B} = \lambda\vec{H}, \quad \vec{H} = g(\psi)\vec{\nabla}\phi + I(\psi)\vec{\nabla}\theta + \beta_*\vec{\nabla}\psi \tag{9}$$

The permeability λ is chosen so $(\vec{\nabla} \times \vec{H})\cdot\vec{\nabla}\psi = 0$. In scalar pressure plasmas $\lambda = 1$. In toroidal configurations it is convenient to interpret $2\pi\ \psi$ as the magnetic flux inside a magnetic surface, θ as the poloidal angle, and ϕ as the toroidal angle. With this interpretation and with $\lambda = 1$, $cg(\psi)/2$ is the total poloidal current outside a flux surface and $cI(\psi)/2$ is the total toroidal current inside a flux surface.[8] In a large aspect ratio tokamak with major and minor radii R and r, $B_\phi = g/R$, $B_\theta \simeq I/r$, $\psi \simeq B_\phi r^2/2$, $1/q = d\psi_p/d\psi \simeq RB_\theta/rB_\phi$, and $\beta_* \simeq (8\pi/B^2)(dP/dr)\sin\theta$, with P the plasma pressure, is closely related to the Pfirsch-Schluter currents. In general toroidal geometry, the poloidal flux function ψ_p is closely connected to the loop voltage V. The relation is[8]

$$\left(\frac{\partial\psi_p}{\partial t}\right)_\psi = \frac{c}{2\pi} V \tag{10}$$

Actually any three quantities ψ, θ, ϕ can be used as magnetic coordinates if they satisfy Eqs. (8) and (9) and have single valued transformation equations $\vec{X}(\psi, \theta, \phi, t)$. By the transformation equations we mean the Cartesian coordinates expressed as functions of the magnetic coordinates and time. A practical method has been given for evaluating functions of the magnetic coordinates in general toroidal scalar pressure equilibria.[9]

To analyze a stochastic magnetic field, by which we mean a toroidal configuration without magnetic surfaces, the exact field is written as[10]

$$\vec{B}_e = \vec{B} + \vec{\nabla} \times \alpha\vec{H} \tag{11}$$

with $\vec{B}$ described by Eqs. (8) and (9). If $\alpha(\psi,\theta,\phi)$ is Fourier decomposed

$$\alpha(\psi,\theta,\phi) = \sum_{n,m} \alpha_{nm}(\psi) \exp[i(n\phi - m\theta)] \tag{12}$$

then magnetic islands occur around magnetic surfaces on which the rotational transform $\iota = d\psi_p/d\psi$ satisfies $\iota = n/m$. The half width in ψ of these islands is

$$\Delta_{nm} = \left|\frac{4}{m\frac{d\iota}{d\psi}} (mg + nI)\alpha_{nm}\right|^{1/2} \tag{13}$$

III) Drift Hamiltonian

The drift Hamiltonian we propose is

$$h(p_\theta, \theta; p_\phi, \phi) = \frac{1}{2}mv_\parallel^2 + \mu B + e\Phi \tag{14}$$

with the Φ the electric potential. The canonical momenta are

$$p_\theta = \frac{c}{e} (I\rho_c + \psi), \quad p_\phi = \frac{c}{e} (g\rho_c - \psi_p) \tag{15}$$

The canonical gyroradius is

$$\rho_c = \lambda\frac{mc}{eB} \; v_\parallel + \alpha. \tag{16}$$

Hamilton's equation are

$$\dot{p}_\theta = -\frac{\partial h}{\partial \theta} \qquad \dot{p}_\phi = -\frac{\partial h}{\partial \phi}$$
$$\dot{\theta} = \frac{\partial h}{\partial p_\theta} \qquad \dot{\phi} = \frac{\partial h}{\partial p_\phi} \tag{17}$$

The inclusion of the function α means the particle motion is being followed in the field $\vec{B}_e$. In non-stochastic fields, α can be set equal to zero. If λg is not zero, one can write a more standard form

$$h = \frac{1}{2m} \left(\frac{B}{\lambda g}\right)^2 \left[p_\theta + \frac{e}{c}(\psi_p - \alpha)\right]^2 + \mu B + e\Phi \tag{18}$$

The proposed drift Hamiltonian agrees with Alfvén's drift velocity Eq. (7) to lowest non-vanishing order in the gyradius as can be demonstrated by somewhat messy partial differentiations. The agreement is exact if $\vec{B}\cdot(\vec{\nabla} \times \vec{B}) = 0$.

In a globally steady state field, the guiding center velocity can be written as[3]

$$\vec{v}_g = \frac{v_\parallel}{B} \frac{1}{1 + \rho_\parallel(\vec{B}\cdot\vec{\nabla} \times \vec{B})/B^2} [\vec{B} + \vec{\nabla} \times \rho_\parallel \vec{B}] \tag{19}$$

with $\rho_\parallel(E, \mu, \vec{x}) = v_\parallel/(eB/mc)$. A Hamiltonian formulation, which agrees with this drift, has been given.[4] The drift Hamiltonian proposed in this paper agrees with Eq. (19) if $\vec{B}$ inside the curl operations is replaced by $\vec{B} - \lambda\beta_*\vec{\nabla}\psi$. By dividing the drift velocity into two parts, in a manner analogous to that carried out in the study of ripple transport,[11] one can show that the β_* term should distort the θ and ϕ position of a particle by an amount $\beta_*\rho_c$, which is of order the gyroradius to system size even in a high beta plasma. One hopes both Hamiltonian formulations are correct with the choice being one of convenience.

IV) Lagrangian Derivation

The relation between the exact, time dependent equations of motion and the guiding center equations is clarified by a Lagrangian analysis. The importance of the Lagrangian is that the form Lagrange's equations is independent of the spacial coordinate system. This basicly follows from the fact that the variational principle for the Lagrangian, confusingly known as Hamilton's principle, does not explicitly depend on the coordinates. Consider the exact Langrangian in a stationary frame of reference.

$$L = \frac{1}{2} mV^2 + \frac{e}{c} \vec{A}\cdot\vec{V} - e\Phi_s \tag{20}$$

If the magnetic field is time dependent, then the exact particle velocity $\vec{V} = d\vec{x}/dt$ can be written

$$\vec{V} = \frac{\partial\vec{x}}{\partial t} + \vec{e}_\alpha \dot{\xi}^\alpha \tag{21}$$

with $\xi' = \psi$, $\xi^2 = \theta$, $\xi^3 = \phi$ and $\vec{e}_\alpha = \partial\vec{x}/\partial\xi^\alpha$. The expression

$$\vec{v}_B = \frac{\partial\vec{x}}{\partial t} \tag{22}$$

is the velocity of the magnetic field, and

$$\vec{v} = \vec{e}_\alpha \dot{\xi}^\alpha \tag{23}$$

is the velocity of the particle relative to the field. The exact Lagrangian can, therefore, be written

$$L = \frac{1}{2} mv^2 + \frac{e}{c} \vec{a}\cdot\vec{v} - e\Phi \tag{24}$$

with

$$\vec{a} = \vec{A} + \frac{mc}{e} \vec{v}_B, \quad \Phi = \Phi_s - \vec{A}\cdot\frac{\vec{v}_B}{c} - \frac{1}{2}\frac{m}{e} v_B{}^2$$

The total field, $\vec{\nabla} \times \vec{a}$, which is the sum of the magnetic and vorticity field, is usually indistinguishable from the magnetic field in plasma problems. The point is

$$\left|\frac{mc\ \vec{v}_B}{e\vec{A}}\right| \simeq \frac{v_B}{v}\frac{\rho}{r} \tag{25}$$

with ρ the gyroradius and r a typical plasma dimension. In plasmas both v_B/v and ρ/r are generally quite small. Consequently, we will not distinguish between $\vec{a}$ and $\vec{A}$, although technically one should. Similarly the potential Φ can be considered to obey $\Phi = \Phi_s - \vec{A}\cdot\vec{v}_B/c$, which is just the Lorentz transformed potential. The term $\vec{A}\cdot\vec{v}_B/c$ contains the so-called $\vec{E} \times \vec{B}$ motion of the field lines.

Ignoring the distinction between $\vec{a}$ and $\vec{A}$, the canonical momenta of the exact particle motion $P_\alpha = \partial L/\partial\dot{\xi}^\alpha$ are

$$P_\alpha = (m\vec{v} + \frac{e}{c}\vec{A})\cdot\vec{e}_\alpha$$

and the Hamiltonian is $H = P_\alpha \dot{\xi}^\alpha - L$

$$H = \frac{1}{2m} (P_\alpha - \frac{e}{c} A_\alpha)\, g^{\alpha\beta}(P_\beta - \frac{e}{c} A_\beta) + e\Phi$$

The metric tensor $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ defined by $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$.

The adiabatic conservation of the magnetic moment $\mu = mv_\perp^2/2B$ means the kinetic energy associated with motion perpendicular to the field lines behaves as if it were a potential energy μB. Intuitively, this implies the Lagrangian[12]

$$\ell = \frac{1}{2} mv_\parallel^2 + \frac{e}{c} \vec{A}\cdot\vec{v} - \mu B - e\Phi$$

should represent the drift motion. Indeed, we will show this Lagrangian reproduces the drift Hamiltonian of Sec. III.

The two adiabaticly conserved actions of the drift problem imply there are only four degrees of freedom or two independent spacial coordinates. We take the independent coordinates to be θ and ϕ The canonical momenta are $P_\alpha = \partial\ell/\partial\xi^\alpha$ with α equal θ and ϕ or

$$P_\alpha = (m\vec{v}_\parallel + \frac{e}{c} \vec{A})\cdot\vec{e}_\alpha$$

Using the tensor identity $\vec{e}_\alpha \cdot \vec{\nabla}\xi^\beta = \delta_\alpha^\beta$ and the expression for the vector potential

$$\vec{A} = \psi\vec{\nabla}\theta - \psi_p\vec{\nabla}\phi,$$

one obtains the p_θ and p_ϕ of Eq. (15) for non-stochastic fields. The drift Hamiltonian $h = p_\alpha \xi^\alpha - \ell$ with α equal θ and ϕ. This Hamiltonian is identical to Eq. (14). Stochastic fields can be included by adding $\alpha\vec{H}$ to $\vec{A}$.

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