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Resonance Behavior in the Presence of Space Charge*

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Abstract

An analysis is presented of the resonance behavior of particle beams in the presence of space charge fields. Since self-consistent requirements are ignored, the results describe onset or early behavior. It is shown that in a beam of uniform current resonances excited by magnetic field errors are stabilized by the detuning effect of the self-field space charge force. This situation is changed when a radiofrequency accelerating field is applied. As beam bunching results after rf turn-on, the space charge force becomes modulated along the bunches, vanishing at the ends. At these regions of small or vanishing space charge, stabilization from non-linear detuning tends to disappear, thus leaving particles susceptible to resonance blow-up. This picture of the effect of beam bunching can be studied by considering the phase space structure for particles at different positions along the bunches. A somewhat unusual conclusion is made on the use of this analysis to model beam capture in a synchrotron at low energy.

1. Introduction

High transverse density particle beams at low energy in a synchrotron are strongly influenced by the self-field space charge forces. Because of the relativistic cancellation of electric and magnetic contributions to this force, space charge effects decrease as $1/\gamma^3$ and they are therefore important only at low energy.

The dominant impact of the space charge force is to introduce a characteristic detuning of particles in the beam. This means that the frequency of betatron oscillations becomes a function of amplitude and that resonant conditions can only be sustained over an amplitude region of phase space which decreases for increasing space charge. Thus a low magnetic field resonance will be restricted to a bounded phase space amplitude range and results in an amplitude modulation rather than instability. In a uniform current beam, neglecting scattering and noise effects, the tune of any given particle remains fixed and the betatron oscillations remain stable and involve only small amplitude modulations around the resonance amplitudes.

When radiofrequency accelerating fields are applied to a beam, this situation is drastically altered. After rf turn-on, the beam becomes bunched, with the current distribution following the bunch structure. The space charge force thus develops a current modulation which peaks at the bunch center and vanishes at the ends. At regions of small or vanishing space charge, nonlinear resonance stabilization tends to disappear, thus leaving particles susceptible to resonance blow-up.

We study the problem of resonance behavior in the presence of space charge by treating 1-dimensional resonances. We find the resonant invariant for a resonance excited for example by magnetic imperfections. We then describe how the phase space structure is influenced by the strength of the space charge self-field. In the case of bunched beams, the modulation of this force causes this structure to be very different for different particles in the beam and in fact will change for a given particle during its synchrotron period as it moves from the front of the bunch, through the central region, to the back and so on. This changing phase space structure introduces beam loss mechanisms, which depend on the rate of synchrotron motion as well as on the resonance strength, the space charge detuning strength and the tune modulations resulting from the synchrotron motion.

In section 2, we calculate the resonant phase space invariant and define the major parameters influencing the phase space behavior and structure. In section 3, particle motion is described by analyzing the changing phase space structure. This analysis can be used to model the behavior of a beam during injection and capture in a synchrotron at low energy.

2. Phase Space Invariant

2.1 Linear Focusing System

We consider a ring with a focusing structure given by the function $K(s)$, where s is the distance along the particle equilibrium orbit, measured with respect to a reference point. In terms of $K(s)$ we can write the betatron oscillation equation,

$$y'' + K(s)\, y = 0,$$

where the derivative is with respect to s, related to the time by $s = \beta ct$. The distance s and time t can be chosen to be equivalent for a system with no time dependent forces.

2.2 The Space Charge Force

We take the case of a Gaussian particle distribution in the transverse phase plane. This adds a force term to the linear equation, and the resulting equation can be expressed as,

$$y'' + Ky - \frac{\lambda r_p}{\beta^2 \gamma^3 \sigma^2}\, y\, H(Y^2) = 0$$

where, λ is the linear particle density,

r_p is the classical proton radius (1.54×10^{-18}m),

β and γ are the usual relativistic parameters,

σ is the rms beam transverse size,

$Y = y/\sqrt{2}\sigma$,

and

$$H(Y^2) = \int_0^1 dt\, e^{-tY^2} = \frac{1-e^{-Y^2}}{Y^2} .$$

The negative sign in the full equation indicates explicitly the defocusing nature of the self-field force.

2.3 Resonance Force Term

To study the effects of magnetic imperfections in the vertical plane, we need an expansion of the radial magnetic field, B_x, around $y = o$. We can write,

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$$\frac{B_x(s,y)}{(B\rho)} = -\sum_{p=0}^{\infty} d_p(s)\, y^p ,$$

where, $(B\rho)$ is the particle magnetic rigidity, and $d_p(s)$ is the distribution of field errors. It can be shown straightforwardly that the relation between the appropriate error field derivative and the error dis-tribution $d_p(s)$ for each value of p is given by

$$d_p(s) = (-1)^{1/2(p+1)} \frac{B^{(p)}}{p!(B\rho)} , \quad p=1,3,\ldots$$

and

$$d_p(s) = (-1)^{1/2(p+2)} \frac{B^{(p)}}{p!(B\rho)} , \quad p=2,4,\ldots$$

where

$$B^{(p)} = \frac{\partial^p B_y}{\partial x^p}\Big|_{x=y=0} .$$

The resonance forcing term modifies the linear equation, with the equation becoming,

$$y'' + K(s)y + d_{p-1}(s)\, y^{p-1} = 0,$$

where we have kept a single resonance term of order p-1 [p = 1, dipole; p = 2, quadrupole; p = 3, sextupole; and so on].

For a resonance in the presence of space charge, we add to the linear equation both resonance and space charge terms, leading to,

$$y'' + Ky - \frac{\lambda r_p}{\beta^2\gamma^3\sigma^2}\, y\, H(Y^2) + d_{p-1}(s) y^{p-1} = 0 .$$

2.4 Normalized Coordinates

We define the space charge strength parameter ξ,

$$\xi = \frac{N r_p}{4\pi\, \epsilon_{rms}\, \beta^2\gamma^3} ,$$

where N is the number of particles in the beam, and ϵ_{rms} is the rms emittance, defined by

$$\epsilon_{rms} = \frac{\sigma^2}{\beta(s)},$$

with $\beta(s)$ the Beta function or Twiss parameter.

To solve the linear part of the equation we introduce the normalized coordinates $(\eta, \dot\eta)$, by

$$\begin{bmatrix} \eta \\ \dot\eta/\nu \end{bmatrix} = \begin{bmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix},$$

with

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds} , \text{ and}$$

differentiation is with respect to the betatron phase, ϕ, given by

$$\phi = \frac{1}{\nu}\int_0^s \frac{ds}{\beta(s)} .$$

In the smooth approximation, the phase ϕ is simply the azimuth θ, i.e. $\phi \to \theta = s/R$, where R is the average. ring radius. The equation of motion in terms of $\eta, \dot\eta$ and ϕ, is

$$\ddot\eta + \nu^2\eta - \frac{2\nu^2\xi\beta(s)}{R}\eta H(Y^2) + \nu^2[\beta(s)]^{\frac{1}{2}(p+2)} d_{p-1}(s)\eta^{p-1} = 0,$$

where

$$Y^2 = \frac{\eta^2}{2\epsilon_{rms}} .$$

2.5 Phase - Amplitude Transformation

We introduce phase and amplitude variables $[I, \psi]$, related to $[\eta, \dot\eta]$ by

$$\eta = \sqrt{I}\cos\psi ,$$

$$\dot\eta = -\nu\sqrt{I}\sin\psi .$$

Inverting and differentiating we obtain the equations of motion for I and ψ in the form,

$$\dot I = -\frac{2\sqrt{I}}{\nu}\sin\psi\,(\ddot\eta + \nu^2\eta) ,$$

and

$$\dot\psi = \nu - \frac{\cos\psi}{\nu\sqrt{I}}(\ddot\eta + \nu^2\eta) .$$

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We define the resonant phase,

$$\tau = \psi - \frac{m}{p}\phi + \text{constant phase} ,$$

having the property of a slow time variation:

$$\dot\tau = \dot\psi - \frac{m}{p} \approx 0 , \text{ with } \dot\psi \approx \nu .$$

In other words, we require that the tune for betatron oscillations ν be close to the order-p resonance value, m/p. Here, m is the ring azimuthal harmonic (in the variable ϕ) of the resonant force term appro-priate to the resonance of order p.

2.7 Phase Averaging

The equations for the amplitude and phase variables result in the resonance equations by applying the method of phase averaging. The relevant phase variables are ψ, the phase space phase and ϕ, the ring azimuth phase. The explicit dependence on ϕ represents the azimuthal harmonic content of the perturbing force in the original equations and is the necessary ingredient for resonance behavior. The phase averaging procedure involves two steps. First, the space charge term has no explicit ϕ dependence except for the β-function variation, whose symmetry tends to suppress resonance excitation. Therefore the space charge term simply oscillates rapidly in ψ and ϕ about some mean value. Averaging over ψ and ϕ as

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Averaging the space charge term over ψ and ϕ, we find in the case of no resonance force term,

$$\dot{I} = 0 ,$$

and

$$\dot{\psi} = \nu - \xi F(\alpha),$$

where

$$\alpha = I/2\varepsilon_{rms},$$

and

$$F(\alpha) = \frac{1}{\pi}\int_0^{2\pi} du \cos^2 u \int_0^1 dt\, e^{-t\alpha \cos^2 u} .$$

Here, we have taken for the average β: $1/2\pi R \int_0^{2\pi R} \beta(s)ds \approx R/\nu$ and we have taken, $\xi > 0$.

The function $F(\alpha)$ can be expressed in terms of the Bessel Functions,

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} \cos n\theta\, d\theta,$$

by

$$F(\alpha) = \frac{2}{\alpha}\left(1 - e^{-\alpha/2} I_o\left(\frac{\alpha}{2}\right)\right).$$

The phase equation in the presence of space charge thus has the property that the oscillating tune, $\dot{\psi}$, becomes a function of amplitude. Thus, a resonant condition on the tune becomes amplitude dependent and resonances become restricted to certain amplitude regions in the phase space.

To average the resonant terms, we must look for the term containing the slow phase, $\psi - (m/p)\phi$. Consider the resonant term in the equation for $\dot{\psi}$:

$$\frac{\cos\psi}{\nu\sqrt{I}} \nu^2 \beta^{1/2(p+2)} d_{p-1}(\phi)\, \eta^{p-1} ,$$

or, with $\eta = I^{1/2}\cos\psi$, we have

$$\nu\beta^{1/2(p+2)} d_{p-1}(\phi)\, I^{1/2(p-2)} \cos^p \psi .$$

Since the only term in ψ which will contribute to the slow phase term is $\cos p\psi$, we make the replacement

$$\cos^p \psi \rightarrow \frac{1}{2^{p-1}} \cos p\psi .$$

Also, we expand $d_{p-1}(\phi)$ in a Fourier series in ϕ and retain only the m^{th} harmonic, which is the only term contributing to the slow phase.
That is,

$$d_{p-1}(\phi) \rightarrow e_{pm} \cos m\phi + f_{pm} \sin m\phi,$$

where

$$e_{pm} = \frac{1}{2\pi}\int_0^{2\pi} d_{p-1}(\phi) \cos m\phi\, d\phi,$$

and

$$f_{pm} = \frac{1}{2\pi}\int_0^{2\pi} d_{p-1}(\phi) \sin m\phi\, d\phi.$$

Making the substitutions for $\cos^p \psi$ and $d_{p-1}(\phi)$ and retaining only the slow phase term, we have

$$\nu\beta^{1/2(p+2)} I^{1/2(p-2)} \frac{1}{2^p} \{e_{pm} \cos(p\psi - m\phi) - f_{pu} \sin(p\psi - m\phi)\} .$$

If we now define

$$\Gamma_p e^{i\gamma_p} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, d_{p-1}(\phi) e^{im\phi} ,$$

we obtain for this term in the phase equation,

$$\nu\beta^{1/2(p+2)} I^{1/2(p-2)} \frac{1}{2^p} \Gamma_p \cos(p\psi - m\phi + \gamma_p).$$

The equation for $\dot{\psi}$ is therefore

$$\dot{\psi} = \nu - \xi F(\alpha) + \nu\beta^{1/2(p+2)} I^{1/2(p-2)} \frac{1}{2^p} \Gamma_p \times \cos(p\psi - m\phi + \gamma_p).$$

In a similar way, the equation for the amplitude I can be written,

$$\dot{I} = 2\nu\beta^{\frac{1}{2}(p+2)} I^{\frac{1}{2}p} \frac{1}{2^p} \Gamma_p \sin(p\psi - m\phi + \gamma_p) .$$

We now introduce the slow phase, $\tau = \psi - m\phi/p + \gamma_p/p$, with the property that $\dot{\tau}$ is small, i.e. that τ is slowly varying when the betatron tune is close to resonance. Using the variable $\alpha = I/2\varepsilon_{rms}$, we have equations for α and τ:

$$\dot{\tau} = \nu - \frac{m}{p} - \xi F(\alpha) + \frac{\nu\beta^{\frac{1}{2}(p+2)}}{2^{\frac{1}{2}(p+2)}} \varepsilon_{rms}^{\frac{1}{2}(p-2)} \alpha^{\frac{1}{2}(p-2)} \Gamma_p \cos p\tau,$$

and

$$\dot{\alpha} = \frac{1}{2^{\frac{1}{2}p}} \nu\beta^{\frac{1}{2}(p+2)} \varepsilon_{rms}^{\frac{1}{2}(p-2)} \alpha^{\frac{1}{2}p} \Gamma_p \sin p\tau .$$

Define the stopband width Δ_e by

$$\Delta_e = \frac{1}{2^{\frac{1}{2}(p+2)}} \nu \beta^{\frac{1}{2}(p+2)} \varepsilon_{rms}^{\frac{1}{2}(p-2)} \Gamma_p \ .$$

Thus, the τ and α equations become,

$$\dot{\tau} = \nu - \frac{m}{p} - \xi F(\alpha) + \Delta_e \alpha^{\frac{1}{2}(p-2)} \cos p\tau,$$

and

$$\dot{\alpha} = 2 \Delta_e \alpha^{\frac{1}{2}p} \sin p\tau \ .$$

2.8 The Resonance Invariant

Using the $\dot{\alpha}$ and $\dot{\tau}$ equations, we can construct an invariant C by requiring that

$$\dot{\tau} = \frac{\partial C}{\partial \alpha}$$

and

$$\dot{\alpha} = -\frac{\partial C}{\partial \tau} \ .$$

In this case, since C is explicitly independent of ϕ, C is an invariant in the sense that

$$\dot{C} = \frac{\partial C}{\partial \alpha}\dot{\alpha} + \frac{\partial C}{\partial \tau}\dot{\tau} = 0 \ .$$

It is seen in a straightforward way that C is given by,

$$C = \Delta_L \alpha - \xi U(\alpha) + \frac{2}{p} \Delta_e \alpha^{p/2} \cos p\tau,$$

where we have written

$$\Delta_L = \nu - \frac{m}{p} \ ,$$

and

$$U(\alpha) = \int_0^\alpha F(\alpha)\, d\alpha \ .$$

The resonance invariant defines a set of curves in the phase space (α,τ) which represent the phase space structure for particle motion near the resonance. There are three terms in the invariant, each represented by a parameter: Δ_L is the distance of the linear unperturbed tune from resonance; ξ is the strength of the space charge force; and Δ_e is the resonance stopband width, proportional to the m^{th} harmonic of the magnetic error field exciting the resonance. It should be noted that in the presence of the space charge force, the resonance tune condition is amplitude dependent. Furthermore, Δ_L need not be small, since the space charge term also contains a linear tune shift which is not included in Δ_L. We therefore have defined ν as the linear unperturbed tune as determined by the external focusing structure. Under space charge conditions, the particle linear tune is depressed by ξ, i.e.

$$\nu_{particle} = \nu - \xi \text{ (small amplitudes)}.$$

Also, the tune amplitude dependence is given by

$$\nu_{particle}(\alpha) = \nu - \xi F(\alpha) \ .$$

3. Particle Behavior Under Resonance and Space Charge Conditions

3.1 Fixed Points

When particle trajectories in the phase space are isolated points, these are called fixed points. They can be defined by $\dot{\alpha} = \dot{\tau} = 0$. For a resonance of order p, the fixed points come in sets of p points. A fixed point is stable if nearby trajectories are elliptical around it, and is unstable if nearby trajectories are hyperbolic and move towards and away from it. If the phase space structure is defined by the set of invariant curves,

$$C = \Delta_L \alpha - \xi U(\alpha) + \frac{2}{p} \Delta_e \alpha^{p/2} \cos p\tau \ ,$$

then the fixed points are obtained from

$$\frac{\partial C}{\partial \alpha} = \frac{\partial C}{\partial \tau} = 0 \ .$$

3.2 Condition for Fixed Point Type

To determine the nature of the fixed point, we must expand the function C to second order in deviations from the fixed point. Let α_F and τ_F be a fixed point. Let $\delta\alpha = \alpha - \alpha_F$, $\delta\tau = \tau - \tau_F$ be small deviations from the fixed point. Then, a small deviation in C can be expressed by

$$\delta C = \frac{\partial^2 C}{\partial \alpha^2}(\delta\alpha)^2 + \frac{\partial^2 C}{\partial \tau^2}(\delta\tau)^2 + \frac{\partial^2 C}{\partial\alpha\partial\tau}(\delta\alpha)(\delta\tau),$$

where derivatives are evaluated at the fixed point. To test whether the fixed point is elliptic or hyperbolic, we rotate the coordinates through the angle w by the relation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \cos w & \sin w \\ -\sin w & \cos w \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\tau \end{bmatrix}.$$

Choosing the angle w to be,

$$\cot 2w = \frac{\dfrac{\partial^2 C}{\partial \alpha^2} - \dfrac{\partial^2 C}{\partial \tau^2}}{\partial^2 C/\partial\alpha\partial\tau} \ ,$$

we remove the pq cross term. After some algebraic manipulation, we can show that the 2^{nd} order equation is elliptic or hyperbolic according to the rule,

$$4 \frac{\partial^2 C}{\partial \alpha^2}\frac{\partial^2 C}{\partial \tau^2} - \left(\frac{\partial^2 C}{\partial\alpha\partial\tau}\right)^2 \begin{cases} >0 \text{ elliptic (stable fixed point)} \\ <0 \text{ hyperbolic (unstable fixed point)} \end{cases} .$$

3.3 Evaluation of Fixed Points for an Order p Resonance

The fixed points (α_F, τ_F) are found from

$$\frac{\partial C}{\partial \alpha} = \frac{\partial C}{\partial \tau} = 0, \text{ or}$$

$$\sin p\tau_F = 0,$$

$$\Delta_L - \xi F(\alpha_F) \pm \Delta_e \alpha_F^{p/2-1} = 0,$$

which does not include the fixed point at $\alpha_F = 0$. To determine the nature of these fixed points, we find the second derivatives,

$$\frac{\partial^2 C}{\partial \alpha^2} = -\xi F'(\alpha) + (\frac{p}{2}-1)\Delta_e \alpha^{p/2-2}\cos p\tau ,$$

$$\frac{\partial^2 C}{\partial \tau^2} = -2p\Delta_e \alpha^{p/2}\cos p\tau ,$$

and

$$\frac{\partial^2 C}{\partial \alpha \partial \tau} = -p\Delta_e \alpha^{p/2-1}\sin p\tau .$$

At the fixed point, $\sin p\tau_F = 0$, and

$$\frac{\partial^2 C}{\partial \alpha^2} = -\xi F'(\alpha_F) \pm (\frac{p}{2}-1)\Delta_e \alpha_F^{p/2-2} ,$$

$$\frac{\partial^2 C}{\partial \tau^2} = \mp 2p\Delta_e \alpha_F^{p/2} ,$$

and

$$\frac{\partial^2 C}{\partial \alpha \partial \tau} = 0 .$$

Thus, the fixed points at α_F are stable or unstable according as,

$$\pm \xi F'(\alpha_F) - (\frac{p}{2}-1)\Delta_e \alpha_F^{p/2-2} \begin{matrix} > \\ < \end{matrix} 0 \begin{Bmatrix} \text{s.f.p} \\ \text{u.f.p} \end{Bmatrix},$$

or, alternatively

$$\xi F'(\alpha_F)\alpha_F(\Delta_L - \xi F'\alpha_F)) +$$

$$+ (\frac{p}{2}-1)(\Delta_L - \xi F(\alpha_F))^2 \begin{matrix} < \\ > \end{matrix} 0 \begin{Bmatrix} \text{s.f.p.} \\ \text{u.f.p.} \end{Bmatrix} .$$

In the case $\xi = 0$, the condition becomes

$$(\frac{p}{2}-1)\Delta_L^2 \begin{matrix} < \\ > \end{matrix} 0 \begin{Bmatrix} \text{s.f.p.} \\ \text{u.f.p.} \end{Bmatrix}.$$

Therefore, for dipole resonances, $p = 1$, we have stable fixed points only; for quadrupole resonances, $p = 2$, we have no fixed points for $\alpha_F \neq 0$; for sextupole or higher order resonances, $p \geq 3$, we have unstable fixed points only. If ξ is not zero, we can see from the above conditions that if $p \geq 2$ the top sign gives an unstable fixed point since $\xi > 0$ and $F'<0$. Thus the set of p unstable fixed points are obtained from,

$$\Delta_L - \xi F(\alpha_F) + \Delta_e \alpha_F^{p/2-1} = 0 .$$

In other words, $\cos p\tau_F = +1$ for the unstable fixed points.

3.4 The Separatrix

The separatrix is the phase space trajectory which passes through a set of unstable fixed points. The unstable fixed points are given by

$$\cos p\tau_F = +1;\ \Delta_L - \xi F(\alpha_F) + \Delta_e \alpha_F^{p/2-1} = 0 .$$

To evaluate the separatrix, we evaluate C_F, the constant, at the fixed points:

$$C_F = \Delta_L \alpha_F - \xi U(\alpha_F) + \frac{2}{p}\Delta_e \alpha_F^{p/2} .$$

Thus, the separatrix equation is,

$$\Delta_L(\alpha - \alpha_F) - \xi(U(\alpha) - U(\alpha_F)) +$$

$$+ \frac{2}{p}\Delta_e(\alpha^{p/2}\cos p\tau - \alpha_F^{p/2}) = 0,$$

or to second order in $\delta\alpha = \alpha - \alpha_F$,

$$r^2(1 + (\frac{1}{2})(\frac{p}{2})(\frac{p}{2}-1)h\cos p\tau) -$$

$$- (\frac{p}{2})rh(1-\cos p\tau) - h(1-\cos p\tau) = 0,$$

$$\text{where } r = \frac{\delta\alpha}{\alpha_F} ,$$

and

$$h = -\frac{4\Delta_e}{\xi\alpha_F F'(\alpha_F)p}\alpha_F^{p/2-1} .$$

For small h, we can solve this equation for r and take the leading term, which goes like $\sqrt{h}$, to obtain,

$$r \simeq \pm\sqrt{h(1-\cos p\tau)} .$$

Thus, for small h, the separatrix is a string of p islands around the origin, with a width,

$$w = \pm\sqrt{2h} .$$

3.5 Resonance Phase Space Structure With No Space Charge

Setting $\xi = 0$, we can write the set of invariant curves defining the resonance phase space structure as,

$$C = \Delta_L \alpha + \frac{2}{p}\Delta_e \alpha^{p/2}\cos p\tau.$$

The fixed points for $\alpha_F \neq 0$ can be found from,

$$\Delta_L \pm \Delta_e \alpha_F^{p/2-1} = 0 .$$

If these exist, they will be stable or unstable according to,

$$(\frac{p}{2} - 1)\,\Delta_L^2 \lessgtr 0 \begin{cases} \text{s.f.p.} \\ \text{u.f.p.} \end{cases} .$$

Consider the case where a particle is on resonance; that is, $\Delta_L = 0$. Then, if $p \geq 2$, the only fixed point is at $\alpha_F = 0$. The phase space structure is then controlled by an unstable fixed point at the origin, with all the trajectories being of the unstable form (that is, trajectories with unlimited amplitude). The set of phase space trajectories are given by

$$\frac{2}{p}\,\Delta_e\,\alpha^{p/2} \cos p\tau = C,$$

and the trajectories passing through the unstable fixed point at $\alpha_F = 0$ are given by

$$\alpha^{p/2} \cos p\tau = 0 \ , \ p \geq 2.$$

This is a set of p straight lines passing through the origin of the $(\sqrt{\alpha},\tau)$ phase plane. Defining the lines for the angle τ in the range $-\pi/2<\tau \leq \pi/2$, we can express the lines by,

$$\tau = \pm \frac{q\pi}{2p},\ q = 1,3,\ldots(p-1)\ \{\text{for } p \text{ even}\} ;$$

and

$$\tau = \begin{cases} \pm \frac{q\pi}{2p}\ ,\ q = 1,3,\ldots(p-2) \\ \pi/2,\ q = p \end{cases} \quad (\text{for } p \text{ odd}).$$

For a dipole resonance, p = 1, the phase space trajectories are given by

$$C = \Delta_L\,\alpha + 2\,\Delta_e\,\sqrt{\alpha}\cos\tau,$$

with a stable fixed point given by,

$$\Delta_L \pm \frac{\Delta e}{\sqrt{\alpha_F}} = 0 .$$

Thus we see that for $\Delta_L \neq 0$, there is a stable fixed point at

$$\sqrt{\alpha_F} = \left|\frac{\Delta_e}{\Delta_L}\right| \ , \text{ with } \tau = \begin{cases} 0, \text{ if } \Delta_L<0 \\ \pi, \text{ if } \Delta_L>0 \end{cases} .$$

The trajectories are circles around this fixed point. At the resonance, $\Delta_L = 0$, and the phase space structure degenerates into the straight lines

$$\sqrt{\alpha}\cos\tau = \text{constant}.$$

Notice that for the dipole resonance, the fixed point at $\sqrt{\alpha} = 0$ vanishes because of the resonance. All other resonance orders retain the $\sqrt{\alpha} = 0$ fixed point. This is easily seen by recognizing that to get the fixed points, $\partial C/\partial\sqrt{\alpha}$ must be set to zero, rather than $\partial C/\partial\alpha$. Of course, for the $\sqrt{\alpha} \neq 0$ fixed points, using the latter is appropriate.

3.6 Resonance Phase Space Structure with Space Charge

If we include space charge, we have for the p unstable fixed points,

$$\Delta_L - \xi\,F(\alpha_F) + \Delta_e\,\alpha_F^{p/2-1} = 0.$$

The p stable fixed points can be found from the expression,

$$\Delta_L - \xi F(\alpha_F) - \Delta_e \alpha_F^{p/2-1} = 0 .$$

If a particle is oscillating in the phase space not close to the fixed points, i.e. not near the resonance, then the behavior is simply

$$\dot{\alpha} = 0 ,$$

and

$$\dot{\psi} = \nu - \xi F(\alpha) ,$$

where ν is the externally applied betatron tune and ξ is the space charge strength. $\dot{\psi}$ is simply a perturbed tune for the amplitude α. Calling this $\nu_p(\alpha)$, we have,

$$\nu_p(\alpha) = \nu - \xi F(\alpha) .$$

At amplitudes near the fixed points, i.e. near the resonance, the phase space structure is generally a string of p islands around the origin. The fixed points define the amplitude at which these islands are located. The amplitude variation can be found from the approximate expression for the separatrix,

$$\frac{\Delta\alpha}{\alpha_F} = \pm\sqrt{h(1-\cos p\tau)}.$$

This means that the amplitude modulation around the fixed points, or the extent of the resonance in amplitude, is given by

$$\frac{(\Delta\alpha)_{max}}{\alpha_F} = \pm\sqrt{2h} ,$$

where

$$h = -\frac{4\,\Delta_e \alpha_F^{p/2-1}}{p\,\xi\alpha_F F'(\alpha_F)} .$$

For large space charge strength, h << 1, and we see that the space charge has stabilized the resonance. Without space charge the resonances fill the entire phase space with unstable trajectories. Adding space charge reduces the impact to a small phase space amplitude region.

Consider the calculation for the case where the unstable fixed points are at $\alpha_F = 1$, corresponding to the beam edge (i.e. $\sqrt{2}\,\sigma_{rms}$). We have,

$$\Delta_L - \xi\,F(1) + \Delta_e = 0 ,$$

or

$$\Delta_L = \xi\,F(1) - \Delta_e .$$

Since it is assumed that $\xi >> \Delta_e$, both the stable and unstable fixed points are very near $\alpha = 1$, and the tune required for this condition is

$$\Delta_L = \xi F(1).$$

In this case the maximum amplitude modulation around $\alpha_F = 1$ is

$$(\Delta\alpha)_{max} = \pm\sqrt{\frac{8\Delta_e}{p\,\xi\,|F'(1)|}} .$$

A good approximation to $F(\alpha)$ is

$$F(\alpha) = \frac{1}{1 + \alpha/2} ,$$

which satisfies $F(0) = 1$, and $F(\infty) \to 2/\alpha$. Using this form, $F(1) = 2/3$ and $F'(1) = 2/9$, where $F'(\alpha) = -\frac{1}{2(1 + \alpha/2)^2}$.

We therefore have:

$$\Delta_L = \frac{2}{3}\xi, \text{ and } (\Delta\alpha)_{max} = \pm\sqrt{\frac{36\Delta_e}{p\,\xi}} .$$

As Δ_L approaches the resonance, $\Delta_L = \xi$, the fixed points shift toward the origin and the islands shrink in width. For $p \geq 2$, and $\Delta_L = \xi$, the fixed points are determined by

$$\xi\left(1 - F(\alpha_F)\right) \pm \Delta_e\,\alpha_F^{\;p/2-1} = 0 ,$$

or

$$\alpha_F = 0.$$

At $\alpha_F = 0$, we have,

$$(\Delta\alpha)_{max} = \pm\,\alpha_F\sqrt{2h} \to 0 .$$

For the quadrupole resonance, $p = 2$, the fixed points are determined by,

$$\Delta_L - \xi F(\alpha_F) \pm \Delta_e = 0 ,$$

and the set of invariant curves are given by,

$$C = \Delta_L\alpha - \xi\,U(\alpha) + \Delta_e\alpha\cos 2\tau .$$

To see the nature of the fixed points, write the fixed point equation in the form,

$$\Delta_L - \xi + \xi\left(1 - F(\alpha_F)\right) \pm \Delta_e = 0 ,$$

noting that $1 - F(\alpha_F) \geq 0$ for $\alpha_F \geq 0$.

Then we have,

for $\Delta_L - \xi > \Delta_e$, no fixed points;
(The origin is a stable fixed point).

for $\Delta_L - \xi < -\Delta_e$, 2 sets of fixed point pairs, one stable, the other unstable (islands strung around the origin);
(The origin is a stable fixed point).

for $-\Delta_e < \Delta_L - \xi < \Delta_e$, one set of stable fixed points.
(The origin is an unstable fixed point).

Thus, the no-space-charge-stopband still exists in a sense. If the linear particle tune (small amplitude) is above the stopband there are no fixed points. That is, we have distorted circles of varying tune in phase space. Below the stopband, islands develop around the origin. We have calculated the width of these islands before, using an expansion about the unstable fixed points for $\alpha_F \neq 0$. We obtained a width given by,

$$(\Delta\alpha)_{max} = \pm\sqrt{\frac{4\Delta_e\alpha_F}{3|F'(\alpha_F)|}} .$$

Again, if the tune is chosen so that the fixed point is at the beam edge, $\alpha_F = 1$, we have

$$(\Delta\alpha)_{max} = \pm\sqrt{\frac{18\Delta_e}{\xi}} .$$

This is a new effect introduced by the space charge force. Inside the stopband, the space charge force adds a stable fixed point to the already existing unstable fixed point at the origin, leading to a "figure-8" type separatrix. Since the separatrix passes through the origin, it is found by taking the constant $C = 0$, that is

$$\Delta_L\alpha - \xi U(\alpha) + \Delta_e\alpha\cos 2\tau = 0.$$

To determine the general nature of the phase space structure, we use the approximate expression for $F(\alpha)$,

$$F(\alpha) = \frac{1}{1 + \alpha/2} .$$

Solving for the fixed points, we obtain

$$\frac{1}{2}\alpha_F = \frac{\xi}{\Delta_L \pm \Delta_e} - 1 .$$

The stable fixed point when the tune is inside the stopband is found using the "-" sign. At the center of the stopband, $\Delta_L = \xi$, and we have for the stable fixed point,

$$\frac{1}{2}\alpha_F = \frac{\Delta_e}{\xi - \Delta_e} \cong \frac{\Delta_e}{\xi} ,$$

and for the separatrix

$$\xi\left(\alpha - U(\alpha)\right) + \Delta_e\,\alpha\cos 2\tau = 0 .$$

Expand about the unstable fixed point at $\alpha = 0$. Then,

$$U(\alpha) = \alpha - \frac{1}{4}\alpha^2 + \ldots$$

Thus, we have for the separatrix,

$$\alpha = -\frac{4\Delta_e}{\xi}\cos 2\tau .$$

Thus this is a "vertical figure-8", with the stable fixed points along $\tau = \pi/2$ and $\tau = 3\pi/2$, with maximum amplitudes along this line given by

$$\alpha_{max} = \frac{4\Delta_e}{\xi} .$$

To the extent that α_{max} is small, i.e. $\xi >> \Delta_e$, the space charge force has bounded the resonant trajectories to regions in amplitude around the origin of order Δ_e/ξ.

For the dipole resonance, $p = 1$, the fixed points are determined by,

$$\Delta_L - \xi F(\alpha_F) \pm \Delta_e/\sqrt{\alpha_F} = 0.$$

It should be noted that in the dipole case there is no added fixed point at the origin. This is a reflection of the fact that dipole resonances directly affect the closed orbit. The set of resonant trajectories are given by,

$$C = \Delta_L\alpha - \xi U(\alpha) + 2\,\Delta_e\,\alpha^{1/2}\cos\tau\ .$$

To see the general nature of the fixed point structure, write the fixed point equation in the form,

$$\Delta_L - \xi + \xi\,[1-F(\alpha_F)] \pm \frac{\Delta e}{\sqrt{\alpha_F}} = 0$$

The point $\alpha_F = 0$ is never a fixed point as long as $\Delta_e \neq 0$. On resonance, $\Delta_L - \xi = 0$, and the fixed point which was at infinite amplitude when there was no space charge has moved to a finite amplitude which can be found from the equation,

$$\sqrt{\alpha_F}\,[1 - F(\alpha_F)] = \frac{\Delta_e}{\xi}\ ,$$

where only the bottom sign in the fixed point equation gives a fixed point, which is a stable fixed point. An approximate solution can be found by noting that for Δ_e/ξ small, the solution must have $\alpha_F < 1$. So, expand about $\alpha_F = 0$, giving the equation

$$|F'(0)|\ \alpha_F^{3/2} = \frac{\Delta_e}{\xi}\ ,$$

or

$$\sqrt{\alpha_F} = \sqrt[3]{\frac{\Delta_e}{\xi|F'(0)|}} = \sqrt[3]{\frac{2\Delta_e}{\xi}}\ .$$

Now, if $\Delta_L - \xi > 0$, this fixed point must move closer to the origin since a solution requires that the term $(-\Delta_e/\sqrt{\alpha_F})$ increase in magnitude.

We can estimate the fixed point structure as a function of $\Delta_L - \xi$ by using the approximate form of F or by expanding F to lowest order in α since all the fixed points will be close to $\alpha = 0$ if $h << 1$. Writing,

$$1-F = \frac{\alpha}{2+\alpha} \approx \frac{\alpha}{2} \quad \text{if } \alpha << 1\ ,$$

and defining

$$h = \frac{\Delta_e}{\xi}\ ,$$

$$\epsilon = \frac{\Delta_L-\xi}{\xi}\ ,$$

$$\chi = \sqrt{\alpha}\ ,$$

we have for the fixed point equation,

$$\chi^3 + 2\epsilon\chi \pm 2h = 0\ .$$

Consider the fixed points as a function of $\epsilon = (\Delta_L-\xi)/\xi$. On resonance, $\epsilon = 0$, and we have the solution,

$$\chi = \sqrt[3]{2h}\ .$$

To complete the analysis of the dipole resonance, the cubic equation for the fixed points has been solved off resonance ($\epsilon \lessgtr 0$) as well, with the conclusion that the space charge force does indeed preserve a stable region of phase space near the origin. In fact, the worst situation is on resonance, where the fixed point moves from infinity (no space charge stabilization) to a distance from the origin $\sqrt[3]{2h}$, i.e. a distance on the order $(\Delta e/\xi)^{1/3}$.

3.7 Particle Tune

The particle tune is generally the rate at which the particle phase rotates in the phase plane. In a linear system, $\dot{\psi}$ is constant and is the linear tune produced by the external focusing system,

$$\dot{\psi} = \nu_L \text{ (linear tune)}\ .$$

If we add a nonlinear force, the tune becomes dependent on the betatron amplitude. Looked at over a long time, i.e. long compared to a revolution, the phase space structure is composed of curves of essentially constant amplitude but with tunes which depend on the trajectory, i.e. the amplitude. In the presence of a space charge force of strength ξ, the small amplitude tune is depressed by an amount ξ and the particle tune becomes amplitude dependent, rising at large amplitudes. We have in this case,

$$\dot{\psi} = \nu(\alpha) = \nu_L - \xi + \xi[1 - F(\alpha)],\ \text{(with space charge)}$$

where $1 - F(\alpha) \geq 0$, $F(0) = 1$, and $\nu(\alpha)$ is the particle tune, a function of amplitude.

This picture must be modified when a resonance appears in the phase space. In this case, over long times compared to a revolution, the phase space structure is altered and in some cases, as we have discussed, changes drastically, even to the point of trajectories becoming unbounded. In general, we can estimate the tune at which these effects emerge in the phase space and the range of tune over which they act: in particular, they occur when

$$\frac{m}{p} - \Delta_e \lesssim \nu(\alpha) \lesssim \frac{m}{p} + \Delta_e\ ,$$

where, m and p are integers, and Δ_e is the stopband width for the resonance of order p.

The linear tune arises from the magnetic focusing structure of the ring. There are other contributions, for example from the coherent tune shift, which arises from image fields induced by the circulating current in the walls of the vacuum chamber or in the magnet poles. The strength or tune of the focusing magnetic structure depends on the particle momentum. Thus, ν_L will in general vary with momentum in a beam of non-zero momentum spread. If C is the chromaticity, and δp is the fractional momentum deviation with respect to a central orbit momentum, then the linear tune shift is

$$\delta\nu_L = C\ \delta p\ .$$

3.8 Behavior of a Uniform Current Beam

A uniform current beam has the characteristic that all particles are mediated by the same space charge strength ξ. Particles of different amplitudes will have different tunes, but the force is the same for all particles. A second point is that, except for scattering and noise effects, the linear tune remains constant for each particle, although the beam will have a spread in linear tune due to the chromaticity.

If the linear tune of a particle is fixed, and if the space charge strength is fixed, to determine stability we need only examine the "fixed" phase space structure. In the case of large current, or large ξ, we have seen that resonances of all orders have only a minor impact on the phase space structure. Thus, even on resonance, the beam will be stabilized by the strong resonance detuning effect of the space charge force.

3.9 Behavior of a Bunched Beam

When a beam is bunched by an applied radiofrequency field, two effects manifest themselves, which change this picture of space charge induced stability. The most important impact is that the space charge strength ξ is a function of the local current density, which is a constant in the non-bunched case. At both ends of the bunch, the local current drops to zero. Thus, the detuning effect of the space charge force disappears for these particles and if they are on a resonance, the phase space structure will be strongly affected as we have previously seen. A second important effect of bunching is the synchrotron motion of particles around the bunch. In other words, particles rotate around the center of the bunch and in particular, those of large synchrotron amplitude move from front to center to back and so on. Thus, a particle can move from a region where it is space charge stabilized to one where it is resonance vulnerable. The problem is to control the tune such that uncorrected resonances are only located around the bunch center where space charge stabilization is effective; while the tunes corresponding to the bunch ends are kept free of such resonances. This becomes increasingly difficult as ξ increases since the bunch tune spread from center to ends is on the order of ξ. Application of this model to low energy beam capture in a synchrotron leads to the unusual conclusion that in the bunching phase if the beam in tune space must be located on an uncorrected resonance, then it is stable if the locally dense portion of the beam is directly on the resonance and the locally dilute portion of the beam is free of this resonance.