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R. PELLAT, M. FREY, M. TAGGER

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R. PELLAT

Centre de Physique Théorique, Ecole Polytecnique, Palaiseau, France.

M. FREY, M. TAGGER

ASSOCIATION EURATOM-CEA SUR LA FUSION
Département de Recherches sur la Fusion Contrôlée
Centre d'Etudes Nucléaires
Boite Postale n° 6. 92260 FONTENAY-AUX-RÓSES (FRANCE)

Abstract : The growth of a Tearing instability in Rutherford's non-linear regime is investigated. Using a singular perturbation technique, we recover to lowest order Rutherford's result. To the following order we show that the mode generates a quasi-linear deformation of the equilibrium flux profile, whose resistive diffusion slows down the growth and shows the possibility of a saturation of the instability.

- I -

Tearing instabilities [1] have been shown to play an important role in many areas of plasma physics, including magnetic fusion experiments, solar flares and magnetospheric activity. A decisive step in the understanding of their non-linear behaviour was accomplished by Rutherford [2]. He showed that when the width of a magnetic island, due to a high-m ($m > 2$) tearing mode, exceeds that of the resistive singular layer, the non-linear $\vec{J} \times \vec{B}$ force becomes more important than plasma inertia to oppose the fluid motion. There then results a regime where the island size grows linearly - rather than exponentially - with time. This result has been extended to a model with anomalous electron viscosity, modelling the braiding of magnetic field lines in the vicinity of the separatrix [3].

This however does not explain the saturation of the instability, observed both in experiments [4] and in numerical simulations [5,6]. Saturation was found in a semi-analytical calculation by White et al. [5]. They observed that in numerical simulations the perturbed current behaves linearly whith the total magnetic flux, inside the magnetic island. Then the ansatz that $J = a + b\,\Psi$, where a and b are constants to be determined by matching with the outer solution, allows them to find the island's growth rate as :

$$\frac{\partial x_T}{\partial t} = A\,\Delta'(x_T)\left[1 - \alpha\, x_T\right]$$

where A is a constant, x_T is the island width, and α is due to the asymmetry of the island in cylindrical geometry. $\Delta'(x_T)$ is the slope-jump of the outer solution, found numerically at the separatrix. Saturation results when the island is wide enough to cancel the slope-jump, which measures the available magnetic energy.

However this work raises many questions, mainly on the validity of the original ansatz and of the matching method. Matching the inner and outer solutions at the separatrix leads the authors to neglect all resistive effects - including Rutherford's current- outside the island. This relates to the fact that the inner region should cover the whole area where $x \sim x_T$, and not only $x < x_T$.

We address this problem in a more systematic manner, using standard singular perturbation techniques [7]. Such techniques have already been used to regularize Rutherford's solution, which was singular at the separatrix [8].

Our smallness parameter is $\varepsilon = \Delta' x_T$, where Δ' is the linear slope-jump. $\varepsilon \sim x_T \left(\frac{1}{\Psi_1}\frac{\partial \Psi_1}{\partial x}\right)$ is thus the boundary-layer parameter. It also appears, as was noted by Drake et al. [9], as the ratio τ_R^i/τ_g, where $\tau_R^i = x_T^2/\eta$ is the skin time through the island and $\tau_g = \left(\frac{1}{x_T}\frac{\partial x_T}{\partial t}\right)^{-1}$ is the growth time of the island [10].

Section II of this paper outlines the main steps of the determination of the inner and outer solutions for the perturbed magnetic flux. Section III gives the final results and the conclusion. Detailed calculations are given in separate appendices.

- II -

We start from the equations of resistive MHD, written as :

$$\frac{\partial}{\partial t}\Psi - \vec{B}.\vec{\nabla}u = \eta(j_z - j_0) \quad (1a)$$
$$j_z = j_z(\Psi) \quad (1b)$$
$$\Delta\Psi = j_z \quad (1c)$$

where Ψ and u are the stream functions of the magnetic field and velocity :

$$\vec{B} = B_z\vec{e}_z + \vec{e}_z \times \vec{\nabla}\Psi$$

$$\vec{V} = \vec{e}_z \times \vec{\nabla}u$$

$j_0(x)$ is the equilibrium current, and eq. (1b) is obtained by neglecting plasma inertia, as was done by Rutherford.

At equilibrium we have $\Psi = \Psi_0(x)$, $B_y = B_0(x) = \frac{\partial\Psi}{\partial x}$, and in the vicinity of the resonant surface (where $B_0 = 0$) we approximate : $\Psi_0(x) \approx B'_y \frac{x^2}{2}$

Rutherford has also shown that when a high-m ($m \geqslant 2$) Tearing mode is growing, its harmonics can be neglected, provided they are sufficiently damped. Accordingly we assume a perturbation given by :

$$\Psi = \Psi_0(x) - \tilde{\Psi}(x,y,t)$$

$$\tilde{\Psi}(x,y,t) = \Psi_1(x,t)\cos ky + \delta\Psi_0(x,t)$$

$\delta\Psi_0$ is the quasi-linear perturbation of the equilibrium flux due to the growth of the unstable mode Ψ_1.

Averaging eq. (1a) over field lines, we get :

$$j_z(\Psi) = j_0(x) - \tilde{j}$$

$$\tilde{j} = \frac{\left\langle \frac{1}{\eta}\frac{\partial\tilde{\Psi}}{\partial t} \Big/ \frac{\partial\Psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial\Psi}{\partial x} \right\rangle} + \left[j_0(x) - \frac{\left\langle j_0 \Big/ \frac{\partial\Psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial\Psi}{\partial x} \right\rangle} \right] \tag{2}$$

where the brackets mean the average over y at constant Ψ. Following Rutherford we solve eq. (2) separately in the island region $(x \sim x_T = 2\,(\Psi_1/B_y')^{1/2}$ and outside of it $(x \gg x_T)$

In the inner region (not to be confused with the inside of the island, $\Psi < \Psi_1$) we use Rutherford's result giving $\frac{\partial x_T}{\partial t} \sim \eta \Delta'$ to order the different terms in equation (2). As pointed out by Drake et al. [9] $\epsilon = 0$ corresponds to the familiar "constant-Ψ" approximation. Assuming $\epsilon = \Delta' x_T$ small but finite allows us to relax this approximation and to treat explicitly the weak variation of $\tilde{\Psi}$ in the island region. We write it as :

$$\begin{aligned} \tilde{\Psi} = {} & \Psi_i(t)\left[1 + \Delta' s x + \varepsilon h_0(x) + \varepsilon^2 h_1(x)\right] \cos ky \\ & + \varepsilon\, \delta\Psi_0(t)\left[1 + g_0(x) + \varepsilon g_1(x)\right] \end{aligned} \tag{3}$$

h and g will appear as integrals of Rutherford's current, which is even in x. Then we require h and g to be even, and to be zero at the resonant surface. s is an integration constant, implying an asymmetry of the island due to that of the outer solution.

We write eq. (2) as :

$$\Delta\tilde{\Psi} = \frac{\left\langle \frac{1}{\eta}\frac{\partial\tilde{\Psi}}{\partial t} \Big/ \frac{\partial\Psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial\Psi}{\partial x} \right\rangle} + \left[j_0(x) - \frac{\left\langle j_0 \Big/ \frac{\partial\Psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial\Psi}{\partial x} \right\rangle} \right] \tag{4}$$

In the vicinity of the island the Laplacian reduces to $\partial^2/\partial x^2$ The term in square brackets (linearized current) writes to lowest order $j_0'\left[x - \langle 1/\tfrac{1}{x}\rangle^{-1}\right]$. With $j_0' \sim B_y'/a$ and $\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t} \sim \Delta' B_y' x_T$ the linearized current is thus of order $(a\Delta')^{-1}$ compared to the first term (Rutherford's current) in the RHS of Eq. (4). Then we must require $a\Delta'$ to be large to neglect this term, as was done by Rutherford. Writing :

$$\frac{\partial \Psi}{\partial x} = B_y' x^* \left[1 + \varepsilon M(x,y,t)\right] + \mathcal{O}(\varepsilon^2) \tag{5a}$$

$$x^{*2} = \frac{2}{B_y'}\left[\Psi + \Psi_i(t)\cos Y + \varepsilon\,\delta\Psi_o(t)\right] \tag{5b}$$

$$M(x,y,t) = \frac{1}{B_y' x^{*2}}\left[(h_o - x h_o')\Psi_i(t)\cos Y + (g_o - x g_o')\delta\Psi_o\right] \tag{5c}$$

$$\frac{1}{\eta}\frac{\partial}{\partial t} \sim \frac{\varepsilon}{x_T^2} \quad , \quad Y = ky$$

where primes denote derivatives with respect to x, and solving Eq. (4) order by order in ε, we get :

$$\varepsilon\frac{\partial^2}{\partial x^2}\left[\Psi_i h_o \cos Y + g_o\,\delta\Psi_o\right] = \frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\,\frac{\langle \cos Y/x^*\rangle}{\langle 1/x^*\rangle} \tag{6a}$$

$$\varepsilon^2\frac{\partial^2}{\partial x^2}\left[\Psi_i h_1 \cos Y + g_1\,\delta\Psi_o\right] = \varepsilon\,\frac{\left\langle \frac{1}{\eta}\frac{\partial}{\partial t}(\Psi_i h_o \cos Y + g_o\,\delta\Psi_o)/x^*\right\rangle}{\langle 1/x^*\rangle} \tag{6b}$$

$$+ \frac{\varepsilon}{\eta}\frac{\partial \Psi_i}{\partial t}\,\frac{1}{\langle 1/x^*\rangle^2}\left\{\left\langle\frac{M}{x^*}\right\rangle\left\langle\frac{\cos Y}{x^*}\right\rangle - \left\langle\frac{M\cos Y}{x^*}\right\rangle\left\langle\frac{1}{x^*}\right\rangle\right\}$$

Eq. (6a) is just that solved by Rutherford. M is the first-order modification of the metrics.

Eqs. (5) and (6) are solved for h_o, g_o and h_1. Detailed calculations and solutions are given in appendix A. Expanding the solutions for $\tilde{X} = \frac{x}{x_T} \gg 1$ (where they will be matched to the outer ones) we obtain :

$$\Psi_i(\varepsilon h_o + \varepsilon^2 h_1) = \varepsilon a_o + (\varepsilon a_1 + \varepsilon^2 a_2)x_T|\tilde{X}| \tag{7a}$$

$$\varepsilon\,\delta\Psi_o\, g_o = \varepsilon a_3 \, Ln|\tilde{X}| \tag{7b}$$

where :

$$\varepsilon a_o = -\frac{2\Psi_i}{\pi B_y'}\,\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\,c_o \quad , \quad \varepsilon a_1 = \frac{1}{\pi}\left(\frac{2\Psi_i}{B_y'}\right)^{1/2}\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\,c_1$$

$$\varepsilon^2 a_2 = \frac{1}{\pi B'_y}\left(\frac{2\Psi_i}{B'_y}\right)^{1/2}\left\{\left(\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}\right)^2 b_o + \frac{\Psi_i^{1/2}}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i^{1/2}\frac{\partial\Psi_i}{\partial t}\right) b_1 + \frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i\frac{\partial\Psi_i}{\partial t}\right) b_2\right\}$$

$$\varepsilon\, a_3 = \frac{\Psi_i}{2B'_y}\;\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}$$

and :

$$c_o = \int_{-1}^{+1} dw\;\frac{\alpha(w)}{\beta(w)}\sqrt{1-w^2} = .67 \qquad c_1 = \int_{-1}^{+\infty} dw\;\frac{\alpha^2(w)}{\beta(w)} = 1.82$$

$$b_o = \int_{-1}^{+\infty} dw\;\frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\,\frac{\alpha(w')}{\beta(w')}\left[\alpha(w)\frac{\partial}{\partial w}\beta(w,w') - 2\beta(w)\frac{\partial}{\partial w}\alpha(w,w')\right] = .41$$

$$b_1 = \int_{-1}^{+\infty} dw\;\frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\;\alpha(w') = -.10$$

$$b_2 = -\int_{-1}^{+\infty} dw\;\frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\;\frac{\alpha(w')}{\beta(w')}\;\beta(w,w') = -.89$$

$$\alpha(w,w') = \int_0^{Y_M(w')}\frac{dY\cos Y}{\sqrt{w+\cos Y}} \qquad \beta(w,w') = \int_0^{Y_M(w')}\frac{dY}{\sqrt{w+\cos Y}}$$

$$\alpha(w) = \alpha(w,w) \qquad \beta(w) = \beta(w,w)$$

$$Y_M(w') = \cos^{-1}(-w') \qquad (w' \leqslant 1)$$

$$Y_M(w') = \pi \qquad (w' \geqslant 1)$$

The functions α and β are expressed in terms of elliptic integrals, and the integrals over w and w' are performed numerically.

In the outer region we solve Eq. (4) by expansion in $\frac{\tilde{\Psi}}{\Psi} \ll 1$. Detailed calculations are given in appendix B. To second order we obtain :

$$\Delta\left[\Psi_1(x)\cos Y\right] - \frac{j'_o}{B_o}\Psi_1\cos Y = \left[\frac{j'_o}{B_o^2}\delta\Psi'_o - \frac{1}{B_o}\frac{1}{\eta}\frac{\partial}{\partial t}\delta\Psi'_o\right]\Psi_1\cos Y \quad \text{(8a)}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\eta}\frac{\partial}{\partial t}\right)\delta\Psi_o = -\frac{\partial}{\partial x}\left(\frac{\Psi_1^2 j'_o}{2B_o^2}\right) + \frac{1}{\eta}\frac{\partial}{\partial t}\left(\frac{\Psi_1^2}{4B_o}\right) \quad \text{(8b)}$$

The linear terms in eq. (8a) give the familiar ideal MHD equation for the outer solution Ψ_1, while far from the island $\left(\frac{\partial^2}{\partial x^2} \ll \frac{1}{\eta}\frac{\partial}{\partial t}\right)$ eq. (8b) gives the ideal result for $\delta\Psi_o$, used by White et al.[5] :

$$\delta\Psi_o = -\frac{\partial}{\partial x}\left(\frac{\Psi_1^2}{4B_o}\right) \tag{9}$$

but this is easily shown to contribute only to order ε^3 to the slope-jump.

We solve eq. (8b) for $\delta\Psi_o$ with the conditions that it matches for small x to eq. (7b) and for large x to eq. (9). For $x \sim x_\tau \varepsilon^{-1/2}$ (which is the resistive depth on a time τ_c) we find :

$$\frac{1}{\eta}\frac{\partial}{\partial t}\delta\Psi_o' = -\frac{1}{\eta^2}\left[\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\frac{\Psi_1^2}{4B_o}\right]\frac{x^3}{2\eta^2t^2}\frac{1}{2i\pi}\int_{p_o-i\infty}^{p_o+i\infty} dp\, e^{p\eta t} p^{-5/2} H(x\sqrt{p})$$

with :

$$H(z) = e^{-z}E_i(z) + e^{z}E_1(z) - \frac{2}{z}$$

Substituting this result into Eq. (8a) we get : $\Psi_1 = \Psi_{1L} + \delta\Psi_1$, where Ψ_{1L} is the linear solution and :

$$\frac{\partial}{\partial x}\delta\Psi_1 = -\frac{\Psi_{1L}}{4B_y'^2}\left[\frac{1}{\eta^2}\frac{\partial^2}{\partial t^2}\Psi_{1L}^2\right]\cdot\frac{1}{\eta^2t^2}\frac{1}{2i\pi}\int_{p_o-i\infty}^{p_o+i\infty} dp\, e^{p\eta t} p^{-5/2}\int_{x\sqrt{p}}^{\infty} dz\, \frac{H(z)}{z} \tag{10}$$

For $x \ll x_\tau \varepsilon^{-1/2}$ this gives :

$$\Psi_1(x,t) = \Psi_1(t)\left[1 + b_\pm X + \frac{j_o'}{B_y'\Delta'}\left(b_\pm\frac{X^2}{2} + X\,Ln|X|\right) + \varepsilon^{3/2}\alpha\,|X|\right] \tag{11}$$

where $X = \Delta' x$ and $b_+ - b_- = 1$, giving the linear outer slope-jump. The quadratic and logarithmic terms in the parenthesis are of order $\frac{j_o'}{B_y'\Delta'} \sim \frac{1}{a\Delta'}$, and can be neglected (they match to the "linearized current" term neglected in the inner solution), and :

$$\varepsilon^{3/2}\alpha = \frac{-3\gamma}{2^{5/4}\Delta' B_y'^2}\,\Psi_1^{-1/2}\left(\frac{1}{\eta}\frac{\partial\Psi_1}{\partial t}\right)^{5/2}$$

with :

$$\gamma = \frac{1}{\Gamma(\frac{5}{2})}\int_0^\infty \frac{dz}{z}\left[H(z) + \frac{2}{z}\right] = 3.63$$

The ordering in $\varepsilon^{3/2}$ comes from the resistive diffusion on a scale length $x_T \varepsilon^{-1/2}$.

Matching the inner solution (eq. 7a) to the outer one (eq. 11), as shown in Appendix C, gives our final result :

$$\Psi_i = \Psi_1 - \varepsilon a_0 \quad , \quad s = \frac{b_+ + b_-}{2} \frac{\Psi_1}{\Psi_i} \quad , \quad \varepsilon = \Delta' x_T = 2\left(\frac{\Psi_i}{B_y'}\right)^{1/2} \Delta'$$

$$\frac{1}{\eta}\frac{\partial \Psi_1}{\partial t} = \frac{.61}{4\pi} \Delta' (B_y' \Psi_1)^2 \left[1 - .16\,\varepsilon - .40\,\varepsilon^{3/2}\right] \qquad (12)$$

where we have re-introduced the factor 4π neglected for simplicity in eq. (4c).

The lowest order we recover Rutherford's result, although the numerical value of c_1 has been corrected and he did not give s explicitly. To the following orders, although the use of a boundary layer technique restricts us to small values of ε, our result shows that saturation can occur for $\varepsilon \sim 1$ (which is indeed the order of magnitude observed in numerical simulations), due to the quasi-linear evolution of the flux (or current) profile. We notice that at saturation the inner terms proportionnal to $\frac{\partial \Psi_1}{\partial t}$ would be cancelled, but that $\delta \Psi_0$ would keep diffusing, from the level reached during the growth of the mode, on a scale-length $x_T (t/\tau_c)^{1/2}$. This slow evolution of Ψ_0 (and of the saturated amplitude Ψ_1) has been observed in numerical calculations [5].

Work is in progress to study a similar effect, due to the diffusion of the heat generated by Rutherford's current.

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Appendix A

Inner equations and solutions

We start from Eq. (4) :

$$\Delta \tilde{\Psi} = \frac{\left\langle \frac{1}{\eta} \frac{\partial \tilde{\Psi}}{\partial t} \Big/ \frac{\partial \Psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial \Psi}{\partial x} \right\rangle} \tag{A-1}$$

where

$$\Psi = \Psi_0 - \tilde{\Psi}$$

$$\tilde{\Psi} = \Psi_i(t) \left[1 + \Delta' s x + \varepsilon h_0(x) + \varepsilon^2 h_1(x) \right] \cos Y$$

$$+ \varepsilon \delta \Psi_0(t) \left[1 + g_0(x) + \varepsilon g_1(x) \right]$$

and we have neglected the linearized current term.

To first order the metrics writes :

$$\frac{\partial \Psi}{\partial x} = B'_y x^* \left[1 + \varepsilon M(x, y, t) \right]$$

where

$$x^{*2} = \frac{2}{B'_y} \left[\Psi + \varepsilon \delta \Psi_0(t) + \Psi_i(t) \cos Y \right]$$

and

$$M(x, y, t) = \frac{1}{B'_y x^{*2}} \left[(h_0 - x h'_0) \cos Y \; \Psi_i + (g_0 - x g'_0) \delta \Psi_0 \right]$$

We solve Eq. (A-1) order by order in $\varepsilon = \Delta' x_T$, using Rutherford's result $\left(\frac{1}{\eta} \frac{\partial}{\partial t} \sim \frac{\Delta'}{x_T} \right)$ to get :

$$\varepsilon \frac{\partial^2}{\partial x^2} \left(\Psi_i h_0 \cos Y + g_0 \delta \Psi_0 \right) = \frac{1}{\eta} \frac{\partial \Psi_i}{\partial t} \; \frac{\left\langle \frac{\cos Y}{x^*} \right\rangle}{\left\langle \frac{1}{x^*} \right\rangle} \tag{A-3}$$

$$\varepsilon \frac{\partial^2}{\partial x^2} \left(\Psi_i h_1 \cos Y + g_1 \delta \Psi_0 \right) = \frac{\left\langle \frac{\varepsilon}{\eta} \frac{\partial}{\partial t} \left(\Psi_i h_0 \cos Y + g_0 \delta \Psi_0 \right) / x^* \right\rangle}{\left\langle 1 / x^* \right\rangle} +$$

$$+\frac{\varepsilon}{\eta}\frac{\partial\Psi_i}{\partial t}\frac{1}{\langle 1/x^*\rangle^2}\left\{\left\langle\frac{M}{x^*}\right\rangle\left\langle\frac{\cos Y}{x^*}\right\rangle-\left\langle\frac{M\cos Y}{x^*}\right\rangle\left\langle\frac{1}{x^*}\right\rangle\right\} \tag{A-4}$$

From Eqs. (A-2) and (A-3) we get :

$$\frac{\partial}{\partial x}\left(\varepsilon M x^{*2}\right)=-\frac{x^*}{B_y'}\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}\frac{\langle\cos Y/x^*\rangle}{\langle 1/x^*\rangle}$$

M, like h and g, is even and is zero at x = 0. This gives :

$$\varepsilon M(x,y,t)=\frac{-1}{B_y' x^{*2}}\Psi_i\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}\int_{-\cos Y}^{w}dw'\,\frac{\alpha(w')}{\beta(w')} \tag{A-5}$$

where $w=\frac{\Psi+\delta\Psi_0}{\Psi_i}$ and :

$$\alpha(w,w')=\int_0^{Y_M(w')}\frac{dY\cos Y}{\sqrt{w+\cos Y}}\quad,\quad\beta(w,w')=\int_0^{Y_M(w')}\frac{dY}{\sqrt{w+\cos Y}}$$

$$\alpha(w)=\alpha(w,w)\quad,\quad\beta(w)=\beta(w,w)$$

$$Y_M(w')=\cos^{-1}(-w')\qquad(w'\leq 1)$$

$$Y_M(w')=\pi\qquad(w'\geq 1)$$

After some algebra we find :

$$\left\langle\frac{\varepsilon M}{x^*}\right\rangle=\frac{1}{(2\Psi_i B_y')^{1/2}}\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}\int_{-1}^{w}dw'\,\frac{\alpha(w')}{\beta(w')}\frac{\partial}{\partial w}\beta(w,w') \tag{A-6}$$

$$\left\langle\frac{\varepsilon M\cos Y}{x^*}\right\rangle=\frac{1}{(2\Psi_i B_y')^{1/2}}\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}\int_{-1}^{w}dw'\,\frac{\alpha(w')}{\beta(w')}\frac{\partial}{\partial w}\alpha(w,w') \tag{A-7}$$

$$\begin{aligned}\left\langle\frac{1}{\eta}\frac{\partial}{\partial t}(\varepsilon\tilde{\Phi})/x^*\right\rangle=&\frac{1}{(2B_y')^{1/2}}\frac{1}{\eta^2}\frac{\partial}{\partial t}\Psi_i^{1/2}\frac{\partial\Psi_i}{\partial t}\int_{-1}^{w}dw'\,\alpha(w')\\&-\frac{1}{(2\Psi_i B_y')^{1/2}}\frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i\frac{\partial\Psi_i}{\partial t}\right)\int_{-1}^{w}dw'\,\frac{\alpha(w')}{\beta(w')}\beta(w,w')\end{aligned} \tag{A-8}$$

where

$$\tilde{\Phi}=\Psi_i h_0\cos Y+g_0\delta\Psi_0$$

Using these results, we integrate and Fourier transform Eq. (A-4) to obtain, for $w \gg 1$ (where the inner and outer solutions will be matched) :

$$\left[\varepsilon^2 \frac{\partial}{\partial x}\left(\Psi_i h_2\right)\right]_{w\to\infty} = \frac{\sqrt{2}}{\pi}\ \frac{\Psi_i}{B_y'^{3/2}}\ \frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i^{1/2}\frac{\partial \Psi_i}{\partial t}\right)\int_{-1}^{\infty} dw\, \frac{\alpha(w)}{\beta(w)} \int_{-1}^{w} dw'\, \alpha(w')$$

$$- \frac{\sqrt{2}}{\pi}\ \frac{\Psi_i^{1/2}}{B_y'^{3/2}}\ \frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i \frac{\partial \Psi_i}{\partial t}\right)\int_{-1}^{\infty} dw\, \frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\, \frac{\alpha(w')}{\beta(w')}\, \beta(w,w')$$

$$+ \frac{\sqrt{2}}{\pi}\ \frac{\Psi_i^{1/2}}{B_y'^{3/2}}\left(\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\right)^2 \int_{-1}^{\infty} dw\ \frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\, \frac{\alpha(w')}{\beta(w')}\ \cdot$$

$$\cdot \left\{ \alpha(w)\frac{\partial}{\partial w}\beta(w,w') - \beta(w)\frac{\partial}{\partial w}\alpha(w,w')\right\} \qquad \text{(A-9)}$$

From Eq. (A-3) we also get :

$$\left(\varepsilon\, \delta\Psi_o \frac{\partial g_o}{\partial x}\right)_{w\to\infty} = \left(\frac{\Psi_i}{2B_y'}\right)^{1/2} \frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\int_{-1}^{w} dw'\ \alpha(w')$$

$$= - \left(\frac{\Psi_i}{2B_y'}\right)^{1/2}\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\int_{w}^{\infty} dw'\, \alpha(w') \qquad \text{(A-10)}$$

$$\approx \frac{\Psi_i}{2B_y'}\ \frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\ \frac{1}{x}$$

where we have used the facts that :

$$\int_0^{\infty} dx \int_0^{T} dY\ \tilde{j}_z(x,Y) \sim \int_{-1}^{\infty} dw\ \alpha(w) = 0$$

and that, for $w \gg 1$, $w \approx \dfrac{B_y' x^2}{\Psi_i}$

For h_o we obtain :

$$\left(\varepsilon\, \Psi_i \frac{\partial h_o}{\partial x}\right)_{w\to\infty} = \frac{1}{\pi}\left(\frac{2\Psi_i}{B_y'}\right)^{1/2}\frac{\partial \Psi_i}{\partial t}\int_{-1}^{\infty} dw \int_0^{Y_M(w)} dY\, \frac{\cos Y}{\sqrt{w+\cos Y}}\ \frac{\alpha(w)}{\beta(w)}\,(1-\varepsilon M)$$

$$= \frac{1}{\pi}\left(\frac{2\Psi_i}{B_y'}\right)^{1/2}\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\int_{-1}^{+\infty} dw\ \frac{\alpha^2(w)}{\beta(w)}$$

$$- \frac{1}{\pi B_y'}\left(\frac{2\Psi_i}{B_y'}\right)^{1/2}\left(\frac{1}{\eta}\frac{\partial \Psi_i}{\partial t}\right)^2 \int_{-1}^{+\infty} dw\ \frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\ \frac{\alpha(w')}{\beta(w')}\frac{\partial}{\partial w}\alpha(w,w')$$

Finally we get $h_o - x\,h'_o$ by Eqs. (A-2) and (A-5), giving :

$$\left[\varepsilon\,\Psi_i\,(h_o - x\,h'_o)\right]_{w\to\infty} = -\frac{2}{\pi}\,\frac{\Psi_i}{B'_y}\,\frac{1}{\eta}\,\frac{\partial\Psi_i}{\partial t}\int_{-1}^{+1} dw\, w\sqrt{1-w^2}\;\frac{\alpha(w)}{\beta(w)}$$

Collecting the results, we have for $w \gg 1$:

$$\Psi_i(\varepsilon h_o + \varepsilon^2 h_1) = \varepsilon a_o + (\varepsilon a_1 + \varepsilon^2 a_2)\, x_T |\tilde{X}|$$

$$\varepsilon\, g_o\, \delta\Psi_o = \varepsilon\, a_3\, \mathrm{Ln}\,|\tilde{X}|$$

with $\quad \tilde{X} = \dfrac{x}{x_T}$

$$\varepsilon a_o = -\frac{2\Psi_i}{\pi B'_y}\,\frac{1}{\eta}\,\frac{\partial\Psi_i}{\partial t}\,c_o$$

$$\varepsilon\, a_1 = \frac{1}{\pi}\left(\frac{2\Psi_i}{B'_y}\right)^{1/2}\frac{1}{\eta}\,\frac{\partial\Psi_i}{\partial t}\,c_1$$

$$\varepsilon^2 a_2 = \frac{1}{\pi B'_y}\left(\frac{2\Psi_i}{B'_y}\right)^{1/2}\left\{\left(\frac{1}{\eta}\frac{\partial\Psi_i}{\partial t}\right)^2 b_o + \frac{\Psi_i^{1/2}}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i^{1/2}\frac{\partial\Psi_i}{\partial t}\right) b_1 + \frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\Psi_i\frac{\partial\Psi_i}{\partial t}\right) b_2\right\}$$

$$\varepsilon\, a_3 = \frac{\Psi_i}{2B'_y}\,\frac{1}{\eta}\,\frac{\partial\Psi_i}{\partial t}$$

and :

$$c_o = \int_{-1}^{+1} dw\,\frac{\alpha(w)}{\beta(w)}\sqrt{1-w^2} = .67$$

$$c_1 = \int_{-1}^{+\infty} dw\,\frac{\alpha^2(w)}{\beta(w)} = 1.82$$

$$b_o = \int_{-1}^{\infty} dw\,\frac{\alpha(w)}{\beta^2(w)}\int_{-1}^{w} dw'\,\frac{\alpha(w')}{\beta(w')}\left\{\alpha(w)\frac{\partial}{\partial w}\beta(w,w') - 2\beta(w)\frac{\partial}{\partial w}\alpha(w,w')\right\}$$

$$= .41$$

$$b_1 = \int_{-1}^{+\infty} dw\,\frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\,\alpha(w') = -.10$$

$$b_2 = -\int_{-1}^{\infty} dw\,\frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} dw'\,\frac{\alpha(w')}{\beta(w')}\,\beta(w,w') = -.89$$

Appendix B

Outer equations and solutions

In the outer region, we solve eq. (4) :

$$\Delta\tilde{\Psi} = \frac{\left\langle \frac{1}{?}\frac{\partial\tilde{\Psi}}{\partial t} / \frac{\partial\Psi}{\partial x} \right\rangle}{\left\langle 1 / \frac{\partial\Psi}{\partial x} \right\rangle} + \left[j_0(x) - \frac{\left\langle j_0 / \frac{\partial\Psi}{\partial x} \right\rangle}{\left\langle 1 / \frac{\partial\Psi}{\partial x} \right\rangle} \right. \tag{B-1}$$

by expansion to second order in $\frac{\tilde{\Psi}}{\Psi}$, with

$$\tilde{\Psi} = \Psi_1(x,t)\cos Y + \delta\Psi_0(x,t) \quad , \quad \Psi = \Psi_0 - \tilde{\Psi}$$

To do this, we must express the average at constant Ψ of x - dependant quantities. Writing :

$$A(x) = \mathcal{A}(\Psi)\big|_{\Psi = \Psi_0(x)}$$

$$= \mathcal{A}(\Psi) + \tilde{\Psi}\frac{\partial\mathcal{A}}{\partial\Psi} + \frac{\tilde{\Psi}^2}{2}\frac{\partial^2\mathcal{A}}{\partial\Psi^2}$$

we obtain after some algebra :

$$\langle A(x)\rangle = A(x) + \left(\langle\tilde{\Psi}\rangle - \tilde{\Psi}\right)\frac{1}{B_0}\frac{\partial A}{\partial x}$$

$$+ \frac{1}{2}\left(\tilde{\Psi}^2 + \langle\tilde{\Psi}\rangle^2 - 2\tilde{\Psi}\langle\tilde{\Psi}\rangle\right)\frac{1}{B_0}\frac{\partial}{\partial x}\frac{1}{B_0}\frac{\partial A}{\partial x} \tag{B-2}$$

$$\langle A(x)\cos Y\rangle = \frac{\Psi_1}{2B_0}\frac{\partial A}{\partial x} \qquad \langle\tilde{\Psi}^2\rangle = \delta\Psi_0^2 + \frac{\Psi_1^2}{2}$$

$$\langle\tilde{\Psi}\rangle = \delta\Psi_0(x) + \frac{1}{2B_0}\Psi_1\Psi_1' - \frac{1}{B_0}\delta\Psi_0'\,\Psi_1\cos Y$$

and finally :

$$\langle A(x)\rangle = A(x) - \frac{1}{B_0}\frac{\partial A}{\partial x}\Psi_1\cos Y + \frac{1}{B_0^2}\frac{\partial A}{\partial x}\left[\frac{\Psi_1\Psi_1'}{2} - \delta\Psi_0'\Psi_1\cos Y\right]$$

$$+ \frac{\Psi_1^2}{2B_0}\frac{\partial}{\partial x}\left[\frac{1}{B_0}\frac{\partial A}{\partial x}\right] \tag{B-3}$$

This procedure avoids us to assume, as in ref. [5] , the conservation of the current between flux lines.

Then Eq. (B-1) gives :

$$\left[\Delta - \frac{j_o'}{B_o}\right]\Psi_1 \cos Y = \left[\frac{j_o''}{B_o^2}\delta\Psi_o' - \frac{1}{B_o}\,\frac{1}{\eta}\frac{\partial}{\partial t}\delta\Psi_o'\right]\Psi_1 \cos Y \qquad \text{(B-4)}$$

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{\eta}\frac{\partial}{\partial t}\right]\delta\Psi_o = -\frac{\partial}{\partial x}\left[\frac{\Psi_1^2 j_o''}{2B_o^2}\right] + \frac{1}{\eta}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\Psi_1^2}{4B_o} \qquad \text{(B-5)}$$

The linear terms of Eq. (B-4) give the usual linear equation for the outer solution, while far from the island $\left(\frac{\partial^2}{\partial x^2} \ll \frac{1}{\eta}\frac{\partial}{\partial t}\right)$ Eq. (B-5) gives the ideal-MHD result :

$$\delta\Psi_o = -\frac{\partial}{\partial x}\left(\frac{\Psi_1^2}{4B_o}\right) \qquad \text{(B-6)}$$

This solution is easily shown to contribute only, for $x \gtrsim \Delta'^{-1}$, to order ε^3 to the slope-jump of Ψ_1.

Then we solve Eqs. (B-4) and (B-5) for $x \ll \Delta'^{-1}$, where we can estimate the non-linear terms :

$$\left[\frac{\partial}{\partial x}\left(\frac{\Psi_1^2 j_o''}{2B_o^2}\right)\Big/\frac{1}{\eta}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\left(\frac{\Psi_1^2}{4B_o}\right)\right] \sim \left[\frac{j_o''\delta\Psi_o'\Psi_1}{B_o^2}\Big/\frac{\Psi_1}{B_o}\frac{1}{\eta}\frac{\partial}{\partial t}\delta\Psi_o'\right] \sim \frac{x_T}{x a \Delta'} \ll 1$$

The solution of Eq. (B-5) must match for large x $(x \gg x_T \varepsilon^{-1/2})$ to Eq. (B-6) and for small x $(x \sim x_T)$ to the inner solution, Eq. (7b). By Laplace transform in time we find :

$$\frac{1}{\eta}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\delta\Psi_o = \frac{C}{2i\pi}\int_{p_o - i\infty}^{p_o + i\infty} dp \; e^{p\eta t}\, p^{-5/2} H(x\sqrt{p}) \qquad \text{(B-7)}$$

where

$$H(z) = e^{z}E_1(z) + e^{-z}E_i(z) - \frac{2}{z}$$

$$C = -\frac{x^3}{2\eta^2 t^2}\,\frac{1}{\eta^2}\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\left(\frac{\Psi_1^2}{4B_o}\right)$$

and to lowest order C is a constant. Integrating Eq. (B-7) would give the value of $\varepsilon\,\delta\Psi_o(t)$ in the inner solution, but this result is not needed here.

Substituting Eq. (B-7) into Eq. (B-4) we get :

$$\Psi_1(x,t) = \Psi_{1L} + \delta\Psi_1$$

where Ψ_{1L} is the linear outer solution, and :

$$\frac{\partial}{\partial x}\,\delta\Psi_1 = \left(-\frac{\Psi_1}{4B_y'^2}\,\frac{1}{\gamma^2}\frac{\partial^2}{\partial t^2}\Psi_1^2\right)\cdot\frac{1}{\gamma^2 t^2}\cdot\frac{1}{2i\pi}\int_{p_0-i\infty}^{p_0+i\infty} dp\, e^{p\gamma t}\, p^{-5/2}\int_{x\sqrt{p}}^{\infty} dz\,\frac{H(z)}{z} \qquad \text{(B-8)}$$

For $X = \Delta' x \sim \varepsilon$ we obtain :

$$\Psi_{1L}(x,t) = \Psi_1(t)\left[1 + b_\pm X + \frac{j_0'}{B_y'\Delta'}\left(b_\pm \frac{X^2}{2} + X\,Ln|X|\right)\right] \qquad \text{(B-9)}$$

where $b_+ - b_- = 1$ (giving the linear slope-jump) and the terms in the parenthesis can be neglected, as they are of order $(a\Delta')^{-1}$ and match to the linearized current term neglected in the inner solution. For $\delta\Psi_1$ we get to lowest order :

$$\frac{\partial}{\partial x}\,\delta\Psi_1 = \frac{\Psi_1}{4B_y'^2}\,\frac{1}{\gamma^2}\frac{\partial^2}{\partial t^2}\Psi_1^2\left[\frac{1}{x} - \frac{\gamma}{\sqrt{\gamma t}}\right] \qquad \text{(B-10)}$$

where

$$\gamma = \frac{1}{\Gamma(5/2)}\int_0^{\infty}\frac{dz}{z}\left[H(z) + \frac{2}{z}\right] = 3.63$$

The term in $\frac{1}{x}$ matches to a similar term in the inner solution, contributing only to order $\varepsilon^2 Ln\,\varepsilon$ to the growth rate, and we neglect it - Using $\gamma t = 2\Psi_1\left(\frac{1}{\gamma}\frac{\partial\Psi_1}{\partial t}\right)^{-1}$ we obtain from Eqs. (B-9) and (B-10) :

$$\Psi_1(x,t) = \Psi_1(t)\left[1 + b_\pm X + \varepsilon^{3/2}\alpha\,|X|\right] \qquad \text{(B-11)}$$

with

$$\varepsilon^{3/2}\alpha = \frac{-3\gamma}{2^{5/2}\Delta' B_y'^2}\,\Psi_1^{-1/2}\left(\frac{1}{\gamma}\frac{\partial\Psi_1}{\partial t}\right)^{5/2}$$

Appendix C

Matching and final result

We match the inner solution taken at $x \gg x_T$ (Eq. (7a)) to the outer solution taken at $x \ll x_T \varepsilon^{-1/2}$ (Eq. (11)). Details of the matching technique can be found in ref. [7]. Then we have :

$$\Psi_i(x,t) = \Psi_i(t)\left[1 + \varepsilon s \tilde{X}\right] + \varepsilon a_0 + \left(\varepsilon a_1 + \varepsilon^2 a_2\right) x_T |\tilde{X}| \tag{C-1}$$

$$\Psi_1(x,t) = \Psi_1(t)\left[1 + b_\pm X + \varepsilon^{3/2} d\, |X|\right] \tag{C-2}$$

$$\tilde{X} = \frac{x}{x_T} \quad , \quad X = \varepsilon \frac{x}{x_T} \quad , \quad b_+ - b_- = 1 \, .$$

Matching these solutions gives :

$$\Psi_i = \Psi_1 - \varepsilon a_0 \tag{C-3}$$

$$b_\pm \Psi_1 \pm \varepsilon^{3/2} d\, \Psi_1 = s\, \Psi_i \pm \left(\varepsilon a_1 + \varepsilon^2 a_2\right) x_T \tag{C-4}$$

hence :

$$s = \frac{b_+ + b_-}{2} \, \frac{\Psi_1}{\Psi_i} \tag{C-5}$$

and :

$$\varepsilon a_1 = \frac{\Delta'}{2} \Psi_1 + \Delta' \varepsilon^{3/2} d\, \Psi_1 - \varepsilon^2 a_2 \tag{C-6}$$

Solving this equation by iteration we get to lowest order :

$$\frac{1}{\eta} \frac{\partial \Psi_i}{\partial t} = \frac{1}{\eta} \frac{\partial \Psi_1}{\partial t} = \frac{\pi \Delta'}{2^{3/2} C_1} \left(B_y' \Psi_1\right)^{1/2} \tag{C-7}$$

which is just Rutherford's result, and to order $\varepsilon^{3/2}$

$$\frac{1}{\eta} \frac{\partial \Psi_i}{\partial t} = \frac{\pi}{2^{3/4} C_1} \Delta' \left(B_y' \Psi_1\right)^{1/2} \Bigg\{ 1 - \frac{c_0}{\sqrt{2}\, C_1} \Delta' \left(\frac{\Psi_1}{B_y'}\right)^{1/2}$$

$$- \frac{\pi}{2^{3/2} C_1^2} \Delta' \left(\frac{\Psi_1}{B_y'}\right)^{1/2} \left[b_0 + b_1 + \frac{3}{2} b_2 \right] -$$

$$- \frac{3\pi^{5/2}}{2^{11/4} C_1^{5/2}} \gamma \Delta'^{3/2} \left(\frac{\psi_1}{B'_y} \right)^{3/4} \Big\}$$

Using Eq. (C-3) we finally get :

$$\frac{1}{\eta} \frac{\partial \psi_1}{\partial t} = .61 \, \Delta' (B'_y \psi_1)^{1/2} \left[1 - .16 \, \varepsilon - .40 \, \varepsilon^{3/2} \right] \qquad \text{(C-8)}$$

with

$$\varepsilon = \Delta' x_T = 2 \, \Delta' \left(\frac{\psi_1}{B'_y} \right)^{1/2}$$