SCHRÖDINGER FLUID: AN OVERVIEW

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ABSTRACT

The relationship of nuclear internal flow and collective inertia, the difference of this flow from that of a classical fluid, and the approach of this flow to rigid flow in independent-particle model rotation are ellucidated by reviewing the theory of Schrödinger fluid and its implications for collective vibration and rotation.

1. INTRODUCTION

The study of Schrödinger fluid is an effort to build up our knowledge in nuclear matter flow from a basic and elementary level. Since one-body dynamics is an important aspect of nuclear dynamics by virtue of the Pauli exclusion principle, our effort began with the investigation of the quantal matter flow in the one-body limit [1-5]. We studied the quantal velocity field associated with the single-particle wave function when the nucleus is in a collective motion. We sought the relationship between such quantal single-particle flow and the collective inertial parameter characterizing it. And, by studying collective rotation, we generalized some of the results to interacting particles and to nonperturbative treatments [6,7].

Recently, the study of the nuclear matter flow along these lines has generated a series of new works [8-12]. We summarize the main points of this approach and discuss some of its underlying ideas in this paper.

2. SINGLE-PARTICLE SCHRÖDINGER FLUID

The single-particle Schrödinger fluid model of nuclei assumes that each nucleon moves in a time-varying one-body potential $V[\vec{r},\alpha(t)]$. The c-number shape parameter $\alpha(t)$ is assumed to be an externally prescribed function of time. Thus, the Hamiltonian for each single-particle is given by

$$H[\vec{r},\vec{p};\ \alpha(t)] = \frac{p^2}{2m} + V[\vec{r};\ \alpha(t)] \quad . \tag{1}$$

In order to cast the Schrödinger equation into the fluid dynamical form, we write the single-particle wave function in the polar form

$$\psi_k[\vec{r},\ \alpha(t),\ t] = \phi_k[\vec{r},\ \alpha(t)]\ \exp\left\{-\ i\ \frac{m}{\hbar}\ S_k[\vec{r},\ \alpha(t)] - \frac{i}{\hbar}\int^t \epsilon_k(\alpha)\ dt'\right\} \quad , \tag{2}$$

where k is the particle index, ϕ and S are real functions and ϕ is assumed to be positive. From Eqs. (1) and (2) we separate the real and imaginary parts of the single-particle Schrödinger equations. There emerges (a) the continuity equation,

$$\nabla\cdot\left(\rho_k\vec{v}_k\right) = -\frac{\partial\rho_k}{\partial t} \quad , \tag{3}$$

involving a density equal to the single-particle probability density and an irrotational velocity field,†

$$\rho_k = \phi_k^2 \quad , \qquad \vec{v}_k = \vec{v}_{\bullet k} \equiv -\nabla S_k \quad ; \tag{4}$$

and (b) a modified Schrödinger equation for ϕ,

$$\left[H - m\left(\partial S_k/\partial t - \frac{1}{2}\nabla S_k\cdot\nabla S_k\right)\right]\phi_k = \epsilon_k\phi_k \quad . \tag{5}$$

On the basis of Eqs. (3)-(5), we follow Madelung [13] and conceptualize the quantum state of the single-particle as a dynamical fluid.

The fluid dynamical equations (3)-(5) imply no condition of incompressible flow. In fact $\vec{v}_{\bullet k}$ is generally compressible ($\nabla\cdot\vec{v}_{\bullet k}\neq 0$). The imposition of any additional incompressibility condition upon $\vec{v}_{\bullet k}$ will represent constraints over and above those implied by the Hamiltonian (1) and must be expected in general to seriously limit the applicability of the Schrödinger fluid. (The specific consequences of the incompressibility condition are discussed in Ref. [4].) Hence, an approach of adequate generality must deal with compressible flow.

On the other hand, as implied by Eq. (4), the velocity $\vec{v}_{\bullet k}$ is irrotational wherever it is differentiable. At the positions where the single-particle density ρ_k vanishes, $\vec{v}_{\bullet k}$ is not differentiable and is singular. Such singularities comprise line vortices and the circulation around any such line vortex is quantized [4]. Therefore, the velocity field $\vec{v}_{\bullet k}$ is actually irrotational in a multiply-connected space which excludes an infinitesimal volume around each line vortex. Because of these singularities, extreme care has to be exercised in handling these velocity fields.

So far the fluid dynamical description is general and is immediately generalizable for the many-body wave function [4].

†We use the symbol ● to denote an irrotational field for which $\nabla\times\vec{v} = 0$.

3. THE ADIABATIC APPROXIMATION AND THE ADIABATIC COLLECTIVE KINETIC ENERGY

In the adiabatic limit ($\dot{\alpha} \to 0$), the single-particle is approximated by

$$\psi_k \simeq u_k + i\mu_k \qquad (6)$$

where $u_k(\vec{r},\alpha)$ is the quasi-static wave function satisfying the unperturbed Schrödinger equation $Hu_k = \epsilon_k u_k$, and μ_k is the first order time-dependent perturbative correction. Both u_k and μ_k are here chosen to be real, without loss of generality. In this approximation, the fluid dynamical quantities are

$$\rho_k = u_k^2 + \mu_k^2 \quad , \qquad (7)$$

$$S_k = \frac{\hbar}{m} \arctan(\mu_k/u_k) \quad , \qquad (8)$$

$$\vec{v}_{\bullet k} = -\frac{\hbar}{m}\frac{1}{\rho_k}\left(\mu_k \nabla u_k - u_k \nabla \mu_k\right) \quad . \qquad (9)$$

We have shown in Ref. [4] that these quantities satisfy Eq. (5) to order $\dot{\alpha}$ and Eq. (3) to order $\dot{\alpha}^2$:

$$\rho_k \nabla^2 S_k + \nabla\rho_k \cdot \nabla S_k = 2\dot{\alpha} u_k\, \partial u_k/\partial\alpha \quad . \qquad (10)$$

The adiabatic approximation enables one to obtain a relationship between the collective kinetic energy and the velocity field of the single-particle flow as we show in the following. The adiabatic collective kinetic energy is given by

$$T_k = \hbar\dot{\alpha} \int \mu_k (\partial u_k/\partial\alpha)\, d\tau \quad , \qquad (11)$$

which is fully equivalent to the familiar cranking model formula of Inglis [14]. We make use of the continuity equation (10) and the vector identity,

$$\phi\nabla \cdot (\mu\phi\vec{v}) = \mu\rho\nabla \cdot \vec{v} + \rho\vec{v} \cdot \nabla\mu + \frac{1}{2}\mu\vec{v} \cdot \nabla\rho \quad , \qquad (12)$$

to transform Eq. (11) to

$$T_k = -\frac{\hbar}{2}\int d\tau \left\{ \vec{v}_{\bullet k} \cdot \left[\frac{1}{2}\mu_k\nabla\rho_k - \rho_k\nabla u_k\right] + \phi_k\nabla \cdot (\mu_k\phi_k\vec{v}_{\bullet k})/u_k \right\}. \qquad (13)$$

The quantity inside the square brackets in this equation is proportional to the single-particle current, since

$$\left(\frac{1}{2}\mu\nabla\rho - \rho\nabla\mu\right)/u = \mu\nabla u - u\nabla\mu = -m\rho\vec{v}_{\bullet}/\hbar \quad . \qquad (14)$$

With the help of this relationship we can then rewrite the collective kinetic energy in terms of the velocity field $\vec{v}_{\bullet k}$:

$$T_k = \frac{m}{2}\int \rho_k v_{\bullet k}^2\, d\tau - \frac{\hbar}{2}\int (\phi_k/u_k)\nabla \cdot (\mu_k\phi_k\vec{v}_{\bullet k})\, d\tau \quad . \qquad (15)$$

The first term of this equation appears in the standard form for the kinetic energy of a classical fluid. The second term is due to the singularities of $\vec{v}_{\bullet k}$. Therefore, the quantal flow of a single-particle in the adiabatic limit has a kinetic energy different from that of a classical fluid.

Moreover, in the study of collective rotations where we are able to extend the results to more general situations (discussed below) we find also that the quantal flow exhibits a kinetic energy which differs in general from the classical value.

4. AN ALTERNATIVE FORM OF THE COLLECTIVE KINETIC ENERGY

Observe that in transforming the collective kinetic energy from the cranking form (11) to the fluidic form (15), the use of the continuity equation is the key step. This observation leads one to consider velocity fields which satisfy the continuity equation (3) other than the irrotational field (4) obtained from the Schrödinger equation. One such solution is the regular solution $\vec{v}_R$ defined as a solution with no singularity. If it exists, such a solution can be introduced in such a way as to transform the kinetic energy (11) or (15) into a simple form. If one follows the parallel manipulations in obtaining (15) from (11), the following formula emerges:

$$T_k = \frac{m}{2}\int \rho_k \vec{v}_{\bullet k} \cdot \vec{v}_{Rk}\, d\tau \quad . \qquad (16)$$

Other alternative forms are also explored in Refs. [2] and [4].

The general sufficient condition for the regular velocity field to exist is not known. We can however guarantee [4] that such a velocity field can be found for the case where the wave function is stationary in a time-dependent curvilinear coordinate system $\vec{\xi} = \vec{\xi}[\vec{r},\alpha(t)]$. Particular examples of this include the rotation of a general system, and the often-studied anisotropic scaling deformations of simple harmonic oscillator (SHO) wave function, such as the breathing and quadrupole deformation modes.

In Figure 1, R is plotted as a function of deformation, near

For quadrupole deformation of SHO this regular field is proportional to the quadrupole field and is identical to all the particle irrotational fields $\vec{v}_{\bullet k}$:

$$\vec{v}_R = -\frac{\dot{\alpha}}{\alpha}\left(x\hat{i} + y\hat{j} - 2z\hat{k}\right) = \vec{v}_{\bullet k} \quad , \tag{17}$$

and for breathing mode, we have

$$\vec{v}_R = \frac{\dot{\alpha}}{\alpha}\vec{r} = \vec{v}_{\bullet k} \quad . \tag{18}$$

For rotation, the regular velocity field is equal to that of rigid rotation:

$$\vec{v}_R = \vec{\Omega} \times \vec{r} \quad , \tag{19}$$

where $\vec{\Omega}$ is the angular velocity of the rotation.

With the regular velocity field defined from the curvilinear coordinates, formula (16) for the collective kinetic energy is indeed interesting. It involves the scalar product of two velocity fields, the velocity field characterizing the time change of the coordinate system, and the single-particle current established inside the nucleus in response to such coordinate changes.

5. VORTICES AT LEVEL CROSSINGS

In order to see the effect of vortices in the collective kinetic energy we consider the case of single-particle level crossings. In nuclear shape vibrations, level crossings can occur between two single-particle states with very different nodal structures. By use of the cranking formula, Griffin [15] has shown that the inertial parameters in the immediate neighborhood of a level crossing can be some 1000 times as large as the typical inertia for the case of no crossing in the independent particle model, and as much as 10 times as large even when a reasonable pairing is taken into account.

In the language of Schrödinger fluid it has been shown [2] for the case of SHO under quadrupole deformation that such a level crossing induces rearrangements of density ripples in the nucleus and such rearrangements establish strong vortices in the flow field of the single-particle states participating in the crossing. Such vortices lead to large contributions to the collective kinetic energy.

The large deviation of the single particle flow from the quadrupole flow $\vec{v}_Q$ [= $\vec{v}_R$ given in (17)] can be illustrated by the following quantity, defined as the "dynamical rippling" R;

$$R^2 = \int d\tau \; |\vec{v}_Q \cdot \nabla u_k - \nabla \cdot (\vec{v}_{\bullet k} u_k)|^2/\dot{\alpha}^2 \quad . \tag{20}$$

In Figure 1, R is plotted as a function of deformation, near the crossing. This quantity shows a sharp peak at the level crossing.

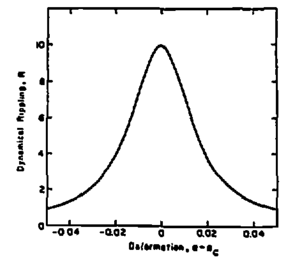


Figure 1. Dynamical rippling R as a function of deformation α, for the single-particle state participating in a crossing at α_c. The coupling between the crossing level is 0.1 MeV.

6. KINETIC ENERGY FOR COLLECTIVE ROTATION

With the regular velocity field given in (19) the collective kinetic energy for rotation is given by

$$T_k = \frac{m}{2}\int \rho_k \vec{v}_{\bullet k} \cdot \vec{\Omega} \times \vec{r} \; d\tau \quad . \tag{21}$$

This leads to the moment of inertia for the nucleus in the following form

$$\mathcal{I} = \left(m/\Omega^2\right)\int \vec{J} \cdot \vec{\Omega} \times \vec{r} \; d\tau \quad , \tag{22}$$

where $\vec{J}$ is the total matter current of the nucleus.

This formula was first derived [2-5] for the first order cranking for the independent-particle model. Its validity is later generalized [6,7]. In Ref. [7] the first order cranking of a general many-body wave function undergoing a collective rotation is considered and it is shown there that exactly the same formula (22) can be obtained.

In Ref. [6] the collective rotation about a fixed axis is studied with the time-dependent variational principle. From the assumption that the many-body wave function be stationary in the body-fixed frame, one can single out two variational parameters which are consistent with the continuity equation: the angle of rotation and the magnitude of the internal current. Then, the variational principle implies a uniform rotation with the collective Hamiltonian given by

$$\mathcal{H}(\theta, p_\theta) = p_\theta^2/2\mathcal{I} + V \qquad (23)$$

where the moment of inertia $\mathcal{I}$ is given in precisely the same form as shown in (22).

The result of this variational approach is much more general than the cranking result, since no assumption of external cranking or adiabaticity is involved in the derivation. Formula (22) for the moment of inertia is hence generalized to any time-dependent descriptions derivable from the variational principle, whose wave function satisfies the condition of being stationary in the body-fixed frame. Therefore, the time-dependent Hartree-Fock with its single-particle wave functions obeying such a stationarity requirement will have a moment of inertia as given by Eq. (22) where the current $\vec{J}$ is computed by the Hartree-Fock wave function, of course.

Note also that the dynamics of the many-body system in the body-fixed frame enters the moment of inertia (22) through the total matter current $\vec{J}$. When this dynamics is treated unperturbationally, $\vec{J}$ will depend in general on all powers of Ω. Thus, Eq. (22) implies in general a moment of inertia varying with the angular velocity of rotation, in contrast to the case of first order cranking where the moment of inertia is independent of the angular velocity.

Since the formula (22) for the moment of inertia is completely different from that of a classical fluid, some common terminology in nuclear collective rotation, such as rigid or irrotational moments, needs to be clarified. In the usual terminology, rigid (or irrotational) moment means the classical moment, $(m/\Omega^2) \int \rho v^2 \, d\tau$, calculated with the rigid (or irrotational) velocity field. Here, we re-ascertain that if the total matter velocity field $\vec{v}_T$ $(= \vec{J}/\Sigma\rho_k)$ is equal to the rigid field the quantal moment of inertia is equal to the classical rigid value, and, similarly, if the total matter velocity field is equal to the irrotational field with no singularity the quantal moment of inertia is equal to the classical irrotational value. For the rigid flow, this assertion is obvious, since when $\vec{v}_T = \vec{\Omega} \times \vec{r}$ the integrand of Eq. (22) becomes the classical form involving the absolute square of the rigid field. For the irrotational flow, the proof is more elaborate and the reader is referred to Ref. [7]. Therefore in these two (very special) cases, the usual terminology can still be used without referring to whether the classical or the quantal description is employed.

7. SCHRÖDINGER FLUID FLOW FOR ROTATION

The viewing of nucleus dynamics as the flow of a Schrödinger fluid enables one to focus attention on the quantal flow of the system. It thereby provides a precise way to determine whether the system is performing a rigid rotation or irrotational rotation: that is, it ought to be judged by the flow and not by the moment of inertia.

In classical fluid dynamics an irrotational moment of inertia will imply an irrotational flow by virtue of Kelvin's theorem [16]. In Schrödinger fluid, no theorem analogous to Kelvin's has yet been found. Therefore, even when an irrotational moment of inertia is obtained, one cannot be sure whether the quantal flow is actually irrotational, if the flow itself is not examined.

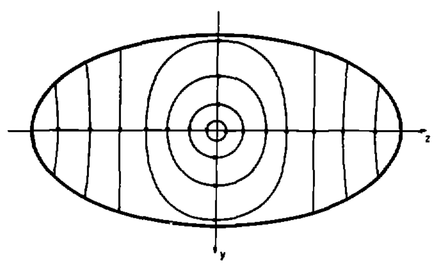
Bohr and Mottelson [17] have shown that the SHO independent-particle system acquires the rigid moment of inertia when the system is in equilibrium deformation. This result is true for any number of nucleons occupying any set of single-particle SHO states, provided that the equilibrium deformation is determined by this set of occupied states. One can then apply their result to one SHO single-particle state. However, as we have shown in Section 2, the quantal flow field of a single-particle state is derived from a velocity potential and has a nonzero curl only on the line vortices. Therefore, even though the single particle at equilibrium exhibits a rigid moment of inertia, it can never have the velocity field of rigid rotation, the curl of which is constant and nonzero throughout the whole space.

Figure 2 shows the vortex structure of two SHO single-particle states. Radomski [8] has calculated the quantal flow field of C^{12} in SHO approximation and found similar vortex structures. These examples clearly show that the nucleus can exhibit a nonrigid internal flow even when its overall moment of inertia is equal to the classical rigid value.

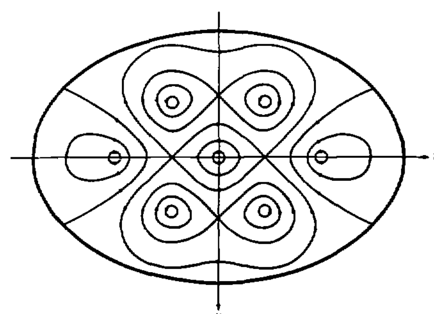
Can the internal flow of a nucleus ever approach rigid rotation? Such a question is important when we consider the result of Bohr [18] that the internal flow in rotation always approaches the rigid flow at equilibrium, in the classical thermodynamic limit. We consider this question without making any thermodynamic assumption.

In fact, Bohr's result suggests examining the internal flow of a large nucleus. At large nucleon number A, the total velocity field $\vec{v}$ is built up from many single-particle velocity fields which have vortices at many different locations in space. At infinite A, these vortices may distribute themselves uniformly in space, and this could lead to a smooth total velocity field. Indeed, it has been shown in Ref. [5] that $\vec{v}_T \to \vec{\Omega} \times \vec{r}$ as $A \to \infty$ for three distinct independent-particle systems at their equilibrium deformations: (a) the Amado-Brueckner box (the uniform potential with periodic boundary conditions applied on the surfaces of a rectangular parallelepiped box), (b) the Hill-Wheeler box, and (c) the SHO. Note that these potentials as regards smoothness represent both the very smooth and the very rapidly varying limits of nuclear one-body potentials.

We should mention that another limit, i.e., the limit of the irrotational flow, has been established long ago by Prange [19] for a nucleus in the limit of very strong pairing correlations.



(a) Quasistatic state, |lmn> = |001>



(b) Quasistatic state |lmn> = |112>

Figure 2. The irrotational velocity field $\vec{v}_{ck}$ for rotation of a single SHO state at its equilibrium deformation. (l, m, n are the number of nodal planes perpendicular to the x, y, z directions, respectively.)

A logical next step is to study how the nuclear internal flow approaches the infinite A limit and the infinite pairing limit. Reference [5] undertakes a less ambitious effort. Instead of the flow field itself, the independent-particle moment of inertia for a large but finite Amado-Brueckner box is separated in Ref. [5] into a smooth part and a fluctuating part. The smooth part $\bar{\mathcal{J}}$ is shown to vary with A according to

$$\bar{\mathcal{J}}/\mathcal{J}_{rig} = 1 - 1.209\, A^{-1/3}\left(\frac{1}{3}\ln A - 2.382\right) \quad . \tag{24}$$

A comparison of this formula and the exact numerical value is shown in Figure 3.

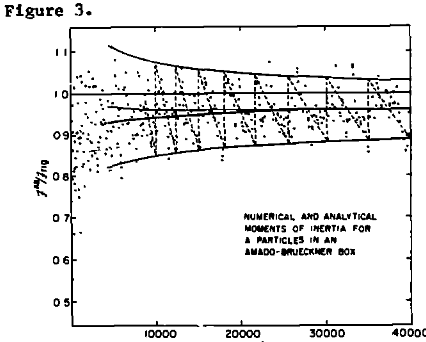


Figure 3. Ratio of the cranking moment of inertia with the rigid value as a function of A for the Amado-Brueckner box with cubical geometry. The heavy solid curve is a smooth average calculated from the exact values (represented by dots); the middle light solid curve shows the analytic smooth average, Eq. (24); and the dashed sawtooth curve shows the result of the analytic fluctuation term obtained in Ref. [5].

8. SUMMARY

We have outlined the main ideas involved in the Schrödinger fluid view of nuclear dynamics. The fluidic form of the collective inertia is identified as an important link between the quantal nuclear internal flow and the conventional treatment of the cranking model. It also clarifies certain differences between the quantal flow and the classical flow. These points are illustrated with collective vibration and rotation. For rotation, the generality of the fluidic moment of inertia is emphasized and the question when the nucleus' internal flow can attain a rigid flow is discussed.

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