TO MAMA, KUNI AND TO THE MEMORY OF MY MOTHER

IC/83/55
INTERNAL REPORT
(Limited distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

MATRIX TRANSFORMATIONS AND SEQUENCE SPACES *

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MIRAMARE - TRIESTE
June 1983

* To be submitted for publication.

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ABSTRACT

In most cases the most general linear operator from one sequence space into another is actually given by an infinite matrix and therefore the theory of matrix transformations has always been of great interest in the study of sequence spaces. The study of general theory of matrix transformations was motivated by the special results in summability theory. This paper is a review article which gives almost all known results on matrix transformations. This also suggests a number of open problems for further study and will be very much useful for research workers.

1. The general theory of matrix transformations was motivated by special and classical results in summability theory which were obtained by Cesàro, Borel, Nörlund, Riesz and others. On the other hand, in most cases the most general linear operator on one sequence space into another is actually given by an infinite matrix. Therefore matrix transformations are of great interest in the study of sequence spaces. The celebrated German mathematician O. Toeplitz first observed in 1911, that the techniques of linear space theory can be used to characterise matrix transformations. Later the Banach-Steinhaus theorem and related results became useful tools in dealing with such problems.

In this work we systematically present almost all the results on matrix transformations in sequence spaces. We first present some important definitions, notations and conventions which will be used in describing the results on matrix transformations.

2. If $\{p_k\}$ is a bounded sequence of strictly positive real numbers, then (see [16], [17], [18], [27])

$$\ell_\infty(p) = \{x : \sup_k |x_k|^{p_k} < \infty\},$$
$$c_0(p) = \{x : |x_k|^{p_k} \longrightarrow 0\},$$
$$c(p) = \{x : |x_k - \ell|^{p_k} \longrightarrow 0 \text{ for some } \ell\},$$
$$\ell(p) = \{x : \sum_k |x_k|^{p_k} < \infty\},$$
$$\omega(p) = \{x : n^{-1}\sum_{k=1}^{n} |x_k - \ell|^{p_k} \longrightarrow 0 \text{ for some } \ell\}.$$

$c_0(p)$ is a linear metric space paranormed by $\|x\| = \sup_k |x_k|^{p_k/M}$ where $M = \max(1, \sup p_k)$. (Note that we use the symbol $\|\cdot\|$ for a paranorm also.) $\ell_\infty(p)$ and $c(p)$ are paranormed by $\|x\| = \sup_k |x_k|^{p_k/M}$ if and only if $\inf p_k > 0$. $\ell(p)$ and $\omega(p)$ are paranormed by $\|x\| = \left(\sum_k |x_k|^{p_k}\right)^{1/M}$ and $\|x\| = \sup \left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k}\right)^{1/M}$, respectively. All the spaces defined above are complete in their topologies. In general $\ell_\infty(p)$, $c_0(p)$, $c(p)$, $\ell(p)$ and $\omega(p)$ are not normed spaces. If $p_k = p$ for all k, then $\ell_\infty(p) = \ell_\infty$, $c_0(p) = c_0$, $c(p) = c$, $\ell(p) = \ell_p$ and $\omega(p) = \omega_p$. ℓ_∞, c and c_0 are, respectively, the Banach spaces of bounded, convergent and null sequences normed by $\|x\| = \sup|x_k|$. ℓ_p and ω_p are Banach spaces normed by $\|x\| = \left(\sum_k |x_k|^p\right)^{1/p}$ and $\|x\| = \sup \left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$, respectively for

$1 \leqslant p < \infty$. If $0 < p < 1$, then ℓ_p and ω_p are complete p-normed spaces p-normed by $\|x\| = \sum_k |x_k|^p$ and $\|x\| = \frac{1}{n}\sum_{k=1}^{n} |x_k|^p$, respectively.

The form of the argument used to prove some particular sequence space to be a linear metric space and complete in its metric topology is such that it can be used to obtain such results for a number of spaces. Therefore in the following theorem we only prove the assertions for $\ell(p)$.

Theorem 1

$\ell(p)$ is a complete linear metric space paranormed by g defined by

$$g(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M}$$

where $M = \max(1, H = \sup p_k)$.

Proof

For any complex number λ and $x, y \in \ell(p)$ we have the inequalities:

$$\left(\sum_k |x_k + y_k|^{p_k}\right)^{1/M} \leq \left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}} + \left(\sum_k |y_k|^{p_k}\right)^{1/M} \qquad (1)$$

and

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^M) \qquad (2)$$

It follows from the above inequalities that $\ell(p)$ is a linear space with respect to pointwise linear operations. Clearly $g(0) = 0$ and $g(x) = g(-x)$ for all $x \in \ell(p)$. It also follows from the above two inequalities that g is a subadditive and

$$g(\lambda x) \leqslant \max(1, |\lambda|)\, g(x) \quad .$$

Therefore the function $(\lambda, x) \mapsto \lambda x$ is continuous at $\lambda = 0$, $x = 0$ and that, when λ is fixed, the function $x \mapsto \lambda x$ is continuous at $x = 0$. If x is fixed, and $\epsilon > 0$, we can choose K such that

$$\sum_{k>K} |x_k|^{p_k} < \frac{\epsilon}{2}$$

and $\delta > 0$, so that $|\lambda| < \delta$ gives

$$\sum_{k \leq K} |\lambda x_k|^{p_k} < \frac{\epsilon}{2} .$$

Therefore $|\lambda| < \min(1, \delta)$ implies that $g(\lambda x) < \epsilon$. Thus the function $\lambda \mapsto \lambda x$ is continuous at $\lambda = 0$ and so $\ell(p)$ is a paranormed space.

To show that $\ell(p)$ is complete, let $\{x^i\}$ be a Cauchy sequence in $\ell(p)$. Then since for each fixed k,

$$|x_k^i - x_k^j| \leq \left(\sum_k |x_k^i - x_k^j|^{p_k}\right)^{\frac{1}{M}} = g(x^i - x^j),$$

it follows that $\{x_k^i\}$ is a Cauchy sequence of complex numbers and therefore $x_k^i \to x_k$ as $i \to \infty$ for each k. Put $x = \{x_k\}$. We first show that $x \in \ell(p)$. We know that $g(x^i)$ is bounded, say, $g(x^i) \leqslant K$. Now for any t,

$$\left(\sum_{k=1}^{t} |x_k^i|^{p_k}\right)^{\frac{1}{M}} \leq g(x^i) \leq K.$$

Letting $i \to \infty$ and then $t \to \infty$ we obtain

$$\left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}} \leq K$$

and this shows that $x \in \ell(p)$. It remains to prove that $g(x^i - x) \to 0$. Let $\epsilon > 0$ be given. Then there is an integer N such that $g(x^i - x^j) < \epsilon$ for $i, j \geqslant N$. Therefore for any t,

$$\left(\sum_{k=1}^{t} |x_k^i - x_k^j|^{p_k}\right)^{\frac{1}{M}} \leq g(x^i - x^j) < \epsilon$$

for $i, j \geqslant N$. Letting $j \to \infty$ we obtain

$$\left(\sum_{k=1}^{t} |x_k^i - x_k|^{p_k}\right)^{\frac{1}{M}} \leq \epsilon \qquad \text{for } i \geqslant N .$$

Since t is arbitrary, we can let $t \to \infty$ and obtain $g(x^i - x) \leqslant \epsilon$ for $i \geqslant N$ and this completes the proof.

3. Let X and Y be any two non-empty subsets of the space of all complex sequences and let $A = (a_{nk})$ $(n,k = 1,2,\ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n. If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, then we say that A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. The sequence Ax is called the A transform of x. By (X,Y) we mean the class of matrices A such that $A : X \to Y$. If in X and Y there is some notion of limit or sum, then we write (X,Y,P) to denote the subset of (X,Y) which preserves the limit or sum.

The result which characterises the class (c,c,P) is the famous Silverman-Toeplitz theorem. Silverman proved the sufficiency of the conditions and the necessity was due to Toeplitz. The necessary and sufficient conditions for A to be in (c,c,P) are known as "Toeplitz conditions"and the matrices in the class (c,c,P) are called "Toeplitz matrices"- these are also called "regular matrices". The characterisation of the class (c,c) is known as Kojima-Schur theorem and the matrices in the class (c,c) are known as conservative or convergence preserving matrices. The class (ℓ_∞,c) was obtained by Schur in 1921. The characterization of this class is known as Schur theorem and the matrices in the class (ℓ_∞,c) are known as Schur matrices.

We present below some important results on matrix transformations.

Theorem 2

$A \in (\ell_\infty,\ell_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty$$

Proof

The sufficiency is easy. For necessity suppose that $A \in (\ell_\infty,\ell_\infty)$. Put $Ax = (A_n(x))$ and observe that (A_n) is a sequence of bounded linear operators on ℓ_∞ such that $\sup_n |A_n(x)| < \infty$. Now the result follows from an application of Banach-Steinhaus theorem.

Theorem 3

$A \in (\ell_\infty(p),\ell_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty$$

for every integer $N > 1$.

Proof

For sufficiency take an integer $N > \max(1, \sup_k |x_k|^{p_k})$. Then for every n,

$$|A_n(x)| \leq \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty$$

and therefore $A \in (\ell_\infty(p),\ell_\infty)$. For necessity suppose that $A \in (\ell_\infty(p),\ell_\infty)$ but there is an integer $N > 1$ such that $\sup_n \sum_k |a_{nk}| N^{1/p_k} = \infty$. Then the matrix $(a_{nk} N^{p_k}) \notin (\ell_\infty,\ell_\infty)$ and so there is an $x \in \ell_\infty$ with $\|x\| = 1$ such that $\sum_k a_{nk} N^{1/p_k} x_k$ is not bounded. Hence although $y = (N^{1/p_k} x_k) \in \ell_\infty(p)$, the sequence $(A_n(y)) \notin \ell_\infty$. This contradicts the fact that $A \in (\ell_\infty(p),\ell_\infty)$ and completes the proof.

Theorem 4

$A \in (c,c)$ if and only if

i) $\sup_n \sum_k |a_{nk}| < \infty$,

ii) $\lim_{n\to\infty} a_{nk} = \alpha_k$,

iii) $\lim_{n\to\infty} \sum_k a_{nk} = \alpha$.

Proof

Suppose that $A \in (c,c)$. Since $e_k = \{\delta_{nk}\}$, where $\delta_{nk} = 1$ $(n=k)$ and $= 0$ $(n \neq k)$, belongs to c, the necessity of ii) holds. Similarly by taking $x = e = (1,1,1,\ldots) \in c$ we get iii). The proof of the necessity of i) is same as that of Theorem 2.

For sufficiency let $x_k \to \ell$ and let the conditions i)-iii) hold. We write

$$\sum_k a_{nk} x_k = \sum_k a_{nk}(x_k - \ell) + \ell \sum_k a_{nk}$$

Since iii) holds, the second term on the right tends to $\ell\alpha$ as $n \to \infty$. From i) and ii) and the fact that $x_k \to \ell$ it follows that the first term on the right tends to $\sum_k \alpha_k (x_k - \ell)$ as $n \to \infty$. Therefore we have

$$\lim_{n\to\infty} \sum_k a_{nk} x_k = \sum_k \alpha_k (x_k - \ell) + \ell \alpha \qquad (3)$$

and this completes the proof.

Theorem 5

$A \in (c,c,P)$ if and only if

i) $\sup_n \sum_k |a_{nk}| < \infty$,

ii) $\lim_{n\to\infty} a_{nk} = 0$,

iii) $\lim_{n\to\infty} \sum_k a_{nk} = 1$.

Proof

Let $A \in (c,c,P)$. Then the conditions hold by Theorem 4. Suppose that the conditions hold. Then $A \in (c,c)$ and (3) reduces to

$$\lim_{n\to\infty} \sum_k a_{nk} x_k = \ell$$

and this completes the proof.

We now give some examples of Toeplitz matrices which are extensively used in the theory of summability and associated with particular authors.

Example 1 - Arithmetic mean

The Toeplitz matrix defined by

$$a_{nk} = \frac{1}{n+1} \quad (0 \le k \le n)$$
$$= 0 \quad (k > n)$$

Example 2 - Cesàro means

For each $r > -1$ the (c,r) matrix is defined by

$$a_{nk} = \frac{A_{n-k}^{r-1}}{A_n^r} \quad (0 \le k \le n)$$
$$= 0 \quad (k > n);$$

where

$$A_n^r = \frac{(r+1)(r+2)\cdots(r+n)}{n!} \quad \text{for } n \ge 1,\ A_0^r = 1.$$

When $r \geqslant 0$, $A_n^{r-1} \geqslant 0$ and using the fact that

$$(1-z)^{-r-1} = \sum_{n=0}^{\infty} A_n^r z^n \quad (|z| < 1)$$

it is easy to show that

$$\sum_{k=0}^{n} A_{n-k}^{s} A_k^{r} = A_n^{s+r+1}$$

from which it follows that $\sum_k |a_{nk}| = \sum_k a_{nk} = 1$ for every n. It is known from elementary analysis that

$$\frac{A_n^r}{n^r} \longrightarrow \frac{1}{\Gamma(r+1)} \quad (n \to \infty)$$

where $\Gamma(x)$ denotes the gamma function.

It follows that for fixed k, $a_{nk} \to 0$ $(n \to \infty)$. Thus each matrix (c,r) which is called the Cesàro matrix of order r is a Toeplitz matrix when $r \geqslant 0$. The case $r = 0$ gives the unit matrix and the case $r = 1$ gives the arithmetic mean.

Example 3 - Euler-Knopp matrix

Define

$$a_{nk} = \begin{cases} \binom{n}{k} r^k (1-r)^{n-k} & (0 \le k \le n), \\ 0 & (k > n), \end{cases}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Then A is Toeplitz when $0 < r < 1$.

Example 4 - Nörlund mean

Let $p_0 > 0$, $p_n \geq 0$ $(n \geq 1)$, $P_n = p_0 + p_1 + \cdots + p_n$. Define

$$a_{nk} = \frac{p_{n-k}}{P_n} \quad (0 \leq k \leq n)$$
$$= 0 \quad (k > n).$$

Then A is a Toeplitz matrix if and only if $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. The matrix A defined above is called the Nörlund matrix. The case $p_0 = p_1 = \cdots = p_n = 1$ gives the arithmetic mean.

Example 5 - Borel matrix

The Toeplitz matrix defined by

$$a_k(t) = \frac{e^{-t} t^k}{k!} \quad (k = 0, 1, \ldots ; t > 0)$$

is called the Borel matrix, after the French mathematician E. Borel. Here we replace a_{nk} by $a_k(t)$ and let $t \to \infty$ continuously. $a_k(t)$ defined above tends to zero as $t \to \infty$ for fixed k and

$$\sum_{k=0}^{\infty} |a_k(t)| = \sum_{k=0}^{\infty} a_k(t) = 1,$$

whence $\sup_t \sum_k |a_k(t)| < \infty$. The essential property of the Borel matrix is that it maps convergent sequences into convergent functions, leaving the limit unchanged.

Example 6 - Riesz mean

Let $p_0 > 0$, $p_n \geq 0$ $(n \geq 1)$, $P_n = p_0 + p_1 + \cdots + p_n$. Define

$$a_{nk} = \frac{p_k}{P_n} \quad (0 \leq k \leq n)$$
$$= 0 \quad (k > n).$$

Then A is a Toeplitz matrix if and only if $P_n \to \infty$ as $n \to \infty$. The matrix A defined above is called the Riesz matrix.

We now prove Schur's theorem and in order to prove this we need the following lemma. These results can be found in Maddox [18].

Lemma 1

If $\sum_k |b_{nk}| < \infty$ for each n and $\sum_k |b_{nk}| \to 0$ as $n \to \infty$, then $\sum_k |b_{nk}|$ converges uniformly in n.

Proof

$\sum_k |b_{nk}| \to 0$ as $n \to \infty$ implies that $\sum_k |b_{nk}| < \infty$ for $n > N(\epsilon)$.

Since $\sum |b_{nk}| < \infty$, $1 \leq n \leq N(\epsilon)$, there is $m = m(n,\epsilon)$ such that $\sum_{k \geq m} |b_{nk}| < \epsilon$.

Hence there is $M = M(\epsilon) \geq 1$ such that $\sum_{k \geq M} |b_{nk}| < \epsilon$ for all n, which means that $\sum_k |b_{nk}|$ converges uniformly in n.

Theorem 6

$A \in (\ell_\infty, c)$ if and only if

i) $\sum_k |a_{nk}|$ converges uniformly in n,

ii) $\lim_{n \to \infty} a_{nk} = \alpha_k$

Proof

Suppose the conditions i) and ii) hold and let $x \in \ell_\infty$. Then $\sum_k a_{nk} x_k$ is absolutely and uniformly convergent in n. Hence $\lim_{n \to \infty} \sum_k a_{nk} x_k = \sum_k \alpha_k x_k$ and thus $A \in (\ell_\infty, c)$. This proves the sufficiency. Take $x = e_k \in \ell_\infty$. Then ii) holds. Now for the necessity of i) observe that $\left(\sum_k a_{nk} x_k\right)$ converges and hence is bounded whenever $x \in \ell_\infty$. Therefore it follows from Theorem 2 that $\sup_n \sum_k |a_{nk}| < \infty$. Write $b_{nk} = a_{nk} - \alpha_k$. Since $\sum_k |\alpha_k| < \infty$ we have that $\left(\sum_k b_{nk} x_k\right)$ converges whenever $x \in \ell_\infty$. We now show that this implies that

$$\sum_k |b_{nk}| \to 0 \quad (n \to \infty) \tag{4}$$

Suppose to the contrary that $\sum_k |b_{nk}| \not\to 0$ as $n \to \infty$. It then follows that $\sum_k |b_{nk}| \to c > 0$ as $m \to \infty$ through some subsequence of the positive integers.

Also we have $b_{mk} \to 0$ ($m \to \infty$, each k). Hence we may determine m_1 such that

$$\left| \sum |b_{m_1 k}| - c \right| < \frac{c}{10} \quad \text{and} \quad |b_{m_1,1}| < \frac{c}{10}.$$

Since $\sum_k |b_{m1,k}| < \infty$ we may choose $k_2 > 1$ such that

$$\sum_{k=k_2+1}^{\infty} |b_{m_1,k}| < \frac{c}{10}$$

It follows that

$$\left| \sum_{k=2}^{k_2} |b_{m_1,k}| - c \right| < \frac{3c}{10}$$

Write $\sum_{k=p}^{q} |b_{mk}| = B(m,p,q)$.

Now choose $m_2 > m_1$ such that $|B(m_2,1,\infty) - c| < \frac{c}{10}$ and $B(m_2,1,k_2) < \frac{c}{10}$. Then choose $k_3 > k_2$ such that

$$B(m_2, k_3+1, \infty) < \frac{c}{10}$$

It follows that

$$|B(m_2, k_2+1, k_3 - c)| < \frac{3c}{10}$$

Continuing in this way we find $m_1 < m_2 < \cdots$, $1 = k_1 < k_2 < \cdots$ such that

$$\left.\begin{array}{l} B(m_r, 1, k_r) < \frac{c}{10} \\ B(m_r, k_{r+1}+1, \infty) < \frac{c}{10} \\ |B(m_r, k_r+1, k_{r+1})| < \frac{3c}{10} \end{array}\right\} \tag{5}$$

Define $x \in \ell_\infty$, $\|x\| = 1$, by

$$x_k = \begin{array}{l} 0 \quad (k=1), \\ = (-1)^r \operatorname{sgn} b_{m_r,k} \quad (k_r < k \leq k_{r+1}),\ r = 1, 2, \cdots \end{array}$$

Write

$$\sum_k b_{m_r,k} x_k = \Sigma_1 + \Sigma_2 + \Sigma_3$$

where Σ_1 is over $1 \leq k \leq k_r$, Σ_2 over $k_r < k \leq k_{r+1}$ and Σ_3 over $k > k_{r+1}$. It follows from (5) and the definition of x that

$$\left| \sum_k b_{m_r,k} x_k - (-1)^r c \right| < \frac{c}{2}.$$

Consequently, it is clear that the sequence $(B_n(x)) = \left\{ \sum_k b_{nk} x_k \right\}$ is not a Cauchy sequence and hence is not convergent. We have thus proved that $(B_n(x))$ is not convergent for all $x \in \ell_\infty$ and this contradicts the fact that $A \in (\ell_\infty, c)$. Therefore (4) must hold. Now by Lemma 1 it follows that $\sum_k |b_{nk}|$ converges uniformly in n. Hence $\sum_k |a_{nk}| = \sum_k |b_{nk} + \alpha_k|$ converges uniformly in n and this completes the proof of the theorem.

Theorem 7

Let $p_k > 0$ for every k. Then $A \in (\ell_\infty(p), c)$ if and only if

i) $\sum_k |a_{nk}| N^{1/p_k}$ converges uniformly in n for all integers $N > 1$,

ii) $\lim_{n \to \infty} a_{nk} = \alpha_k$.

Proof

By (i) $\sum_k a_{nk} x_k$ converges uniformly in n for each $x \in \ell_\infty(p)$. Therefore

$$\lim_{n \to \infty} \sum_k a_{nk} x_k = \sum_k \alpha_k x_k$$

and hence sufficiency holds. The necessity of ii) is trivial. If i) does not hold, then $(a_{nk} N^{1/p_k}) \notin (\ell_\infty, c)$ for some integer $N > 1$, whence as in Theorem 3 there is $x \in \ell_\infty(p)$ such that $\left\{ \sum_k a_{nk} x_k \right\} \notin c$. This completes the proof of the theorem.

3. Several Banach algebras of infinite matrices arise in the theory of summability and we now discuss some of these algebras. There are several ways of combining infinite matrices to give some kind of product matrix. The most natural one is the usual matrix product. If $A = (a_{nk})$ and $B = (b_{nk})$ are two infinite matrices, the matrix product is defined by

$$(AB)_{nk} = \sum_i a_{ni} b_{ik} .$$

$(\ell_\infty, \ell_\infty)$ is a Banach algebra with respect to the matrix product and the norm defined by

$$\|A\| = \sup_n \sum_k |a_{nk}| .$$

(c,c) is also a Banach algebra with respect to the matrix product and the norm defined as above; it is clearly a subalgebra of $(\ell_\infty, \ell_\infty)$. Since a_{nk} and $\sum_k a_{nk}$ are all linear functionals of unit norm, this shows that (c,c) is a closed subalgebra of $(\ell_\infty, \ell_\infty)$. It has been shown by Wilansky and Zeller [34] that (c,c) contains its inverses, that is, $A \in (c,c)$ and $A^{-1} \in (\ell_\infty, \ell_\infty)$ imply that $A^{-1} \in (c,c)$.

Recall that a Banach algebra is said to be semisimple if the radical contains only zero. We now show that $(\ell_\infty, \ell_\infty)$ and (c,c) are semisimple. This was first proved by Wilansky and Zeller [35].

Theorem 8

$(\ell_\infty, \ell_\infty)$ and (c,c) are semisimple.

Proof

Given $A \in (\ell_\infty, \ell_\infty)$, $A \neq 0$, let $x \in c$ and $Ax \neq 0$. For example, if, for certain n,k, we have $a_{nk} \neq 0$, we may take $x = \delta^k$, a sequence of zeros save for a one in the k^{th} place. Define $B \in (c,c)$ as follows: B consists entirely of zeros except for a single column and $BAx = x$. Then $I - BA$ carries x into 0, hence is not injective and has no inverse. This proves that A is not in the radical of $(\ell_\infty, \ell_\infty)$ or (c,c) and hence that these spaces are semisimple.

For $A \in (c,c)$ define

$$\chi(A) = \lim_n \sum_k a_{nk} - \sum_k \left(\lim_n a_{nk} \right)$$

Then χ is a multiplicative linear functional and $\chi(A)$ is called the characteristic of the matrix A. A matrix A is called coregular if $\chi(A) \neq 0$ and conull otherwise. It follows from Theorem 5 that $\chi(A) = 1$ if A is Toeplitz, and hence Toeplitz matrices form a subset of coregular matrices.

(ℓ_∞, c) is a subset of (c,c), but it follows from Theorem 6 that matrices in (ℓ_∞, c) have characteristic zero and hence are conull. Thus we obtain the following remarkable theorem in the theory of summability which was originally proved by Steinhaus.

Theorem 9

$(c,c,P) \cap (\ell_\infty, c) = \phi$. The proof of the original theorem of Steinhaus may be found in Cooke [3]. Several results on Steinhaus type may be found in Maddox [15].

It is easy to show that (ℓ_∞, c) is a two-sided ideal in, and a maximal linear subspace of (c,c), but it is not an ideal in $(\ell_\infty, \ell_\infty)$. The set of all coregular matrices is an open subset of (c,c) but is not a linear subspace; for if $\sum_k a_{nk} \to \alpha$ then $2 \sum_k a_{nk} \to 2\alpha$. (c,c,P) is a subset of (c,c) but is obviously not a subspace. But we have

Theorem 10

As a subset of (c,c), (c,c,P) is a closed convex semigroup with identity with respect to the matrix product.

Proof

Let $A^{(m)} \in (c,c,P)$, $A \in (c,c)$ and $\|A^{(m)} - A\| \to 0$ as $m \to \infty$. Also

$$|a_{nk}| \leq \|A^{(m)} - A\| + |a_{nk}^{(m)}|$$

whence $a_{nk} \to 0$ $(n \to \infty,\ k \text{ fixed})$, since $a_{nk}^{(m)} \to 0$ $(n \to \infty,\ k \text{ fixed})$. Again

$$\left| \sum_k a_{nk} - 1 \right| \leq \|A^{(m)} - A\| + \left| \sum_k a_{nk}^{(m)} - 1 \right|$$

and so $\sum_k a_{nk} \to 1$ $(n \to \infty)$, whence $A \in (c,c,P)$. Thus (c,c,P) is closed in (c,c). Let $A,B \in (c,c,P)$ and $\lambda + \mu = 1$. Then obviously $\lambda A + \mu B \in (c,c,P)$ and hence (c,c,P) is convex.

Let $A,B \in (c,c,P)$ and $C = AB$. Then $C_n(x) = A_n(Bx)$ for $x \in c$. Hence

$$\lim_n C_n(x) = \lim_n A_n(Bx) = \lim_n B(x) = \lim_n x_n$$

and so $AB \in (c,c,P)$. The identity matrix is an element of (c,c,P) and is the identity element of the semigroup. Besides the matrix product there is another type of product, called the convolution, of infinite matrices. If

$A = (a_{nk})$ and $B = (b_{nk})$ are two infinite matrices, then the convolution $A * B$ is defined by

$$(A*B)_{nk} = \sum_{i=0}^{k} a_{ni}\, b_{n,k-i} .$$

If we consider convolution, then (c,c) becomes a commutative Banach algebra with identity.

Theorem 11

(c,c) is a commutative Banach convolution algebra with identity.

Proof

We have only to show closure under convolution and the submultiplicative property of the norm. If $C = A * B$, then by the definition of the convolution we have

$$c_{nk} = a_{n0}b_{nk} + a_{n1}\, b_{n,k-1} + \cdots + a_{nk}b_{n0}$$
$$\longrightarrow a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$$
$$(\text{as } n \to \infty, \quad k \text{ fixed})$$

Also by the elementary theorem on the Cauchy product of absolutely convergent series, we have for each n,

$$\sum_k |c_{nk}| \leq \sum_k |a_{nk}| \sum_k |b_{nk}| , \quad \sum_k c_{nk} = \sum_k a_{nk} \sum_k b_{nk}$$

Hence $\|C\| \leq \|A\|\ \|B\|$ and $\sum_k c_{nk} \to ab$ as $n \to \infty$. Therefore $C \in (c,c)$ and the norm is submultiplicative.

4. We now characterize the matrices in the class (ℓ_1,ℓ_p) .

Theorem 12

$A \in (\ell_1,\ell_p)$ if and only if

i) $M = \sup_k \sum_n |a_{nk}|^p < \infty \qquad (1 \leq p < \infty)$

ii) $\sup_{n,k} |a_{nk}| < \infty \qquad (p = \infty).$

Proof

We prove only i). If $x \in \ell_1$, then by Minkowski's inequality we have

$$\left(\sum_n \left|\sum_k a_{nk}x_k\right|^p\right)^{\frac{1}{p}} \leq \sum_k \left(\sum_n |a_{nk}x_k|^p\right)^{\frac{1}{p}}$$
$$= \sum_k |x_k| \left(\sum_n |a_{nk}|^p\right)^{\frac{1}{p}}$$
$$\leq M^{\frac{1}{p}} \|x\| < \infty .$$

(The interchange of summation over n and k is justified by absolute convergence.) Thus $Ax \in \ell_p$ and this proves the sufficiency. For necessity suppose that $A \in (\ell_1,\ell_p)$, so that

$$\sum_i |A_i(x)|^p < \infty \quad \text{on } \ell_1$$

where $A_i(x) = \sum_k a_{ik}\, x_k$. Since $\sum_k a_{ik}\, x_k$ converges for each i, whenever $x \in \ell_1$, it follows from Banach-Steinhaus theorem that $\sup_k |a_{ik}| < \infty$ for each i. Define

$$q_n(x) = \left(\sum_{i=1}^{n} |A_i(x)|^p\right)^{\frac{1}{p}} \qquad (n = 1, 2, \cdots)$$

and note that each q_n is a seminorm on ℓ_1. Moreover, since each A_i is a bounded linear functional on ℓ_1, each q_n is bounded on ℓ_1. Thus we have a sequence (q_n) of continuous seminorms on ℓ_1 such that

$$\sup_n q_n(x) = \left(\sum_{i=1}^{\infty} |A_i(x)|^p\right)^{\frac{1}{p}} < \infty$$

for each $x \in \ell_1$. It follows from Banach-Steinhaus theorem that there is a constant H such that

$$\left(\sum_{i=1}^{\infty} |A_i(x)|^p\right)^{\frac{1}{p}} \leq H\ \|x\|$$

on ℓ_1. Putting $x = e_k$ we obtain i) and this completes the proof.

Theorem 13

$A \in (\ell_1,\ell_1,P)$ if and only if

i) $\sup_k \sum_n |a_{nk}| < \infty$

ii) $\sum_n a_{nk} = 1$ for all k.

Proof

Since i) holds, $A \in (\ell_1, \ell_1)$. By ii) it follows that

$$\sum_n A_n(x) = \sum_n \sum_k a_{nk} x_k$$
$$= \sum_k x_k \sum_n a_{nk} = \sum_k x_k$$

(the change of order of summation is justified by the absolute convergence of $\sum_n A_n(x)$). This proves the sufficiency. For necessity suppose that $A \in (\ell_1, \ell_1, P)$. Then i) holds. Also

$$\sum_n A_n(x) = \sum_n \sum_k a_{nk} x_k = \sum_k x_k$$

Now taking $x_k = 1 (k = r)$, $= 0$ $(k \neq r)$ we get

$$\sum_n a_{nr} = 1$$

Since r is arbitrary ii) holds and this completes the proof.

Theorem 14

Let $1 < p < \infty$ and let $A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1)$. Then $A \in (\ell_p, \ell_p)$.

Proof

By Hölder's inequality we have

$$\left|\sum_k a_{nk} x_k\right| \leq \sum_k |a_{nk}|^{\frac{1}{p}} |a_{nk}|^{\frac{1}{q}} |x_k|$$
$$\leq \left(\sum_k |a_{nk}| |x_k|^p\right)^{\frac{1}{p}} \left(\sum_k |a_{nk}|\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore

$$\left|\sum_k a_{nk} x_k\right|^p \leq \left(\sum_k |a_{nk}| |x_k|^p\right) \left(\sum_k |a_{nk}|\right)^{\frac{p}{q}}$$

Now

$$\|Ax\|^p = \sum_n \left|\sum_k a_{nk} x_k\right|^p$$
$$\leq \sum_n \sum_k |a_{nk}| |x_k|^p \|A\|_\infty^{\frac{p}{q}}$$
$$\leq \|A\|_\infty^{p/q} \sum_k |x_k|^p \sum_n |a_{nk}|$$
$$\leq \|A\|_\infty^{\frac{p}{q}} \|x\|^p \|A\|_1 .$$

This completes the proof.

The necessity of the above theorem is a long standing problem. Recently, the problem has been solved for the case $p = 2$ by Crone [4]. But the complete characterization of some special matrices to be bounded operators on ℓ_p has been discussed by several mathematicians. To mention a few: the works of Brown et al. [2] and Jakimovski et al. [10]. Cesàro operators were discussed by Brown et al. [2] and Hausdorff matrices were discussed by Jakimovski et al. [10].

5. We now characterize the classes $(\ell(p), \ell_\infty)$ and $(\ell(p), c)$. For this we need the following inequality (see [12] and [13]).

For any $B > 0$ and any two complex numbers a and b,

$$|ab| \leq B\left(|a|^q B^{-q} + |b|^p\right) \tag{6}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

To contract the notation throughout the proofs we shall write

$$C(n, N, m, s) = \sum_{k=m}^{s} |a_{nk}|^{q_k} N^{-q_k}$$

for all integers $n, m \geq 1$, $1 \leq s \leq \infty$ and $N > 1$. We shall simply write $C(n,N)$ instead of $C(n,N,1,\infty)$ and put $C(N) = \sup_n C(n,N)$ for every integer $N > 1$.

Theorem 15

i) Let $1 < p_k \leq \sup p_k = H < \infty$ for every k. Then $A \in (\ell(p), \ell_\infty)$ iff there is an integer $B > 1$ such that $C(B) < \infty$.

ii) Let $0 < p_k \leq 1$ for every k. Then $A \in (\ell(p), \ell_\infty)$ iff $\sup_{n,k} |a_{nk}|^{p_k} < \infty$.

Proof

We only prove i), leaving ii) as an exercise for the readers. Let $x \in \ell(p)$. By using the inequality (6) we see that

$$|A_n(x)| \leq B\big(C(n,B) + g^H(x)\big) \leq B\big(C(B) + g^H(x)\big)$$

where $g^H(x) = \sum_k |x_k|^{p_k}$. This proves the sufficiency. For necessity suppose that $A \in (\ell(p), \ell_\infty)$ but that $C(N) = \infty$ for every integer $N > 1$. Then $\sum_k a_{nk} x_k$ converges for every n and every $x \in \ell(p)$ and each A_n defined by $A_n(x) = \sum a_{nk} x_k$ is an element of $\ell^*(p)$. Since $\ell(p)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $\ell(p)$, by uniform boundedness principle there exists a number G independent of n and x and a number $\delta < 1$ such that

$$|A_n(x)| \leq G \tag{7}$$

for every $x \in S[0,\delta]$ and every n, where by $S[0,\delta]$ we denote the closed sphere in $\ell(p)$ with centre at the origin $0 = (0,0,\ldots)$ and radius δ. Now choose an integer $Q > 1$ such that

$$Q\delta^H > G$$

By our supposition we have $C(Q) = \infty$ and so two cases are possible: either $C(n,Q) < \infty$ for every $n \geq 1$, or there exists $n \geq 1$ such that $C(n,Q) = \infty$. In the first case there exists $n \geq 1$ such that $C(n,Q) > 2$ and there exists $k_0 > 1$ such that

$$C(n, Q, k_0+1, \infty) < 1,$$

whence $C(n,Q,1,k_0) > 1$. In the second case we may choose $k_0 > 1$ such that $C(n,Q,1,k_0) > 1$, so that in either case there exists $n \geq 1$ and $k_0 > 1$ such that

$$S \equiv C(n, Q, 1, k_0) > 1 \tag{8}$$

We now define a sequence $x = (x_k)$ as follows

$$x_k = \delta^{\frac{H}{p_k}} (\operatorname{sgn} a_{nk}) |a_{nk}|^{q_k - 1} S^{-1} Q^{-\frac{q_k}{p_k}} \quad \text{for } 1 \leq k \leq k_0,$$
$$= 0 \quad \text{for } k > k_0.$$

It is easy to see that $g(x) \leq \delta$ and that

$$|A_n(x)| = S^{-1} Q \sum |a_{nk}|^{q_k} Q^{-q_k} \delta^{\frac{H}{p_k}}$$
$$\geq Q\delta^H > G$$

which contradicts (7) and completes the proof.

Theorem 16

Let $p = (p_k)$ be as in Theorem 15. Then $A \in (\ell(p), c)$ iff

i) the conditions of Theorem 15 holds,

ii) $\lim_{n\to\infty} a_{nk} = \alpha_k$ for each k.

Proof

We only prove the case $1 \leq p_k \leq H \leq \infty$ for every k. Since $e_k = \{\delta_{nk}\} \in \ell(p)$ the necessity of ii) follows. For the sufficiency observe that for every integer $m \geq 1$ and every n we have

$$C(n,B,1,m) \leq C(B) < \infty .$$

Hence

$$\lim_{m\to\infty} \lim_{n\to\infty} C(n, B, 1, m) \leq C(B),$$

that is,

$$\sum_k |\alpha_k|^{q_k} B^{-q_k} \leq C(B).$$

Therefore the series $\sum \alpha_k x_k$ and $\sum a_{nk} x_k$ converge for every n and every $x \in \ell(p)$. For each $x \in \ell(p)$ we can choose an integer $m \geq 1$ such that

$$\sum_{k=m+1}^{\infty} |x_k|^{p_k} < 1 .$$

By using the inequality (6) it is easy to check that

$$\sum_{k=m+1}^{\infty} |a_{nk} - \alpha_k| |x_k| \leq 2B(2C+1)\Big(\sum_{k=m+1}^{\infty} |x_k|^{p_k}\Big)^{\frac{1}{H}}$$

It now follows that

$$\lim_{n\to\infty} \sum_k a_{nk} x_k = \sum_k \alpha_k x_k$$

This completes the proof.

In our next theorem we characterize the class (ω_p, c).

Theorem 17

(a) Let $0 < p < 1$. Then $A \in (\omega_p, c)$ if and only if

i) $\lim_{n\to\infty} a_{nk} = \alpha_k$ (k fixed),

ii) $M(A) = \sup_n \sum_{r=0}^{\infty} 2^{\frac{r}{p}} A_r^1(n) < \infty$

where $A_r^1(n) = \max_r |a_{nk}|$ for each n the maximum is taken for k such that

$$2^r \leq k < 2^{r+1} .$$

(b) Let $p \geq 1$. Then $A \in (\omega_p, c)$ iff

i) $\lim_{n\to\infty} a_{nk} = \alpha_k$ (k fixed),

ii) $\sup_n \sum_{r=0}^{\infty} 2^{\frac{r}{p}} A_r^p(n) < \infty,$

iii) $\lim_{n\to\infty} \sum_k a_{nk} = \alpha,$

where

$$A_r^p(n) = \Big(\sum_r |a_{nk}|^q\Big)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

the sum is taken over k, for each n, and with k satisfying $2^r \leq k < 2^{r+1}$.

Proof

We only prove (a). (b) may be proved in a similar manner and is left as an exercise for the reader. Suppose that the conditions hold. Since ii) holds,

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{nk} x_k| &= \sum_{r=0}^{\infty} \sum_r |a_{nk} x_k| \\ &\leq \sum_{r=0}^{\infty}\Big(\sum_r |a_{nk} x_k|^p\Big)^{\frac{1}{p}} \leq \sum_{r=0}^{\infty} A_r^1(n) \cdot 2^{\frac{r}{p}} \|x\|^{\frac{1}{p}} \\ &\leq M(A) \|x\|^{\frac{1}{p}} < \infty,\end{aligned}$$

whenever $x \in \omega_p$. So the series $\sum_{k=1}^{\infty} a_{nk} x_k$ is absolutely convergent for each n.

Now let $\frac{1}{p} + \frac{1}{q} = 1$, $0 < p < 1$; so that $q < 0$. Then for any positive integer s and for any $m \geq 2^s$, we have, for all n,

$$\sum_{k=m}^{\infty} |a_{nk}| \leq \sum_{r=s}^{\infty} \sum_r |a_{nk}| \leq \sum_{r=s}^{\infty} A_r^1(n)\, 2^r \leq M(A)\, 2^{\frac{s}{q}}.$$

Since $q < 0$ it follows that $\sum_{k=1}^{\infty} |a_{nk}|$ is uniformly convergent in n. This, together with i), implies that

$$\lim_n \sum_k a_{nk} = \sum_k \alpha_k \qquad (9)$$

Now take $x \in \omega_p$ and suppose

$$n^{-1} \sum_{k=1}^{n} |x_k - \ell|^p \to 0 \quad (n \to \infty).$$

Write

$$\sum_k a_{nk} x_k = \sum_k a_k x_k + \ell \sum_k (a_{nk} - a_k) + \sum_k (a_{nk} - a_k)(x_k - \ell) \tag{10}$$

and note that i) and ii) imply

$$\sum_{r=0}^{\infty} 2^{\frac{r}{p}} \max_r |a_k| \le M(A).$$

Hence $\sum_k |a_k x_k| < \infty$ andthe last sum in (10) has limit zero as $n \to \infty$. Thus by (9), we now have $\sum_k a_{nk} x_k \to \sum_k a_k x_k$ as $n \to \infty$, for every $x \in \omega_p$.

Necessity

Necessity of i) is trivial and we now prove the necessity of ii). Suppose that $A_n(x) = \sum_k a_{nk} x_k$ exists for each $n \geq 1$ whenever $x \in \omega_p$. Then for each n and each $r \geq 0$, the functionals $f_{rn}(x) = \sum_r a_{nk} x_k$ are in the dual space ω_p^* ; they are trivially linear and continuous since

$$|f_{rn}(x)| \le A_r^1(n)\ 2^{\frac{r}{p}} \|x\|^{\frac{1}{p}}$$

It follows from Banach-Steinhaus theorem that for each n,

$$\lim_s \sum_{r=0}^{s} f_{rn}(x) = A_n(x) \text{ is in } \omega_p^*,$$

whence

$$|A_n(x)| \le \|A_n\|\ \|x\|^{\frac{1}{p}} \tag{11}$$

For each n we take any integer $s \geq 0$ and define $x \in \omega_p$ by $x_k = 0$ for $k \geq 2^{s+1}$, $x_{N(r)} = 2^{r/p}$ sgn $a_{n,N(r)}$, $x_k = 0$ $(k \neq N(r))$ for $0 \leq r \leq s$, where $N(r)$ is such that $|a_{n;N(r)}| = \max_r |a_{nk}|$. By (11)we get

$$\sum_{r=0}^{s} 2^{\frac{r}{p}} A_r^1(n) \le \|A_n\|,$$

whence for each n,

$$\sum_{r=0}^{s} 2^{\frac{r}{p}} A_r^1(n) \le \|A_n\| < \infty \tag{12}$$

Now the argument used in the sufficiency to prove that the series defining $A_n(x)$ was absolutely convergent gives

$$|A_n(x)| \le \sum_{r=0}^{\infty} 2^{\frac{r}{p}} A_r^1(n)\ \|x\|^{\frac{1}{p}},$$

so that

$$\|A_n\| \le \sum_{r=0}^{\infty} 2^{\frac{r}{p}} A_r^1(n). \tag{13}$$

Together,(12) and (13) imply

$$\|A_n\| = \sum_{r=0}^{\infty} 2^{\frac{r}{p}} A_r^1(n).$$

Finally by Banach-Steinhaus theorem, the existence of $\lim A_n(x)$ on ω_p implies that

$$\sup_n \|A_n\| = \sup_n \sum_{r=0}^{\infty} 2^{\frac{r}{p}} A_r^1(n) < \infty,$$

which is ii). This completes the proof.

We now characterize the class $(c_0(p), c_0(q))$ which was first discussed by Maddox [19]. See also [29] and [30] which also characterize operators on generalized entire sequences.

Theorem 18

Let p be any positive sequence and let q be a bounded sequence. Then a matrix $A = (a_{nk}) \in (c_0(p), c_0(q))$ if and only if

i) $|a_{nk}|^{q_n} \to 0$ $(n \to \infty$, each $k)$,

ii) $\lim_N \limsup_n \left\{ \sum_k |a_{nk}| N^{-1/p_k} \right\}^{q_n} = 0$.

Proof

It is understood that ii) involves the existence of a positive integer M such that

$$\sup_n \sum_k |a_{nk}| M^{-\frac{1}{p_k}} < \infty \qquad (14)$$

Let us write $H = \sup q_k$ and observe that

$$|a_k + b_k|^{q_k} \le c\left(|a_k|^{q_k} + |b_k|^{q_k}\right)$$

where $c = \max(1, 2^{H-1})$. For the sufficiency we take $\epsilon > 0$ and let $x \in c_0(p)$. Since $c_0(p)$ is a (proper) subset of ℓ_∞ we have $\|x\| = \sup|x_k| < \infty$. By ii) there exists N such that

$$\sup_n \left(\sum_k |a_{nk}| N^{-\frac{1}{p_k}}\right)^{q_n} < \epsilon \qquad (15)$$

Also, there exists r such that $|x_k| < N^{-1/p_k}$ for all $k > r$. Hence

$$|A_n(x)|^{q_n} \le c\left(\|x\|^{q_n}\left[\sum_{k=1}^{r} |a_{nk}|^{\frac{q_n}{H}}\right]^H + \left[\sum_k |a_{nk}| N^{-\frac{1}{p_k}}\right]^{q_n}\right)$$

Taking $\limsup_n$ in this last inequality we see that since $q \in \ell_\infty$, i) and (15) imply $Ax \in c_0(q)$. The necessity of i) is obtained by taking $x = (e_k)$. In proving the necessity of ii) we may suppose, since $c_0(q) = c_0(q/H)$, that $\sup q_k \le 1$.

Now $(c_0(p), c_0(q)) \subset (c_0(p), \ell_\infty)$ and observe that $A \in (c_0(p), \ell_\infty)$ if and only if (14) holds for some integer M. If $p \in \ell_\infty$ then the necessity of this result is readily shown by an application of the uniform boundedness

principle; otherwise an ad hoc proof is required. Let us denote the $\limsup_n$ in ii) by ε_N, so that the sequence $(B_M, B_{M+1}, \ldots)$ is decreasing. The limit in ii) certainly exists; suppose if possible that it is equal to $3a$, where $a > 0$. Then there exists $n(1) > 1$ such that

$$\sum |a_{n(1)k}|(M+1)^{-\frac{1}{p_k}} > (2a)^{\frac{1}{q_{n(1)}}}$$

Hence by (14) there exists $k(1) > 1$ such that

$$\sum_{k > k(1)} |a_{n(1)k}|(M+1)^{-\frac{1}{p_k}} < \left(\frac{a}{4}\right)^{\frac{1}{q_{n(1)}}}$$

Define $x_k = (\operatorname{sgn} a_{n(1)k})(M+1)^{-1/p_k}$ for $1 \le k \le k(1)$.

Suppose now that

$$n(1) < n(2) < \cdots < n(i) \text{ and } k(1) < k(2) < \cdots < k(i)$$

have been chosen and that x_k has been defined for $k \le k(i)$. Choose $n(i+1) > n(i)$ such that, for $n = n(i+1)$

$$\sum |a_{nk}|(M+i+1)^{-1/p_k} > (2a)^{\frac{1}{q_n}}$$

and $\sum_{k=1}^{k(i)} |a_{nk}|^{q_n} < \frac{1}{4} a$.

Next, choose $k(i+1) > k(i)$ such that, for $n = n(i+1)$,

$$\sum_{k > k(i+1)} |a_{nk}|(M+i+1)^{-\frac{1}{p_k}} < \left(\frac{a}{4}\right)^{\frac{1}{q_n}}$$

Define $x_k = (\operatorname{sgn} a_{n(i+1)k})(M+i+1)^{-1/p_k}$ for $k(i) < k \le k(i+1)$, so that $x \in c_0(p)$. Now for $n = n(i+1)$ write the sum of $a_{nk}x_k$ over $k(i) < k \le k(i+1)$ as

$$\sum - \sum_1^{k(i)} - \sum_{k > k(i+1)}$$

Then

$$\left|\sum_{1+k(i)}^{k(i+1)} a_{nk}x_k\right|^{q_n} > 2a - \frac{1}{4}a - \frac{1}{4}a = \frac{3}{2}a \qquad (16)$$

Finally writing $A_n(x)$ in terms of three suitable sums over k we have by (16) that

$$|A_n(x)|^{q_n} > \frac{3}{2}a - \frac{1}{4}a - \left(\sum_{k>k(i+1)} |a_{nk}x_k|\right)^{q_n}$$

$$> \frac{3}{2}a - \frac{1}{4}a - \frac{1}{4}a$$

$$= a \qquad (17)$$

since $|x_k| < (M+i+1)^{-1/p_k}$ for $k > k(i+1)$. But (17) is contrary to $Ax \in c_0(q)$, whence (ii) is necessary.

6. Before characterizing some more matrix transformations we introduce below the concepts of Banach limits and almost convergence. An application of Hahn-Banach theorem to the vector space of all real bounded sequences gives rise to the concept of Banach limits and this was first introduced by Banach [1]. Lorentz [14] continued the study of Banach limits in greater detail and introduced a new concept called almost convergence.

Note that if $x \in c$, then $\ell(x) = \lim_n x_n$ is defined and ℓ is a linear functional on c. Banach first recognized certain non-negative linear functionals on ℓ_∞ which remain invariant under shift operators and which are extensions of ℓ; such functionals were later termed 'Banach limits'. To be precise, a functional $L : \ell_\infty \to \mathbb{R}$ is called a Banach limit if

(1) L is linear,

(2) $L(x) \geq 0$ if $x_n \geq 0$ for all n,

(3) $L(x) = L(\sigma x)$ where σ denotes the shift operator, that is $\sigma(x) = \sigma(\{x_n\}) = \{x_{n+1}\}$,

(4) $L(e) = 1$.

Let α denote the set of all Banach limits.

<u>Theorem 19</u>

$$\alpha \neq \phi.$$

<u>Proof</u>

Let $q : \ell_\infty \to \mathbb{R}$ be defined by

$$q(x) = \limsup_n \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Then

$$-q(-x) = \liminf_n \frac{x_1 + x_2 + \cdots + x_n}{n}$$

If $x \in c$ we have

$$\ell(x) = \lim_n x_n = \lim_n \frac{x_1 + x_2 + \cdots + x_n}{n} = q(x)$$

Moreover $q(x+y) \leq q(x) + q(y)$ and $q(\lambda x) = \lambda q(x)$ for $\lambda \geq 0$. By applying Hahn-Banach theorem we conclude that there is an extension L of ℓ from c to ℓ_∞ and

$$-q(-x) \leq L(x) \leq q(x)$$

for all $x \in \ell_\infty$. Now

$$q(x - \sigma x) = \limsup_n \frac{x_{n+1} - x_1}{n} = 0$$

Thus L is a Banach limit and this completes the proof.

Theorem 20

If $L \in \alpha$, then for $x \in \ell_\infty$,

$$\liminf_n x_n \leq L(x) \leq \limsup_n x_n .$$

Proof

Since $\lim_n \inf x_n = \lim_k \inf_{n \geq k} x_n$, $\lim_n \sup x_n = \lim_k \sup_{n \geq k} x_n$ and since Banach limit remains invariant under the shift operator, we need only show that

$$\inf x_n \leq L(x) \leq \sup x_n .$$

For this, given $\epsilon > 0$ choose n_0 such that

$$\inf x_n \leq x_{n_0} \leq \inf x_n + \epsilon,$$

whence

$$x_n + \epsilon - x_{n_0} > 0 \text{ for all } n.$$

From Axioms (2) and (4) we get

$$L(x) + \epsilon \geq x_{n_0} \geq \inf x_n,$$

so that $L(x) \geq \inf x_n$. Similarly $L(x) \leq \sup x_n$.

In order to characterize the sequences for which all Banach limits coincide, it is convenient to consider another sublinear functional. Let $p : \ell_\infty \to \mathbb{R}$ be defined by

$$p(x) = \inf \limsup_j \frac{1}{k} \sum_{i=1}^{k} x_{n_i+j}$$

where the infimum is taken over all finite sets of integers $n_1, n_2, \ldots, n_k$. Observe that

$$-p(-x) = \sup \liminf_j \frac{1}{k} \sum_{i=1}^{k} x_{n_i+j}$$

Theorem 21

If $L \in \alpha$ and $x \in \ell_\infty$, then

$$\liminf x_n \leq -p(-x) \leq L(x) \leq p(x) \leq \limsup x_n .$$

Proof

It clearly follows that $\lim\inf x_n \leq -p(-x)$ and $p(x) \leq \lim\sup x_n$.

We apply Theorem 20 to the sequence $\left\{ \frac{1}{k} \sum_{i=1}^{k} x_{n_i} + j \right\}$ for fixed $k, n_1, n_2, \ldots n_k$ and we get

$$\liminf_j \frac{1}{k} \sum_{i=1}^{k} x_{n_i+j} \leq L\left(\frac{1}{k} \sum_{i=1}^{k} x_{n_i+j}\right) = L(x)$$

$$\leq \limsup_j \frac{1}{k} \sum_{i=1}^{k} x_{n_i+j}$$

This completes the proof.

Theorem 22

p is a sublinear functional on ℓ_∞.

Proof

We need only prove $p(x + y) \leq p(x) + p(y)$ for $x, y \in \ell_\infty$. Let $x, y \in \ell_\infty$ and let $\epsilon > 0$ be given. There are $k, n_1, n_2, \ldots n_k$ such that

$$\limsup_j \frac{1}{k} \sum_{i=1}^{k} x_{n_i+j} < p(x) + \frac{\epsilon}{2}$$

and $\ell, m_1, m_2, \ldots, m_\ell$ such that

$$\limsup_j \frac{1}{\ell} \sum_{i=1}^{\ell} x_{m_i+j} < p(y) + \frac{\epsilon}{2}$$

Consider the $k\ell$ integers $n_r + m_s$, $r = 1, \ldots, k$; $s = 1, 2, \ldots \ell$. Using the subaddivity property of lim sup we get

$$p(x+y) \leq \limsup_j \frac{1}{kl} \sum_{r=1}^{k} \sum_{s=1}^{l} (x_{n_r+m_s+j} + y_{n_r+m_s+j})$$

$$\leq \limsup_j \frac{1}{kl} \sum_{r=1}^{k} \sum_{s=1}^{l} x_{n_r+m_s+j}$$

$$+ \limsup_j \frac{1}{kl} \sum_{r=1}^{k} \sum_{s=1}^{l} y_{n_r+m_s+j}$$

Continuing with the first term only, it is no larger than

$$\frac{1}{l} \sum_{s=1}^{l} \limsup_j \frac{1}{k} \sum_{r=1}^{k} x_{n_r+m_s+j}$$

But each

$$\limsup \frac{1}{k} \sum_{r=1}^{k} x_{n_r+m_s+j} < p(x) + \frac{\epsilon}{2},$$

so that the average also has this property. Similarly the second term is less than $p(y) + \frac{\epsilon}{2}$. Thus $p(x+y) \leq p(x) + p(y) + \epsilon$ for every $\epsilon > 0$ and this completes the proof.

We are now ready to give an alternative proof of Theorem 19 by using the functional p instead of q. Note that p is a sublinear functional on ℓ_∞ which dominates the linear functional ℓ on c. Therefore by Hahn-Banach Theorem there is a linear functional L on ℓ_∞ which extends ℓ and which is such that

$$-p(-x) \leq L(x) \leq p(x)$$

for all $x \in \ell_\infty$. In order to show that $L \in \alpha$ we need only to prove that L is invariant under the shift operator. For any $x \in \ell_\infty$ observe that

$$L(\sigma x) - L(x) = L(\sigma x - x)$$

$$\leq p(\sigma x - x)$$

$$\leq \limsup_j \frac{1}{k} \sum_{i=1}^{k} (x_{n_i+1+j} - x_{n_i+j})$$

So if we take $n_i = i$, $i = 1,2,\ldots,k$, we get

$$p(\sigma x - x) \leq \limsup_j \frac{1}{k} \sum_{i=1}^{k} (x_{i+1+j} - x_{i+j})$$

$$= \frac{1}{k} \limsup_j (x_{k+1+j} - x_{1+j})$$

If M is an upper bound for $|x_n|$, for all n, then for arbitrary k we have

$$p(\sigma x - x) \leq \frac{2M}{k},$$

so that $p(\sigma x - x) \leq 0$ and $L(\sigma x) \leq L(x)$. Similarly

$$L(x) - L(\sigma x) \leq p(\sigma x - x) \leq 0.$$

Hence $L(\sigma x) = L(x)$ and this completes the proof.

We now characterize the sequences for which all Banach limits coincide. If $x \in c$, then clearly $L(x) = \ell(x) = \lim_n x_n$ for all $L \in \alpha$. However, there are non-convergent sequences also for which all Banach limits coincide. For example, if $x = (1,0,1,0,\ldots)$, then

$$L(x) = \frac{1}{2}[L(x) + L(\sigma x)] = \frac{1}{2} L(x + \sigma x)$$

$$= \frac{1}{2} L(e) = \frac{1}{2}$$

for all $L \in \alpha$. We first obtain the following result for the case of uniqueness of Banach limits.

Theorem 23

All Banach limits of $x \in \ell_\infty$ coincide if and only if $p(x) = -p(-x)$.

Proof

The sufficiency follows from Theorem 21. For the necessity, let $-p(-x) < p(x)$. Then $x \notin c$. There are distinct extensions of ℓ from c to $(c \cup \{x\})$, since in the proof of Hahn-Banach theorem the extension can have any value at x on the interval.

$$\left(\sup_{x \in c} [-p(-y-x) - \ell(y)], \inf_{x \in c} [p(y+x) - \ell(y)] \right)$$

and these end points reduce to $-p(-x)$ and $p(x)$, respectively.

A sequence $x \in \ell_\infty$ is said to be <u>almost convergent</u> if all its Banach limits coincide and the set of all almost convergent sequences is denoted by $\hat{c}$. Lorentz [14] obtained the following characterization for almost convergent sequences. Subsequently Banach limits and almost convergence have also been discussed in [5], [8], [20], [24], [25], [28].

<u>Theorem 24</u>

$x \in \hat{c}$ if and only if there is a complex number s such that

$$\lim_{p \to \infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s$$

holds uniformly in n.

<u>Proof</u>

Let $\epsilon > 0$ be given. Suppose that x is almost convergent. There are $k, n_1, n_2, \ldots n_k$ such that

$$\limsup_j \frac{1}{k} \sum_{i=1}^{k} x_{n_i + j} < s + \epsilon .$$

Thus, if j is sufficiently large, say $j \geq N$,

$$\frac{1}{k} \sum_{i=1}^{k} x_{n_i + j} < s + \epsilon .$$

Replacing j by $j - N$ and n_i by $m_i = n_i + N$, for $i = 1,2,\ldots,k$, we obtain

$$\frac{1}{k} \sum_{i=1}^{k} x_{m_i + j} < s + \epsilon , \quad j = 0,1,2,\ldots$$

Now for any $n = 1,2,\ldots$ and $p = 1,2,\ldots$

$$\frac{1}{p} \sum_{j=1}^{p} \frac{1}{k} \sum_{i=1}^{k} x_{m_i + j + n} < s + \epsilon .$$

The sum can be evaluated as

$$\frac{1}{pk} \sum_{i=1}^{k} \sum_{j=m_i+1}^{m_i+p} x_{j+n}$$
$$= \frac{1}{pk} \sum_{i=1}^{k} \sum_{j=1}^{p-1} \left[x_{j+n} + \left(x_{m_i+1+j+n} - x_{j+n} \right) \right]$$
$$= \frac{1}{p} \sum_{j=0}^{p-1} x_{j+n} + \frac{1}{p} \left[\frac{1}{k} \sum_{i=0}^{k} \sum_{j=0}^{p-1} \left(x_{m_i+1+j+n} - x_{j+n} \right) \right]$$

The quantity in brackets is bounded for large p, uniformly in n. In fact, if kp exceeds m_i for all $i = 1,2,\ldots,k$, then cancellations occur in the inner sum, leaving only $2m_i$ terms. Thus, if M is a bound for $|x_n|$ for all $n = 1,2,\ldots$, the quantity in brackets does not exceed in absolute value the quantity

$$\frac{2M}{k} \sum_{i=1}^{k} m_i .$$

It follows that p can be chosen so large that

$$\frac{1}{p} \sum_{j=0}^{p-1} x_{j+n} < s + 2\epsilon$$

uniformly in n. The reverse inequality is obtained in a similar argument using $-p(-x) = s$.

Conversely, we have

$$s - \epsilon < \frac{1}{p} \left(x_n + x_{n+1} + \cdots + x_{n+p-1} \right) < s + \epsilon$$

for all n and all sufficiently large p. It then readily follows that

$$s - \epsilon \leq -p(-x) \leq p(x) \leq s + \epsilon$$

So that x is almost convergent and this completes the proof.

One would normally expect that almost convergence must be related to some concept like absolute almost convergence in the same manner as convergence is related to absolute convergence. This concept was first introduced by G. Das *) and was later discussed by Nanda [22] and Das <u>et al</u>. [6]. To describe this concept write for any sequence $x \in \ell_\infty$,

$$d_{p,n} = d_{p,n}(x) = \frac{1}{p} \sum_{i=0}^{p-1} x_{n+i} \quad (p > 0, n \geq 0)$$

---

*) See the proceedings of the British Mathematical Colloquium, Birmingham University, 1968.

Given an infinite series

$$\sum a_n$$

which we will denote by a, let

$$x_n = a_0 + a_1 + \dots + a_n .$$

We now extend the definition of $d_{p,n}$ to $p = 0$ by taking

$$d_{0,n} = d_{0,n}(x) = x_{n-1} .$$

We then write for $p,n \geqslant 0$,

$$\varphi_{p,n} = \varphi_{p,n}(a) = d_{p+1,n} - d_{p,n}$$

A straightforward calculation shows that

$$\varphi_{0,n} = a_n$$

$$\varphi_{p,n} = \frac{1}{p(p+1)} \sum_{i=1}^{p} i\, a_{n+i} \qquad (p \geq 1)$$

The series a or the sequence x is said to be absolutely almost convergent if $\sum_p |\varphi_{p,n}|$ converges uniformly in n. We denote the set of all absolutely almost ~~convergent~~ series by $\hat{\ell}$. It is clear that if x is absolutely almost convergent, then it is almost convergent. The converse of this is false; indeed, even if x is convergent, it need not be absolutely almost convergent. It may be noted that absolute almost convergence implies absolute summability $|c,1|$; but it is well-known that there are convergent sequences which are not summable $|c,1|$.

The definition of $\hat{\ell}$ has also been extended to a more general space $\hat{\ell}(r)$. To define this, let $r = \{r_p\}$ be a bounded sequence of positive numbers. Then

$$\hat{\ell}(r) = \Big\{ a : \quad \sum_p |\varphi_{p,n}|^{r_p} \text{ converges uniformly in } n \Big\}$$

and

$$\hat{\hat{\ell}}(r) = \Big\{ a : \quad \sup_n \sum_p |\varphi_{p,n}|^{r_p} < \infty \Big\}.$$

If $\{r_p\}$ is constant (which we will denote also by r), we write $\hat{\ell}_r$ and $\hat{\hat{\ell}}_r$ in place of $\hat{\ell}(r)$ and $\hat{\hat{\ell}}(r)$, respectively. We omit the suffix r in the case $r = 1$ and this agrees with the definition of $\hat{\ell}$ already given.

If $\alpha_{p,n}$ is any non-negative real-valued function of two integer variables p and n, there is no relation of implication between the two assertions:

$$\sum_p \alpha_{p,n} \quad \text{converges uniformly in } n, \quad (*)$$

$$\sup_n \sum_p \alpha_{p,n} < \infty \ (**).$$

But in case of $\varphi_{p,n}$ (*) implies (**). In other words we have the following theorem.

Theorem 25

$$\hat{\ell}(r) \subset \hat{\hat{\ell}}(r) \quad .$$

Proof

Suppose that $a \in \hat{\ell}(r)$. By the definition, there is an integer K such that

$$\sum_{p \geq K} |\varphi_{p,n}|^{r_p} \leq 1 \tag{18}$$

Hence it is enough to show that, for fixed r, $|\varphi_{p,n}|^{r_p}$ is bounded, or, equivalently, that $\varphi_{p,n}$ is bounded. Now it follows from (18) that $|\varphi_{p,n}| \leqslant 1$ for $p \geqslant K$ and all n. But, if $p \geqslant 1$,

$$a_{p+n} = (p+1)\,\varphi_{p,n} - (p-1)\,\varphi_{p-1,n} \tag{19}$$

Applying (19) with any fixed $p \geqslant K + 1$, we deduce that a_i is bounded. Hence $\varphi_{p,n}$ is bounded for all p,n and this proves the theorem.

We now define matrices for the spaces $\hat{\ell}(r)$ and $\hat{\hat{\ell}}(r)$. For this write $M = \max(1, \sup \gamma_p)$. Define

$$g_r(a) = \sup_n \Big(\sum_p |\varphi_{p,n}|^{r_p} \Big)^{1/M}.$$

This exists for $a \in \hat{\ell}(r)$ in virtue of Theorem 25 and for $a \in \hat{\hat{\ell}}(r)$ from the defining property of $\hat{\hat{\ell}}(r)$. We have

Theorem 26

$\hat{\ell}(r)$ is a complete linear metric space paranormed by g_r. $\hat{\hat{\ell}}(r)$ has the same property if $\inf \gamma_p > 0$.

Proof

The proof uses standard techniques and is therefore left to the reader. But there is one difference between $\hat{\ell}(r)$ and $\hat{\hat{\ell}}(r)$ in the proof and we now mention that. As one step in the proof we have to show that, for fixed a, $\lambda a \to 0$ as $\lambda \to 0$. If $a \in \hat{\ell}(r)$, then given $\epsilon > 0$ there is K such that

$$\sum_{p \geq K} |\varphi_{p,n}|^{r_p} < \epsilon \tag{20}$$

If $\lambda \leq 1$,

$$\sum_{p \geq K} |\varphi_{p,n}(\lambda a)|^{r_p} \leq \sum_{p \geq K} |\varphi_{p,n}(a)|^{r_p} < \epsilon$$

and since, for fixed K,

$$\sum_{p=0}^{K-1} |\varphi_{p,n}(\lambda a)|^{r_p} \longrightarrow 0$$

as $\lambda \to 0$, the result follows. If we are given only that $a \in \hat{\hat{\ell}}(r)$ we cannot assert (20). We now assume that r_p is bounded away from zero. Then there is some constant $\delta > 0$ such that $r_p \geq \delta$ (for all p). Hence for $|\lambda| \leq 1$, $|\lambda|^{r_p} \leq |\lambda|^{\delta}$, so that

$$g_r(\lambda a) \leq |\lambda|^{\delta} g_r(a)$$

and this completes the proof.

The following theorem gives the inclusion between ℓ_r and $\hat{\ell}_r$ spaces.

Theorem 27

If $r \geq 1$, then $\ell_r \subset \hat{\ell}_r$ and the inclusion is proper.

Proof

Let $a \in \ell_r$. If $p \geq 1$, we have by Hölder's inequality when $r > 1$ and trivially when $r = 1$,

$$|\varphi_{p,n}|^r \leq \frac{1}{p(p+1)^r} \sum_{i=1}^{p} |a_{n+i}|^r$$

Hence

$$\sum_{p=1}^{\infty} |\varphi_{p,n}|^r \leq \sum_{i=1}^{\infty} i^r |a_{n+i}|^r \sum_{p=i}^{\infty} \frac{1}{p(p+1)^r}$$
$$\leq \sum_{i=1}^{\infty} |a_{n+i}|^r.$$

Thus since $\varphi_{0,n} = a_n$, we deduce that

$$\sum_p |\varphi_{p,n}|^r \leq \sum_{i=n}^{\infty} |a_i|^r$$

Hence the hypothesis of Lemma 1 are satisfied with $a_{p,n} = |\varphi_{p,n}|^r$ and therefore the result follows from Lemma 1.

To show that the inlcusion is proper, we take

$$a_n = 0 \qquad (n = 0)$$
$$= (-1)^n n^{-\frac{1}{r}} \qquad (n \geq 1).$$

Then it is easy to show that $a \notin \ell_r$, but $a \in \hat{\ell}_r$.

7. We are now ready to characterize the matrix transformations connecting $\hat{c}$. Our first result is the class $(\hat{c}, c)$ which is due to Lorentz [14].

Theorem 28

Let $A \in (c,c,P)$. Then $A \in (\hat{c}, c)$, if and only if

$$\lim_{m \to \infty} \sum_n |a_{m,n} - a_{m,n+1}| = 0 \tag{21}$$

Proof

Assume that the condition (21) holds. Let x be almost convergent and $\text{Lim}\, x_n = s$. For any $\epsilon > 0$ we can find a natural number p such that

$$\frac{1}{p}(x_n + x_{n+1} + \cdots + x_{n+p-1}) = s + \alpha_n, \quad |\alpha_n| < \epsilon, \quad n = 0, 1, \cdots$$

Multiplying by a_{mn} and adding we get

$$\frac{1}{p} \sum_n a_{mn}(x_n + x_{n+1} + \cdots + x_{n+p-1}) = sA_m + \sum_n a_{mn}\alpha_n \tag{22}$$

where $A_m = \sum_n a_{mn} \to 1$. Also $a_{mn} \to 0$ as $m \to \infty$, we have

$$\frac{1}{p}\sum_n a_{mn}\left(x_n+x_{n+1}+\cdots+x_{n+p-1}\right)$$
$$= o(1)+\sum_{n=p-1}^{\infty} x_n\,\frac{1}{p}\left(a_{m,n-p+1}+\cdots+a_{mn}\right)$$
$$= y_m+\sum_{n=p-1}^{\infty} x_n\left[\frac{1}{p}\left(a_{m,n-p+1}+\cdots+a_{mn}\right)-a_{mn}\right]+o(1) \tag{23}$$

where $y_m=\sum_n a_{mn}x_n$. The absolute value of the sum on the right hand side of (23) does not exceed

$$\frac{1}{p}\sum_{n=p-1}^{\infty}\left|\left(a_{m,n-p+1}+\cdots+a_{mn}\right)-p\,a_{mn}\right|\,\|x\|$$
$$\leq \frac{1}{p}\,\|x\|\sum_{\rho=0}^{p-1}\sum_{n=p-1}^{\infty}\left|a_{m,n-\rho}-a_{mn}\right|$$
$$\leq \frac{1}{p}\,\|x\|\sum_{\rho=0}^{p-1}\rho\sum_{n=0}^{\infty}\left|a_{mn}-a_{m,n+1}\right|$$
$$= \frac{p-1}{2}\,\|x\|\sum_{n=0}^{\infty}\left|a_{mn}-a_{m,n+1}\right|$$

From (22) and (23) we now have

$$y_m = sA_m+\sum_{n=p}^{\infty} a_{mn}\alpha_n + o(1).$$

Now $sA_m = s + o(1)$, $\left|\sum_n a_{mn}\alpha_n\right| \leq M\epsilon$ with $M=\sup\sum_n' |a_{mn}|$. Thus for sufficiently large n we certainly have $|y_m - s| \leq (M+1)\epsilon$. Therefore $\lim y_m = s$. This means that (21) is sufficient.

Now assume that (21) does not hold. We now construct a sequence $\{x_n\}$ for which $\operatorname{Lim} x_n = 0$ but which is not summable by the matrix A. By the hypothesis, there exists an $\epsilon > 0$, such that for an infinity of m,

$$\sum_n\left|a_{mn}-a_{m,n+1}\right| > 8\epsilon$$

For every such m we either have

$$\sum_l\left|a_{m,2l}-a_{m,2l+1}\right| > 4\epsilon$$

or

$$\sum_l\left|a_{m,2l+1}-a_{m,2l+2}\right| > 4\epsilon$$

By recurrence we now construct three increasing sequences of natural numbers $\{m_k\}$, $\{p_k\}$ and $\{q_k\}$ where $q_{-1} = 0 < p_1 < q_1 < p_2 < \ldots$ will hold. We first choose m_1, p_1, q_1, such that

$$|a_{m_1,0}| < \frac{\epsilon}{2},\quad \sum_{n=q_1+1}^{\infty}|a_{m_1,n}| < \frac{\epsilon}{2},$$
$$\sum_{l=0}^{\frac{q_1-p_1-1}{2}}\left|a_{m_1,p_1+2l}-a_{m_1,p_1+2l+1}\right| > 2\epsilon.$$

If the numbers m_ν, p_ν, q_ν, $\nu = 1,2,\ldots,k-1$ are already known, m_k, p_k, q_k (where $q_{k-1} < p_k < q_k$ and one of the numbers p_k, q_k even, the other, odd) are chosen such that

$$\sum_{n=0}^{q_k-1}|a_{m_k,n}| < \frac{\epsilon}{2},$$
$$\sum_{l=0}^{\frac{q_k-p_k-1}{2}}\left|a_{m_k,p_k+2l}-a_{m_k,p_k+2l+1}\right| > 2\epsilon,$$
$$\sum_{n=q_k+1}^{\infty}|a_{m_k,n}| < \frac{\epsilon}{2}.$$

Define the sequence $\{x_n\}$ as follows:

$$x_{p_k+2l} = (-1)^k\,\mathrm{sgn}\left(a_{m_k,p_k+2l}-a_{m_k,p_k+2l+1}\right)$$
$$x_{p_k+2l+1} = -x_{p_k+2l}$$
$$x_n = 0 \quad \text{for } q_{k-1} < n < p_k,\ k = 1,2,\cdots,\ l = 0,1,\cdots,\frac{q_k-p_k-1}{2}.$$

Under these conditions we have

$$|y_{m_k}| = \left|\sum_n a_{m_k,n}\, x_n\right|$$
$$> \sum_{\ell=0}^{\frac{q_k - p_k - 1}{2}} \left| a_{m_k, p_k+2\ell} - a_{m_k, p_k+2\ell-1}\right| - \frac{\epsilon}{2} - \frac{\epsilon}{2}$$
$$> \epsilon$$

and

$$\operatorname{sgn} y_{m_k} = (-1)^k .$$

Hence it follows that the sequence $\{y_m\}$ diverges. It is easy to show that $\operatorname{Lim} x_n = 0$ and this completes the proof.

In order to describe some more matrix transformations connecting $\hat{c}$ and $\hat{\hat{\ell}}_p$ we write

$$a(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i,k} ,$$

$$b(n,k,m) = \frac{1}{m(m+1)} \sum_{i=0}^{m} i\, a_{n+i,k} \qquad (m \geq 1)$$

and

$$b(n,k,0) = a_{nk} .$$

<u>Theorem 29</u>

(a) $A \in (\ell_\infty, \hat{c})$ if and only if

(i) $\sup_m \sum_k |a(n,k,m)| < \infty$,

(ii) $\lim_{m\to\infty} a(n,k,m) = \alpha_k$ uniformly in n,

(iii) $\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| = 0$ uniformly in n.

(b) $A \in (\ell_\infty(p), \hat{c})$ if and only if (i) and (ii) of part (a) hold and

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k|\, N^{\frac{1}{p_k}} = 0$$ uniformly in n for every integer $N > 1$.

Part (a) is due to Duran [7] and (b) is due to Nanda [22]. Proof of Part (a) is similar to Theorem 6 and that of Part (b) is similar to Theorem 7.

<u>Theorem 30</u>

(a) $A \in (c, \hat{c})$ if and only if

(i) Conditions (i) and (ii) of Theorem 29 hold.

(ii) $\lim_{m\to\infty} \sum_k a(n,k,m) = \alpha$ uniformly in n.

(b) $A \in (c, \hat{c}, P)$ if and only if

(i) Condition (i) of Theorem 29 holds,

(ii) $\lim_{m\to\infty} a(n,k,m) = 0$ uniformly in n,

(iii) $\lim_{m\to\infty} \sum_k a(n,k,m) = 1$ uniformly in n.

This theorem is due to King [11] and the proof is similar to that of Theorem 4 and and Theorem 5.

<u>Theorem 31</u>

(a) $A \in (\ell_1, \hat{\hat{\ell}}_p)$ if and only if

$$\sup_{n,k} \sum_m |b(n,k,m)|^p < \infty$$

(b) $A \in (\ell_1, \hat{\hat{\ell}}_p, P)$ if and only if

(i) $\sup_{n,k} \sum_m |b(n,k,m)| < \infty$

(ii) $\sum_m b(n,k,m) = 1$ for all n and k.

This is due to Das <u>et al</u>. [6] and the proof is similar to that of Theorem 12 and Theorem 13.

Theorem 32

$A \in (\ell(p), \hat{c})$ if and only if

(i) there is an integer $B > 1$ such that for all n,

$$C(n,B) < \infty, \quad (1 < p_k < \infty),$$

$$\sup_{m,k} |a(n,k,m)|^{p_k} < \infty, \quad (0 < p_k \leq 1)$$

(ii) $\lim_{m\to\infty} a(n,k,m) = \alpha_k$ uniformly in n.

This theorem is due to Nanda [21] and the proof is similar to that of Theorems 15 and 16.

8. PROBLEMS FOR FURTHER STUDY

Stieglitz and Tietz [33] presented a number of results on matrix transformations from a sequence space X into a sequence space Y in a tabular form. This table can be further extended and this will give rise to several new problems for further study. We now quote some definitions which will be needed in the sequel.

We write

$$Sx = \left(\sum_{k=0}^{n} x_k\right),$$

$$\Delta^{\alpha} x_n = \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} x_{n+k}.$$

We define

$$m_s = \{x : Sx \in \ell_\infty\},$$

$$c_s = \{x : Sx \in c\},$$

$$(c_0)_s = \{x : Sx \in c_0\},$$

$$q^{\alpha} = c \cap \left\{x : \sum_n \binom{n+\alpha-1}{n} |\Delta^{\alpha} x_n| < \infty\right\} \quad (\alpha = 1, 2, \cdots),$$

$$bv = \left\{x : \sum_n |x_n - x_{n-1}| < \infty\right\},$$

$$bv_0 = bv \cap c_0.$$

Let $\{p_n\}$ be a bounded sequence of positive real numbers and $\{v_n\}$ any fixed sequence of non-zero complex numbers satisfying

$$\liminf |v_n|^{\frac{1}{n}} = r \quad (0 < r \leq \infty).$$

Define a function $\Lambda : \mathbb{C} \to \mathbb{C}$ by

$$\Lambda(z) = \sum \frac{z^n}{v_n}.$$

Obviously Λ is an analytic function in the disc $\{z : |z| < r\}$. We define (see Srivastava et al. [32]).

$$D_0^{\Lambda}(p) = \left\{f : f(z) = \sum a_n z^n, \ |a_n v_n|^{p_n} \to 0 \text{ as } n \to \infty\right\},$$

$$D_\infty^{\Lambda}(p) = \left\{f : f(z) = \sum a_n z^n, \ \sup_n |a_n v_n|^{p_n} < \infty\right\},$$

$$D^{\Lambda}(p) = \left\{f : f(z) = \sum a_n z^n, \ \sum_n |a_n v_n|^{p_n} < \infty\right\}.$$

The above sets generalize several spaces of analytic functions and several known sequence spaces (see [23,24]). If $p_n = 1$ for all n, then $D_0^{\Lambda}(p) = D_0^{\Lambda}$, $D_\infty^{\Lambda}(p) = D_\infty^{\Lambda}$ and $D^{\Lambda}(p) = D_1^{\Lambda}$ and these spaces were introduced and studied by Srivatastava [31]. If $p_n = p$ for all n we write D_p^{Λ} for $D^{\Lambda}(p)$.

Hahn [9] characterized all matrix transformations (X,Y) by considering $X = c_0, c, \ell_\infty, \ell_p, bv, bv_0, c_s$ and $Y = c, \ell_\infty$. A table has been prepared by Stieglitz and Tietz [33] by taking $X,Y = \ell_\infty, c_0, c, m_s, c_s, (c_0)_s, \ell_p, q^{\alpha}, bv$ and bv_0. It will be interesting to extend the table by taking $X,Y = \ell_\infty(p), c(p), c_0(p), \ell(p), \omega(p), \hat{c}, \hat{c}_0, \hat{\ell}_p, \hat{\hat{\ell}}_p, D_0^{\Lambda}, D_\infty^{\Lambda}, D_p^{\Lambda}, D_0^{\Lambda}(p), D_\infty^{\Lambda}(p)$ and $D^{\Lambda}(p)$ along with $\ell_\infty, c, c_0, m_s, c_s, (c_0)_s, \ell_p, q^{\alpha}$ and bv_0. This will provide some new problems for further study and it will be interesting to try to solve these problems.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy, and the Indian Institute of Technology, Kharagpur, India, for granting him leave of absence. Thanks are also due to Professor G. Das and Professor Sribatsa Nanda whose help and encouragement have always been a source of inspiration for him.

APPENDIX

In the appendix we present the table given in [33] and list the results. In the table the number, say 48, stands for the result on the transformation from the space occurring in the row containing 48 into the space occurring in the coloumn containing 48. In the list of results N,K and K_* denote arbitrary finite subsets of the set of all positive integers.

X \ Y	m	c	c_0	m_s	c_s	$(c_0)_s$	ℓ_r	ℓ	q^β	bv	bv_0
m	1	10	21	32	41	52	63	72	81	92	101
c	1	11	22	32	42	53	63	72	82	92	102
c_0	1	12	23	32	43	54	63	72	83	92	103
m_s	2	13	24	33	44	55	64	73	84	93	104
c_s	3	14	25	34	45	56	65	74	85	94	105
$(c_0)_s$	4	15	26	35	46	57	66	75	86	95	106
ℓ_p	5	16	27	36	47	58	67	76	87	96	107
ℓ	6	17	28	37	48	59	68	77	88	97	108
q^α	7	18	29	38	49	60	69	78	89	98	109
bv	8	19	30	39	50	61	70	79	90	99	110
bv_0	9	20	31	40	51	62	71	80	91	100	111

1. $A \in (m,m) = (c,m) = (c_0,m) \Leftrightarrow (1.1).$

(1.1) $\sup_n \sum_k |a_{nk}| < \infty .$

2. $A \in (m_s,m) \Leftrightarrow (2.1)$ and (2.2) .

(2.1) $\lim_k a_{nk} = 0$ for all n ,

(2.2) $\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty .$

3. $A \in (c_s,m) \Leftrightarrow (2.2)\ ,\ (3.1) \Leftrightarrow (3.2)$.

(3.1) $\sup_n |\lim_k a_{nk}| < \infty ,$

(3.2) $\sup_n \sum_k |a_{nk} - a_{n,k-1}| < \infty .$

4. $A \in ((c_0)_s\ ,\ m) \Leftrightarrow (2.2)$.

5. $A \in (\ell_p\ ,\ m) \Leftrightarrow (5.1)$.

(5.1) $\sup_n \sum_k |a_{nk}|^q < \infty, 1 < p < \infty\ ,\ \frac{1}{p} + \frac{1}{q} = 1$.

6. $A \in (\ell,\ m) \Leftrightarrow (6.1)$.

(6.1) $\sup_{n,k} |a_{nk}| < \infty$

7. $A \in (q^\alpha,m) \Leftrightarrow (7.1)\ ,\ (7.2)$.

(7.1) $\sup_n |\sum_k a_{nk}| < \infty ,$

(7.2) $\sup_{n,\ell} \left| \binom{\ell + \alpha - 1}{\ell}^{-1} \sum_{k=0}^{\ell} a_{nk} \binom{\ell - k + \alpha - 1}{\ell - k} \right| < \infty .$

8. $A \in (bv, m) \Leftrightarrow (7.1), (8.1) \Leftrightarrow (8.2)$.

(8.1) $\sup_{n,\ell} \left| \sum_{k=0}^{\ell} a_{nk} \right| < \infty$,

(8.2) $\sup_{n,\ell} \left| \sum_{k=\ell}^{\infty} a_{nk} \right| < \infty$.

9. $A \in (bv_0, m) \Leftrightarrow (8.1)$.

10. $A \in (m, c) \Leftrightarrow (10.1), (10.2) \Leftrightarrow (1.1), (10.1), (10.3)$

$\Leftrightarrow (10.1), (10.4)$.

(10.1) $\lim_n a_{nk}$ exists for all k,

(10.2) $\lim_n \sum_k |a_{nk}| = \sum_k |\lim_n a_{nk}|$,

(10.3) $\lim_n \sum_k |a_{nk} - \lim_n a_{nk}| = 0$,

(10.4) $\sum_k |a_{nk}|$ converges uniformly in n.

11. $A \in (c, c) \Leftrightarrow (1.1), (10.1), (11.1)$.

$A \in (c, c, P) \Leftrightarrow (1.1), (11.2), (11.3)$.

(11.1) $\lim_n \sum_k a_{nk}$ exists,

(11.2) $\lim_n a_{nk} = 0 \quad \forall k$,

(11.3) $\lim_n \sum_k a_{nk} = 1$.

12. $A \in (c_0, c) \Leftrightarrow (1.1)$ and (10.1).

13. $A \in (m_s, c) \Leftrightarrow (2.1), (13.1), (13.2) \Leftrightarrow (2.1)$

$(2.2), (13.1), (13.3) \Leftrightarrow (2.1), (10.1), (13.4)$

$\Leftrightarrow (2.1), (13.1), (13.4)$.

(13.1) $\lim_n (a_{nk} - a_{n,k+1})$ exists for all k,

(13.2) $\lim_n \sum_k |a_{nk} - a_{n,k+1}| = \sum_k |\lim_n (a_{nk} - a_{n,k+1})|$,

(13.3) $\lim_n \sum_k |a_{nk} - a_{n,k+1} - \lim_n (a_{nk} - a_{n,k+1})| = 0$,

(13.4) $\sum_k |a_{nk} - a_{n,k+1}|$ converges uniformly in n.

14. $A \in (c_s, c) \Leftrightarrow (2.2), (10.1)$.

$A \in (c_s, c, P) \Leftrightarrow (2.2), (14.1)$.

(14.1) $\lim_n a_{nk} = 1$ for all k.

15. $A \in ((c_0)_s, c) \Leftrightarrow (2.2), (13.1)$.

16. $A \in (\ell_p, c) \Leftrightarrow (5.1), (10.1), \; p > 1$.

17. $A \in (\ell, c) \Leftrightarrow (6.1), (10.1)$.

18. $A \in (q^\alpha, c) \Leftrightarrow (7.2), (10.1), (11.1)$.

19. $A \in (bv, c) \Leftrightarrow (8.1), (10.1), (11.1)$.

20. $A \in (bv_0, c) \Leftrightarrow (8.1), (10.1)$.

21. $A \in (m, c_0) \Leftrightarrow (21.1)$.

(21.1) $\lim_n \sum_k |a_{nk}| = 0$,

22. $A \in (c, c_0) \iff (1.1)\ ,\ (11.2)\ ,\ (22.1)$.

(22.1) $\lim_n \sum a_{nk} = 0$.

23. $A \in (c_0, c_0) \iff (1.1)\ ,\ (11.2)$.

24. $A \in (m_s, c_0) \iff (2.1)\ ,\ (24.1)$.

(24.1) $\lim_n \sum_k |a_{nk} - a_{n,k+1}| = 0$.

25. $A \in (c_s, c_0) \iff (2.2)\ ,\ (11.2)$.

26. $A \in \left((c_0)_s, c_0\right) \iff (2.2)\ ,\ (26.1)$.

(26.1) $\lim_n (a_{nk} - a_{n,k+1}) = 0$ for all k .

27. $A \in (\ell_p\ ,\ c_0) \iff (5.1)\ ,\ (11.2)\ ,\ p > 1$.

28. $A \in (\ell\ ,\ c_0) \iff (6.1)\ ,\ (11.2)$.

29. $A \in (q^\alpha\ ,\ c_0) \iff (7.2)\ ,\ (11.2)\ ,\ (22.1)$.

30. $A \in (bv\ ,\ c_0) \iff (8.1)\ ,\ (11.2)\ ,\ (22.1)$

$\iff (8.2)\ ,\ (11.2)\ ,\ (22.1)$.

31. $A \in (bv_0\ ,\ c_0) \iff (8.1)\ ,\ (11.2)$.

32. $A \in (m\ ,\ m_s) = (c\ ,\ m_s) = (c_0, m_s) \iff (32.1)$.

(32.1) $\sup_m \sum_k \left| \sum_{n=0}^{m} a_{nk} \right| < \infty$.

33. $A \in (m_s, m_s) \iff (2.1)\ ,\ (33.1)$.

(33.1) $\sup_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right| < \infty$.

34. $A \in (c_s, m_s) \iff (33.1)\ ,\ (34.1) \iff (34.2)$.

(34.1) $\sup_m \left| \lim_k \sum_{n=0}^{m} a_{nk} \right| < \infty$,

(34.2) $\lim_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k-1}) \right| < \infty$.

35. $A \in \left((c_0)_s\ ,\ m_s\right) \iff (33.1)$.

36. $A \in (\ell_p\ ,\ m_s) \iff (36.1)\ ,\ p > 1$.

(36.1) $\sup_m \sum_k \left| \sum_{n=0}^{m} a_{nk} \right|^q < \infty\ ,\ \frac{1}{p} + \frac{1}{q} = 1$.

(37) $A \in (\ell, m_s) \iff (37.1)$.

(37.1) $\sup_{m,k} \left| \sum_{n=0}^{m} a_{nk} \right| < \infty$,

(38) $A \in (q^\alpha\ ,\ m_s) \iff (38.1)\ ,\ (38.2)$.

(38.1) $\sup_m \left| \sum_{n=0}^{m} \sum_k a_{nk} \right| < \infty$,

(38.2) $\sup_{m,\ell} \left| \binom{\ell + \alpha - 1}{\ell}^{-1} \sum_{n=0}^{m} \sum_{k=0}^{\ell} \binom{\ell - k + \alpha - 1}{\ell - k} a_{nk} \right| < \infty$,

(39) $A \in (bv\ ,\ m_s) \iff (38.1)\ ,\ (39.1) \iff (39.2)$.

(39.1) $\sup_{m,\ell} \left| \sum_{n=0}^{m} \sum_{k=0}^{\ell} a_{nk} \right| < \infty$,

(39.2) $\sup_{m,\ell} \left| \sum_{n=0}^{m} \sum_{k=\ell}^{\infty} a_{nk} \right| < \infty$.

(40) $A \in (bv_0 , m_s) \iff (39.1)$.

(41) $A \in (m , c_s) \iff (41.1) \iff (41.2)$

$\iff (41.3) , (41.4)$.

(41.1) $\lim_m \sum_k \left| \sum_{n=0}^{m} a_{nk} \right| = \sum_k \left| \sum_n a_{nk} \right|$,

(41.2) $\lim_m \sum_k \left| \sum_{n=m}^{\infty} a_{nk} \right| = 0$,

(41.3) $\sum_k \left| \sum_{n=0}^{m} a_{nk} \right|$ converges uniformly in m ,

(41.4) $\sum_n a_{nk}$ converges for all k .

42. $A \in (c , c_s) \iff (32.1) , (41.4) , (42.1)$.

$A \in (c , c_s, P) \iff (32.1) , (42.2) , (42.3)$.

(42.1) $\sum_n \sum_k a_{nk}$ converges ,

(42.2) $\sum_n a_{nk} = 0$ for all k ,

(42.3) $\sum_n \sum_k a_{nk} = 1$.

43. $A \in (c_0 , c_s) \iff (32.1) , (41.4)$.

44. $A \in (m_s , c_s) \iff (2.1) , (44.1) \iff (2.1)$,

$(44.2) \iff (2.1) , (41.4) , (44.3) \iff (2.1) , (44.3),$

(44.4).

(44.1) $\lim_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right| = \sum_k \left| \sum_n (a_{nk} - a_{n,k+1}) \right|$,

(44.2) $\lim_m \sum_k \left| \sum_{n=m}^{\infty} (a_{nk} - a_{n,k+1}) \right| = 0$,

(44.3) $\sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right|$ converges uniformly in m ,

(44.4) $\sum_n (a_{nk} - a_{n,k+1})$ converges for all k .

45. $A \in (c_s , c_s) \iff (34.2) , (41.4),$

$A \in (c_s , c_s, P) \iff (34.2) , (45.1)$.

(45.1) $\sum_n a_{nk} = 1$ for all k .

46. $A \in ((c_0)_s , c_s) \iff (33.1) , (44.4)$.

47. $A \in (\ell_p , c_s) \iff (36.1) , (41.4) , \; p > 1$.

48. $A \in (\ell , c_s) \iff (37.1) , (41.4)$.

49. $A \in (q^\alpha , c_s) \iff (38.2) , (41.4) , (42.1)$.

50. $A \in (bv , c_s) \iff (39.1) , (41.4) , (42.1)$,

$\iff (39.2) , (41.4) , (42.1)$.

51. $A \in (bv_0 , c_s) \iff (39.1) , (41.4)$.

52. $A \in (m , (c_0)_s) \iff (52.1)$.

(52.1) $\lim_m \sum_k \left| \sum_{n=0}^{m} a_{nk} \right| = 0$.

53. $A \in (c , (c_0)_s) \iff (32.1) , (42.2) , (53.1)$.

(53.1) $\sum_n \sum_k a_{nk} = 0$.

54. $A \in (c_0 , (c_0)_s)$ <=> (32.1) , (42.2) .

55. $A \in (m_s , (c_0)_s)$ <=> (2.1) , (55.1) .

(55.1) $\lim_m \sum_K | \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) | = 0$.

56. $A \in (c_s , (c_0)_s)$ <=> (33.1) , (42.2) .

57. $A \in ((c_0)_s , (c_0)_s)$ <=> (33.1) , (57.1) .

(57.1) $\sum_n (a_{nk} - a_{n,k+1}) = 0$ for all k .

58. $A \in (\ell_p , (c_0)_s)$ <=> (36.1) , (42.2) ; $p > 1$.

59. $A \in (\ell , (c_0)_s)$ <=> (37.1) , (42.2) .

60. $A \in (q^\alpha , (c_0)_s)$ <=> (38.2) , (42.2) , (53.1) .

61. $A \in (bv , (c_0)_s)$ <=> (39.1) , (42.2) , (53.1)
<=> (39.2) , (42.2) , (53.1) .

62. $A \in (bv_0 , (c_0)_s)$ <=> (39.1) , (52.2) .

63. $A \in (m , \ell_r) = (c , \ell_r) = (c_0 , \ell_r)$ <=> (63.1) ,
<=> (63.2) ; $r > 1$.

(63.1) $\sup_K \sum_n | \sum_{k \in K} a_{nk} |^r > \infty$,

(63.2) $\sum_n | \sum_{k \in K} a_{nk} |^r$ converges for all k .

64. $A \in (m_s , \ell_r)$ <=> (2.1) , (64.1) ; $r > 1$.

(64.1) $\sup_K \sum_n | \sum_{k \in K} (a_{nk} - a_{n,k+1}) |^r < \infty$.

65. $A \in (c_s , \ell_r)$ <=> (65.1) , $r > 1$.

(65.1) $\sup_K \sum_n | \sum_{k \in K} (a_{nk} - a_{n,k-1}) |^r < \infty$.

66. $A \in ((c_0)_s , \ell_r)$ <=> (64.1) , $r > 1$.

67. $A \in (\ell_p , \ell_r)$, $p > 1$, $r > 1$ unknown,

68. $A \in (\ell , \ell_r)$ <=> (68.1) , $r > 1$.

(68.1) $\sup_k \sum_n |a_{nk}|^r < \infty$.

69. $A \in (q^\alpha , \ell_r)$ <=> (69.1) , (69.2); $r > 1$.

(69.1) $\sum_n | \sum_k a_{nk} |^r$ convergent,

(69.2) $\sup_\ell \sum_n | \binom{\ell + \alpha - 1}{\ell}^{-1} \sum_{k=0}^{\ell} a_{nk} \binom{\ell - k + \alpha - 1}{\ell - k} |^r < \infty$.

70. $A \in (bv , \ell_r)$ <=> (69.1) , (70.1) <=> (70.2), $r > 1$.

(70.1) $\sup_\ell \sum_n | \sum_{k=0}^{\ell} a_{nk} |^r < \infty$,

(70.2) $\sup_\ell \sum_n | \sum_{k=\ell}^{\infty} a_{nk} |^r < \infty$.

71. $A \in (bv_0 , \ell_r)$ <=> (70.1) , $r > 1$.

72. $A \in (m , \ell) = (c , \ell) = (c_0 , \ell)$ <=> (72.1) ,
<=> (72.2) <=> (72.3) <=> (72.4) .

(72.1) $\sup_{N,K} \left| \sum_{n \in N} \sum_{k \in K} a_{nk} \right| < \infty$,

(72.2) $\sup_{N} \sum_{k} \left| \sum_{n \in N} a_{nk} \right| < \infty$,

(72.3) $\sup_{K} \sum_{n} \left| \sum_{k \in K} a_{nk} \right| < \infty$,

(72.4) $\sum_{n} \left| \sum_{k \in K_s} a_{nk} \right|$ converges for all K_s .

73. $A \in (m_s, \ell) \iff$ (2.1) , (73.1) .

(73.1) $\sup_{N,K} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k+1}) \right| < \infty$,

74. $A \in (c_s, \ell) \iff$ (74.1) .

(74.1) $\sup_{N,K} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k-1}) \right| < \infty$.

75. $A \in ((c_0)_s, \ell) \iff$ (73.1) .

76. $A \in (\ell_p, \ell) \iff$ (76.1) , $p > 1$.

(76.1) $\sup_{N} \sum_{k} \left| \sum_{n \in N} a_{nk} \right|^q < \infty$.

77. $A \in (\ell, \ell) \iff$ (77.1) .

(77.1) $\sup_{K} \sum_{n} |a_{nk}| < \infty$.

78. $A \in (q^\alpha, \ell) \iff$ (78.1) , (78.2) .

(78.1) $\sum_{n} \left| \sum_{k} a_{nk} \right|$ convergent,

(78.2) $\sup_{\ell} \sum_{n} \left| \binom{\ell + \alpha - 1}{\ell}^{-1} \sum_{k=0}^{\ell} a_{nk} \binom{\ell - k + \alpha - 1}{\ell - k} \right| < \infty$.

79. $A \in (bv, \ell) \iff$ (78.1) , (79.1) $\iff$ (79.2) .

(79.1) $\sup_{\ell} \sum_{n} \left| \sum_{k=0}^{\ell} a_{nk} \right| < \infty$,

(79.2) $\sup_{\ell} \sum_{n} \left| \sum_{k=\ell}^{\infty} a_{nk} \right| < \infty$.

80. $A \in (bv_0, \ell) \iff$ (79.1) .

81. $A \in (m, q^\beta) \iff$ (10.1) , (10.4) , (81.1) .

(81.1) $\sup_{N,K} \left| \sum_{n \in N} \sum_{k \in K} \binom{n + \beta - 1}{n} \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} a_{n+r,k} \right| < \infty$.

82. $A \in (c, q^\beta) \iff$ (1.1) , (10.1) , (11.1) , (81.1) .

83. $A \in (c_0, q^\beta) \iff$ (1.1) , (10.1) , (81.1) .

84. $A \in (m_s, q^\beta) \iff$ (2.1) , (13.1) , (13.4) , (84.1) .

(84.1) $\sup_{N,K} \left| \sum_{n \in N} \sum_{k \in K} \binom{n + \beta - 1}{n} \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} (a_{n+r,k} - a_{n+r,k+1}) \right| < \infty$.

85. $A \in (c_s, q^\beta) \iff$ (2.2) , (10.1) , (84.1) .

86. $A \in ((c_0)_s, q^\beta) \iff$ (2.2) , (13.1) , (84.1) .

87. $A \in (\ell_p , q^\beta) \iff$ (5.1) , (10.1) , (87.1) .

(87.1) $\sup_N \sum_k | \sum_{n \in N} \binom{n+\beta-1}{n} \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} a_{n+r,k} |^q < \infty$.

88. $A \in (\ell, q^\beta) \iff$ (6.1) , (10.1) , (88.1) .

(88.1) $\sup_k \sum_n \binom{n+\beta-1}{n} | \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} a_{n+r,k} | < \infty$.

89. $A \in (q^\alpha , q^\beta) \iff$ (7.2) , (10.1) , (11.1) , (89.1) , (89.2).

(89.1) $\sum_n \binom{n+\beta-1}{n} | \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} \sum_k a_{n+r,k} |$ convergent,

(89.2) $\sup \sum_n \binom{n+\beta-1}{n} | \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} \binom{\ell+\alpha-1}{\ell}^{-1} \sum_{k=0}^{\ell} a_{n+r,k} \binom{\ell-k+\alpha-1}{\ell-k} | < \infty$.

90. $A \in (bv , q^\beta) \iff$ (8.1), (10.1), (11.1), (89.1), (90.1) $\iff$ (8.2), (10.1), (11.1), (90.2) .

(90.1) $\sup_\ell \sum_n \binom{n+\beta-1}{n} | \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} \sum_{k=0}^{\ell} a_{n+r,k} | < \infty$,

(90.2) $\sup_\ell \sum_n \binom{n+\beta-1}{n} | \sum_{r=0}^{\beta} (-1)^r \binom{\beta}{r} \sum_{k=\ell}^{\infty} a_{n+r,k} | < \infty$.

91. $A \in (bv_0 , q^\beta) \iff$ (8.1), (10.1), (90.1) .

92. $A \in (m , bv) = (c , bv) = (c_0 , bv) \iff$ (92.1) .

(92.1) $\sup_{N,K} | \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n-1,k}) | < \infty$.

93. $A \in (m_s , bv) \iff$ (92.1), (93.1) .

(93.1) $\sup_{N,K} | \sum_{n \in N} \sum_{k \in K} \left[(a_{nk} - a_{n,k+1}) - (a_{n-1,k} - a_{n-1,k+1}) \right] | < \infty$.

94. $A \in (c_s , bv) \iff$ (94.1) .

(94.1) $\sup_{N,K} | \sum_{n \in N} \sum_{k \in K} \left[(a_{nk} - a_{n-1,k}) - (a_{n,k-1} - a_{n-1,k-1}) \right] | < \infty$.

95. $A \in ((c_0)_s , bv) \iff$ (93.1) .

96. $A \in (\ell_p , bv) \iff$ (96.1) , $p > 1$.

(96.1) $\sup_N \sum_k | \sum_{n \in N} (a_{nk} - a_{n-1,k}) |^q < \infty$.

97. $A \in (\ell , bv) \iff$ (97.1) .

(97.1) $\sup_k \sum_n | a_{nk} - a_{n-1,k} | < \infty$.

98. $A \in (q^\alpha , bv) \iff$ (98.1), (98.2) .

(98.1) $\sum_n | \sum_k (a_{nk} - a_{n-1,k}) |$ convergent,

(98.2) $\sup \sum_n | \binom{\ell+\alpha-1}{\ell}^{-1} \sum_{k=0}^{\ell} (a_{nk} - a_{n-1,k}) \binom{\ell-k+\alpha-1}{\ell-k} | < \infty$.

99. $A \in (bv , bv) \iff$ (98.1), (99.1) $\iff$ (99.2) $\iff$ (99.1), (99.3) .

(99.1) $\sup_{\ell} \sum_{n} \left| \sum_{k=0}^{\ell} (a_{nk} - a_{n-1,k}) \right| < \infty$,

(99.2) $\sup_{\ell} \sum_{n} \left| \sum_{k=\ell}^{\infty} (a_{nk} - a_{n-1,k}) \right| < \infty$,

(99.3) $\sum_{k} a_{nk}$ converges for all n.

100. $A \in (bv_o, bv)$ <=> (99.1).

101. $A \in (m, bv_o)$ <=> (21.1), (92.1).

102. $A \in (c, bv_o)$ <=> (11.2), (22.1), (92.1).

103. $A \in (c_o, bv_o)$ <=> (11.2), (92.1).

104. $A \in (m_s, bv_o)$ <=> (2.1), (24.1), (93.1).

105. $A \in (c_s, bv_o)$ <=> (11.2), (94.1).

106. $A \in ((c_o)_s, bv_o)$ <=> (26.1), (93.1).

107. $A \in (\ell_p, bv_o)$ <=> (11.2), (96.1).

108. $A \in (\ell, bv_o)$ <=> (11.2), (97.1).

109. $A \in (q^{\alpha}, bv_o)$ <=> (11.2), (22.1), (98.1), (98.2).

110. $A \in (bv, bv_o)$ <=> (11.2), (22.1), (99.1)

<=> (11.2), (22.1), (99.2).

111. $A \in (bv_o, bv_o)$ <=> (11.2), (99.1).

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