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HAMILTONIAN STRUCTURES OF SOME NON-LINEAR EVOLUTION EQUATIONS *

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ABSTRACT

The Hamiltonian structure of the O(2,1) non-linear sigma model, generalized AKNS equations, are discussed. By reducing the O(2,1) non-linear sigma model to its Hamiltonian form some new conservation laws are derived. A new hierarchy of non-linear evolution equations is proposed and shown to be generalized Hamiltonian equations with an infinite number of conservation laws.

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I. INTRODUCTION

Both from mathematical and from physical points of view the theory of Hamiltonian systems is beautiful and elegant. The great importance of the Hamiltonian theory comes from its deep physical insight. As is well known, the early development of quantum mechanics and statistical mechanics is based entirely on Hamiltonian equations. Moreover, the great significance of the Hamiltonian formalism also comes from the fact that many branches of mathematics are involved in the recent development of the theory, such as the theory of representation of Lie group, theory of canonical operators and symplectic manifolds of finite or infinite dimensions. In the past fifteen years the emergence of the soliton theory gives a new impulsion to the study of the Hamiltonian theory. Since the pioneer work [1)-5)], many important non-linear equations, arising from various branches of physics, such as the Yang-Mills equations, Ernst equations, isotropic Heisenberg equations, Thirring model, and non-linear sigma models, are found to be completely integrable Hamiltonian equations which can be solved by the powerful IST technique [6)].

In the soliton theory, the non-linear evolution equations (NLEEs) are usually derived as an isospectral deformation equation. To be more precise, let

$$\psi_x = U\psi \quad , \qquad \psi_t = V\psi \tag{1.1}$$

be a couple of linear equations with $\Psi \equiv (\psi_1,\ldots,\psi_N)^T$ being an N-vector, and U and V being two $N \times N$ matrices whose entries are dependent on some potential $u = u(x,t) = (u^1,\ldots,u^M)$ and a spectral parameter λ. In most cases the matrices U and V are taken from some Lie algebra such as $sl(N)$. Eq.(1.1) can be written as

$$d\psi = \Omega\psi \quad , \tag{1.2}$$

where $\Omega = Udx + Vdt$ is a one-form related to the spectral problem (1.1). Taking the exterior derivatives on both sides of (1.2), we obtain the following important "zero-curvature" condition

$$d\Omega = \Omega \wedge \Omega \tag{1.3}$$

where $\wedge$ represents the wedge product. In terms of matrices U and V, this integrability condition (1.3) can be written equivalently to

$$U - V + [U , V] = 0 \quad , \tag{1.4}$$

where $[U,V] = UV - VU$ denotes the usual commutator of matrices U and V. It is known that for a given matrix U there always exist different forms of V, usually taking the form of $V = \sum \lambda^{-k} V_k$, such that Eq.(1.4) reduces to a hierarchy of λ-independent non-linear evolution equations

$$u_t = JL^n f(u) \tag{1.5a}$$

or

$$Ku_t = L^n f(u) \quad , \tag{1.5b}$$

where J, K and L are some linear differential operators. Following the terminology in soliton literature we call these equations the soliton equations, since the first such equation – the celebrated KdV equation – exhibits soliton solutions. The extensive investigation undergone in the past decade reveals many intriguing features of soliton equations. One feature among others is the fact that there always exists an infinite set $\{h_n\}$ such that

$$L^n f(u) = (\delta/\delta u)\, h_n(u) \quad , \tag{1.6}$$

where $\delta/\delta u$ stands for the variational derivative. In terms of these h_n, the above hierarchy (1.5) can be written as the form of generalized Hamiltonian equations

$$u_t = J\delta H/\delta u \tag{1.7a}$$

or

$$Ku_t = \delta H/\delta u \quad . \tag{1.7b}$$

To transform an evolution equation $u_t = F(u)$ into its Hamiltonian form is not only a matter of beauty, some rewards can be received from doing so. In fact, as we show in Sec.III, by reducing O(2,1) non-linear sigma model to its Hamiltonian form we are able to find three new conservation laws.

The organization of this paper is as follows. In Sec.II some basic notions on generalized Hamiltonian equations are briefly sketched; in Sec.III the non-linear O(2,1) sigma model is shown to possess three new conservation laws by reducing the model to its Hamiltonian form. In Sec.IV, by following a simple approach presented in Refs.7 to 10, the Hamiltonian form of the hierarchy of NLEEs relating to the generalized $N \times N$ Zakharov-Shabat spectral problem is derived. Finally in Sec.V a new hierarchy of equations with the form (1.5b) is proposed, the first couple of equations in this new hierarchy read:

$$u_t = u_x + 2v$$
$$v_t = 2\varepsilon uv, \quad \varepsilon = \pm 1 \tag{1.8}$$

By setting $v = \exp w$ the above couple of equations can be reduced to a single equation

$$w_{tt} - w_{tx} + 4\varepsilon \exp w = 0$$

and it is shown that Eq.(1.6) holds in this case, thus the whole hierarchy of equations takes the form of generalized Hamiltonian equations.

II. PRELIMINARIES

As is well known, the ordinary Hamiltonian equation for continuous media reads

$$\partial p_i/\partial t = -\delta H/\delta q_i \quad , \quad \partial q_i/\partial t = \delta H/\delta p_i \quad , \quad i = 1, \dots, n \tag{2.1}$$

where $\delta/\delta p$ and $\delta/\delta q$ denote the variational derivatives. Setting $u = (p_1, \dots, p_n, q_1, \dots, q_n)$ and

$$\delta/\delta u = (\delta/\delta p_1, \dots, \delta/\delta q_n)^T \quad , \quad J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where T represents the transpose and I_n is the identity matrix of order n, we may rewrite Eq.(2.1) in the concise form

$$u_t = J_n\, \delta H/\delta u \quad . \tag{2.2'}$$

In this ordinary case the Poisson bracket of Eq.(2.1) is defined by

$$\{F, G\} = \sum_i \left(\frac{\delta F}{\delta q_i} \frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i} \frac{\delta G}{\delta q_i} \right)$$

which can be written as

$$\{F, G\} = (\delta F/\delta u)^T J_n (\delta G/\delta u) \tag{2.3'}$$

To give the definition of generalized Hamiltonian equations let $u = (u^1, \dots, u^M)^T$, $u^i = u^i(x_1, \dots, x_N, t)$ be an N-vector, and

$$u^r_{i_1 \cdots i_k} = \frac{\partial^k u^r}{\partial x_{i_1} \cdots \partial x_{i_k}} , \quad u^{(k)} = \{ u^r_{i_1 \cdots i_k} \}$$

by its x-derivatives of order k. Set

$$D_i = \frac{\partial}{\partial x_i} + \sum_{r k i_1 \cdots i_k} u^r_{i_1 \cdots i_k i} \frac{\partial}{\partial u^r_{i_1 \cdots i_k}}$$

the operator of total differentiation with respect to x_i. The variational derivatives are defined by $\delta/\delta u = (\delta/\delta u^1, \ldots, \delta/\delta u^M)^T$ with

$$\delta/\delta u^r = \sum_{k i_1 \cdots i_k} (-1)^k D_{i_1} \cdots D_{i_k} \partial/\partial u^r_{i_1 \cdots i_k}$$

An operator $K = K(u)$, which depends on the function u, is called symplectic if (i) it is linear and skew symmetric, that is $J^* = -J$, where $*$ stands for the formal conjugation

$$(A_{ij})^* = (A^*_{ji}) , \quad (\sum a_i D^i)^* = \sum (-D)^i (a_i \cdot)$$

(ii) the bracket defined by $\{f,g,h\}_1 = f.K'[g]h$ satisfies the Jacobi identity

$$\{f,g,h\} + \{g,h,f\} + \{h,f,g\} \overset{D}{\sim} 0 ,$$

where $f.g = \sum_{i=1}^{M} f^i g^i$, $f \overset{D}{\sim} g$ means $f - g = \sum D_i h_i$ for some vector $h = (h_1, \ldots, h_N)$, and $K'[g]h$ refers to the Gateaux derivative

$$J'(u)f = (d/d\varepsilon) J(u + \varepsilon f)\Big|_{\varepsilon = 0}$$

An operator $J = J(u)$ is called cosymplectic if (i) it is linear and skew symmetric; (ii) the bracket defined by $\{f,g,h\}_2 = f.J'[Jg]h$ satisfies the Jacobi identity. The equation

$$u_t = J\delta H/\delta u \tag{2.2a}$$

or

$$Ku_t = \delta H/\delta u \tag{2.2b}$$

is called a generalized Hamiltonian equation if the operator J is cosymplectic

or K is symplectic. The Poisson bracket of the generalized Hamiltonian equation (2.2a)is defined by an equation similar to equation (2.3')

$$\{F, G\} = (\delta F/\delta u)^T J (\delta G/\delta u) \tag{2.3}$$

A scalar function $f = f(u, u^{(1)}, \ldots, u^{(k)})$, which depends on $u(x,t)$ and its space derivatives $u^{(k)}$ is called a conserved density of equation (2.2) if $f_t \overset{D}{\sim} 0$ holds when $u(x,t)$ is taken to be solutions of Eq.(2.2).

Some typical examples of the generalized Hamiltonian equations are as follows:

Korteweg-de Vries (KdV) equation $u_t = uu_x + u_{xxx}$

$$J = D = d/dx, \quad H = u^3/6 - u_x^2/2$$

Sine-Gordon equation $q_{tt} - q_{xx} = \sin q$

$$J = J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad H = (q_x^2 + p^2)/2 - \cos q .$$
$$(p = q_t , \quad u = (q, p)^T)$$

Regular long wave equation $u_t = u_x + uu_x + u_{txx}$

$$J = D(1 - D^2)^{-1}, \quad H = u^2/2 + u_x^2/2 .$$

For a given non-linear evolution equation

$$u_t = F(u, u^{(1)}, \ldots, u^{(p)}) \tag{2.4}$$

a function $\eta = \eta(u) = (\eta^1, \ldots, \eta^M)$ is called a generalized symmetry of the equation if the equation (2.4) remains form invariant under the infinitesimal transformation $u' = u + \varepsilon\eta$, where ε is an infinitesimal parameter. The condition for invariance can be written as

$$\eta_t = (d/d\varepsilon) F(u + \varepsilon\eta)\Big|_{\varepsilon = 0} \tag{2.5}$$

To write this condition more precisely we introduce the operator

$$V(f) = \begin{bmatrix} V_1(f^1) & \cdots & V_M(f^1) \\ \cdots & \cdots & \cdots \\ V_1(f^M) & \cdots & V_M(f^M) \end{bmatrix}$$

where

$$V_A(f^B) = \sum_{j\geq 0} (\partial f^B/\partial u_j^A) D^j \quad . \qquad \text{(when } u \text{ is real)}$$

and

$$V_A(f^B) = \sum_j (\partial f^B/\partial u_j^A) D^j + (\partial f^B/\partial \bar{u}_j^A) C D^j \qquad \text{(when } u \text{ is complex)} ,$$

where $\bar{f}$ stands for the complex conjugation of f and $Cf = \bar{f}$. With this operator the condition (2.5) can be written as

$$[\eta , F] \equiv V(\eta)F - V(F)\eta = 0 \, .$$

We need also the following transformation formula (chain rule) of variational derivatives:

$$\begin{matrix} u = u(v, v_1, \cdots, v_\ell) \\ v_i = D^i v \end{matrix} \Longrightarrow \delta/\delta v = V^*(u(v))\, \delta/\delta u \tag{2.6}$$

This transformation formula (2.6) plays a key role in the recurrent method [7)-10)] which we shall follow in Secs.IV and V.

Note that the two brackets $\{ , \}$ and $[,]$ introduced above are related by the following important formula (see, for example Refs.11-14):

$$\left[J\delta F/\delta u , J\delta G/\delta u \right] = J\, \delta/\delta u \{F, G\}$$

from which we deduce the relationship between conserved densities and symmetries of generalized Hamiltonian equations in the case that the operator J is invertible [15)-17)]

$$\begin{matrix} G \text{ is a conserved density} \\ \text{of Eq.(1.7a)} \end{matrix} \Longleftrightarrow \begin{matrix} \eta = J\delta G/\delta u \text{ is a symmetry} \\ \text{of (1.7a)} \end{matrix} \tag{2.7}$$

III. NEW CONSERVATION LAWS OF O(2,1) NON-LINEAR SIGMA MODELS

The O(2,1) non-linear sigma model [18),19)] is defined by the Lagrangian $L = \frac{1}{2} w_u w_v$ subjected to the constraint $w^2 \equiv (w^1)^2 + (w^2)^2 - (w^3)^2 = -1$, where $w = (w^1, w^2, w^3)$ and the subscripts u and v refer to the differentiation. From the Euler-Lagrange equation and the constraint it follows that

$$w_{uv} - (w_u \cdot w_v)\, w = 0 \quad . \tag{3.1}$$

By taking the parametrization [18)]

$$w = \frac{1}{r} \left(i(z - \bar{z}),\ 1 - z\bar{z},\ 1 + z\bar{z} \right) \quad .$$

where $r = z + \bar{z}$, the Lagrange and Eq.(3.1) reduce respectively to

$$L = (z_u \bar{z}_v + \bar{z}_u z_v)/r^2$$

$$z_{uv} = 2 z_u z_v / r \quad . \tag{3.2}$$

In Ref.18 it is shown that this equation exhibits an infinite number of conservation laws

$$\partial_v \chi_0 = 0 \tag{3.3a}$$

$$\partial_v \chi_1 + \partial_u (i b a^{-1} |a|) = 0 \quad , (i = \sqrt{-1}) \tag{3.3b}$$

$$\partial_v \chi_2 + \partial_u \left[\tfrac{1}{2} b a^{-1} (2\chi_0 - (a - \bar{a}) + a^{-1} a_u) \right] = 0 \tag{3.3c}$$

$$\partial_v \chi_{k+3} + \partial_u (b a^{-1} \chi_{k+1}) = 0 \quad , k = 0, 1, \cdots \tag{3.3d}$$

where

$$a = z_u / r, \quad b = z_v / r$$

and χ_r can be calculated in the following recurrent way:

$$\chi_0 = -\tfrac{1}{2} |a|^{-1} |a|_u \, . \quad \chi_1 = -\tfrac{1}{2} i |a|^{-1} (\ell - \chi_{0u} - \chi_0^2)$$

$$\chi_{k+2} = \tfrac{1}{2} i |a|^{-1} \Big(\chi_{k+1,u} + \sum_{s=0}^{k+1} \chi_{k+1-s} \chi_s \Big) , \quad k = 0, 1, \cdots$$

with

$$\ell = \tfrac{1}{2} \left[(a - \bar{a}) - a^{-1} a_u \right]_u + \tfrac{1}{4} \left[(a - \bar{a}) - a^{-1} a_u \right]^2$$

The calculation of χ_r, even the direct verification of Eq.(3.3b) is rather tedious. We show that there are some simple conservation laws which do not involve the modulus $|a|$ and $|b|$:

$$\left[(\bar{z}_u - z_u)/r^2\right]_v + \left[(\bar{z}_v - z_v)/r^2\right]_u = 0 \tag{3.4a}$$

$$\left[(z\bar{z}_u + \bar{z}z_u)/r^2\right]_v + \left[(z\bar{z}_v + \bar{z}z_v)/r^2\right]_u = 0 \tag{3.4b}$$

$$\left[(z^2\bar{z}_u - \bar{z}^2 z_u)/r^2\right]_v + \left[(z^2\bar{z}_v - \bar{z}^2 z_v)/r^2\right]_u = 0 \tag{3.4c}$$

Although the direct verification of (3.4a) to (3.4c) by using (3.2) is a simple matter, we consider it worthy to give the derivation of these conservation laws. To do this we first make the substitution of variables u and v:

$$x = u + v \quad , \quad t = u - v$$

which reduces to the Lagrangian and Eq.(3.2) respectively to

$$L = (z_x\bar{z}_x - z_t\bar{z}_t)/r^2 \quad , \tag{3.5}$$

$$z_{xx} - z_{tt} = 2(z_x{}^2 - z_t{}^2)/r \quad . \tag{3.6}$$

Following the usual way we introduce the momentum

$$p = \partial L/\partial z_t = -\bar{z}_t/r^2 \tag{3.7}$$

and the Hamiltonian

$$H = z_t\,\partial L/\partial z_t + \bar{z}_t\,\partial L/\partial \bar{z}_t - L = -(z_t\bar{z}_t + z_x\bar{z}_x)/r^2$$

With the new variables z and p Eq.(3.6) can be written as

$$\begin{bmatrix} z_t \\ p_t \end{bmatrix} = \begin{bmatrix} -2r^2\bar{p} \\ 2rp\bar{p} - \bar{z}_{xx}/r^2 + 2\bar{z}_x^2/r^3 \end{bmatrix} \equiv F \tag{3.8}$$

or equivalently written in the following Hamiltonian form :

$$\begin{bmatrix} z \\ p \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \delta H/\delta z \\ \delta H/\delta p \end{bmatrix} \tag{3.9}$$

According to the definition a symmetry $f = f(z)$ of Eq.(3.2) satisfies

$$(z+\bar{z})f_{uv} + (f+\bar{f})\,2z_u z_v/r = 2f_u z_v + 2f_v z_u \tag{3.10}$$

and a vector $\eta = (\eta^1, \eta^2)$ would be a symmetry of Eq.(3.8) if

$$V(\eta)F - V(F)\eta = 0 \quad .$$

From Eq.(3.7) we see that there exists a 1-1 correspondence between symmetries f and symmetries η.

$$f \Longleftrightarrow \eta = \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix}$$

with

$$\eta^1 = f \quad , \quad \eta^2 = (d/d\varepsilon)p(z+\varepsilon f)\Big|_{\varepsilon=0} = -\bar{f}_t/r^2 + 2\bar{z}_t(f+\bar{f})/r^3 \tag{3.11}$$

Now we proceed to search for the symmetries f of Eq.(3.2). We consider only the symmetries of the form $f = f(z,\bar{z})$ and suppose that f is smooth with respect to its variables z and $\bar{z}$. Substituting $f = f(z,\bar{z})$ into Eq.(3.2) and comparing the coefficients of $\bar{z}_u\bar{z}_v, z_u z_v$ and $\bar{z}_v z_u + z_v\bar{z}_u$ on both sides of the resulting equation, we find that $f_{\bar{z}} = 0$ and

$$f_{zz}(\bar{z}+z)^2 - 2(z+\bar{z})f_z + 2(f+\bar{f}) = 0 \quad .$$

Suppose that $f = \sum a_i z^i$ we then have $i(i-1)a_i - 2ia_i + 2a_i = 0$ or $(i-1)(i-2)a_i = 0$, from which we deduce that $a_i = 0$ for $i \geqslant 3$, and the general solution is seen to be

$$f = i\alpha_0 + \alpha_1 z + i\alpha_2 z^2 \quad , \quad i = \sqrt{-1} \quad , \tag{3.12}$$

where α_k are arbitrary real constants. From Eq.(3.11) we see that the corresponding symmetry of Eq.(3.8) reads

$$\eta^1 = i\alpha_0 + \alpha_1 z + i\alpha_2 z^2 \; , \; \eta^2 = -p(\alpha_1 + 2i\alpha_2 z)$$

It is easy to see that

$$\begin{bmatrix} i\alpha_0 + \alpha_1 z + i\alpha_2 z^2 \\ -p(\alpha_1 + 2i\alpha_2 z) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta H/\delta z \\ \delta H/\delta p \end{bmatrix}$$

where $H = h + \bar{h}$ and $h = (i\alpha_0 + \alpha_1 z + i\alpha_2 z^2)p$. According to the relation (2.7) between symmetries and conserved densities, we obtain the following three conserved densities:

$$H_1 = p - \bar{p}, \quad H_2 = zp + \bar{z}\bar{p}, \quad H_3 = z^2 p - \bar{z}^2\bar{p} .$$

In fact one can easily verify that

$$H_{1t} = \left[(z_x - \bar{z}_x)/r^2 \right]_x$$

$$H_{2t} = \left[-(z\bar{z}_x + \bar{z}z_x)/r^2 \right]_x$$

$$H_{3t} = \left[(\bar{z}^2 z_x - z^2\bar{z}_x)/r^2 \right]_x$$

Noting that $H_t = G_x \Longleftrightarrow (H - G)_u = (H + G)_v$ we are finally led to the conservation laws (3.4a) to (3.4c). If we continue to search for the symmetry with the form $f = f(z,\bar{z},z_u,\bar{z}_u,z_v,\bar{z}_v)$ then the similar procedure leads to the symmetry

$$\eta^1 = f = \frac{1}{2}(z_u + z_v) = z_x$$

and the corresponding $\eta^2 = p_x$. Since

$$\begin{bmatrix} z_x \\ p_x \end{bmatrix} = \begin{bmatrix} \delta H/\delta p \\ -\delta H/\delta z \end{bmatrix} , \quad \begin{bmatrix} \bar{z}_x \\ \bar{p}_x \end{bmatrix} = \begin{bmatrix} \delta H/\delta \bar{p} \\ -\delta H/\delta \bar{z} \end{bmatrix}$$

where $H = pz_x + \bar{p}\bar{z}_x$, and accordingly we obtain the following conservation laws:

$$(pz_x + \bar{p}\bar{z}_x)_t = -((z_x\bar{z}_x)/r^2 + r^2 p\bar{p})_x$$

which is equivalent to

$$((z_v\bar{z}_v)/r^2)_u + (z_u\bar{z}_u)/r^2)_v = 0 \quad ;$$

this conserved density $(z_u\bar{z}_u)/r^2$ is just $|a|^2$ as mentioned in Eq.(3.3a). One may naturally expect to find the more general symmetries with the form

$f = f(z,\bar{z},z_u,\bar{z}_u,z_v,\bar{z}_v,z_{uu},\bar{z}_{uu},z_{vv},\bar{z}_{vv})$, (note that since $z_{uv} = 2z_u z_v/r$ the two terms z_{uv} and $\bar{z}_{uv}$ can be excluded from the above expression). Unfortunately if we suppose that f is smooth with respect to its variables it seems that no such symmetry exists.

As another remark, we find that among the four symmetries of Eq.(3.8)

$$\eta_1 = \begin{pmatrix} i \\ 0 \end{pmatrix} , \quad \eta_2 = \begin{pmatrix} z \\ -p \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} iz^2 \\ -2izp \end{pmatrix}, \quad \eta_4 = \begin{pmatrix} z_x \\ p_x \end{pmatrix}$$

only η_4 commutes with others. In fact we have

$$[\eta_1, \eta_2] = -\eta_1, \quad [\eta_1, \eta_3] = -2\eta_2 , \quad [\eta_2, \eta_3] = -\eta_3$$

and

$$[\eta_i, \eta_4] = 0 , \quad (i = 1, 2, 3)$$

For example

$$[\eta_2, \eta_3] = V(\eta_2)\eta_3 - V(\eta_3)\eta_2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} iz^2 \\ -2izp \end{bmatrix} - \begin{bmatrix} 2iz & 0 \\ -2ip & -2iz \end{bmatrix} \begin{bmatrix} z \\ -p \end{bmatrix}$$

$$= \begin{bmatrix} iz^2 \\ 2izp \end{bmatrix} - \begin{bmatrix} 2iz^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -iz^2 \\ 2izp \end{bmatrix} = -\eta_3 .$$

IV. HAMILTONIAN STRUCTURES OF N × N AKNS EQUATIONS

Consider the spectral problem

$$\psi_x = U\psi , \quad U = \lambda A + P \tag{4.1}$$

with

$$\psi = (\psi_1, \cdots, \psi_N)$$

$$A = \mathrm{diag}(a_1, \cdots, a_N) , \quad a_i \neq a_j$$

$$a_i \text{ — const.}$$

and

$$P = (P_{ij}), \quad P_{ii} = 0 .$$

Choosing $V = \sum \lambda^{n-j} V_j$, $V_j = V_j(P)$ and substituting U and V into the integrability condition (1.4) we can solve for V_j and obtain the following hierarchy of equations [20),21)]:

$$p_t = JL^n\,[C,P^T]_A \tag{4.2}$$

where

$$C = \operatorname{diag}\,(c_1,c_2,\dots,c_N),\quad c_i = \text{const.}$$

$$JQ = [A,Q^T],\quad LQ = (-Q_x - [P^T,Q]_F + [P^T,I\,[P^T,Q]_D])_A$$

where $S_D = \operatorname{diag}\,(s_{11},s_{22},\dots,s_{NN})$, $S_F = S - S_D$ and for a matrix Q_F the matrix Q_{FA} is defined by the equation $[A,Q_{FA}] = Q_F$.

We shall prove that there exists a series $\{H_n\}$ such that

$$L^n\,[C,P^T]_A = \delta H_n/\delta P \tag{4.3}$$

To this end we introduce first

$$Y^{(i,j)} \equiv \psi_i/\psi_j \tag{4.4}$$

and set

$$H^{(i)} = \sum_{j=1}^{N} P_{ij}\,Y^{(j,i)} \tag{4.5}$$

then expand $H^{(i)}$ into a series of λ^{-1}

$$H^{(i)} = \sum_{n=1}^{\infty} \lambda^{-n} H_n^{(i)}, \tag{4.6}$$

we show that (4.3) holds with

$$H_n = \sum_{i=1}^{N} c_i H_n^{(i)} . \tag{4.7}$$

It is easy to see that Eqs.(4.3), (4.5) and (4.6) are equivalent to a single equation

$$(L-\lambda)\frac{\delta H^{(i)}}{\delta P} = [P^T, E_i]_A \tag{4.8}$$

where $E_i = (\delta_{ik}\delta_{i\ell})$. Since the discussions for different i are the same we shall, for simplicity of notations, prove (4.8) in the case $i = 1$, i.e.

$$(L-\lambda)\,\delta H/\delta P = [P^T, E_1]_A$$

$$H \equiv H^{(1)} = \sum_{j=1}^{N} P_{1j}\,Y_j ,\quad Y_j \equiv Y^{(j,1)} .$$

Note first that from (4.1) and (4.4) we have

$$Y_{jx} = P_{j1} + \sum_{k=2}^{N}(P_{jk}Y_k - P_{1k}Y_jY_k) + (a_j - a_1)\lambda Y_j \tag{4.9}$$

In order to calculate the variational derivative $\delta H/\delta P$ we make the transformation

$$P = (P_{ij}) \longrightarrow \bar P = (\bar P_{ij})$$

with

$$\bar P_{12} = H = \sum_{j=2}^{N} P_{1j}Y_j$$

$$\bar P_{j1} = Y_j ,\quad j = 2,\dots,N$$

$$\bar P_{ij} = P_{ij}Y_j \quad (i=1,\ j\geqslant 3 \ \text{ or } \ i,j\geqslant 2)$$

We can also solve for P_{ij} in terms of $\bar P_{ij}$ as follows:

$$P_{12} = (H - \sum_{i=3}^{N}\bar P_{1i})/Y_2 ,$$

$$P_{j1} = Y_{jx} + HY_j - \sum_{k=2}^{N}\bar P_{jk} + (a_1 - a_j)\lambda Y_j ,$$

$$P_{ij} = \bar P_{ij}/Y_j ,\quad i=1,\ j\geqslant 3 \ \text{ or } \ i,j\geqslant 2 ,$$

from which we obtain

$$V(P_{12}) = \frac{1}{Y_2} ,\quad V_{\bar P_{1i}}(P_{12}) = -\frac{1}{Y_2} ,\quad V_{Y_i}(P_{12}) = -(P_{12}/Y_2)\delta_{i2} ,$$

$$V_{\bar P_{ij}}(P_{12}) = 0 ,\quad (i,j\geqslant 2) ;$$

$$V_H(P_{j1}) = Y_j ,\quad V_{\bar P_{1i}}(P_{j1}) = 0 ,\quad V_{Y_i}(P_{j1}) = \delta_{ij}(D + H + (a_1 - a_j)\lambda) ,$$

$$V_{\bar P_{k\ell}}(P_{j1}) = -\delta_{jk} ,\quad (k,\ell\geqslant 2) ;$$

$$V_H(P_{1j}) = 0 ,\quad V_{\bar P_{1k}}(P_{1j}) = \delta_{jk}/Y_j ,\quad V_{Y_k}(P_{1j}) = \delta_{kj}(-P_{1j}/Y_j) ,$$

$$V_{\bar P_{k\ell}}(P_{1j}) = 0 ,\quad (k,\ell\geqslant 2) ;$$

$$V_H(P_{ij}) = 0 ,\quad V_{Y_k}(P_{ij}) = \delta_{jk}(-P_{ij}/Y_j) ,\quad V_{\bar P_{1j}}(P_{ij}) = 0 ,$$

$$V_{\bar P_{k\ell}}(P_{ij}) = \delta_{ik}\delta_{j\ell}/Y_j ,\quad (k,\ell\geqslant 2) ;$$

By the chain rule (2.6) we have

$$\frac{\delta}{\delta H} = V_H^*(P_{12})\frac{\delta}{\delta P_{12}} + \sum_{i=2}^{N} V_H^*(P_{i1})\frac{\delta}{\delta P_{i1}} + \sum_{j=3}^{N} V_H^*(P_{1j})\frac{\delta}{\delta P_{1j}} + \sum_{i,j=2}^{N} V_H^*(P_{ij})\frac{\delta}{\delta P_{ij}} .$$

and similar expressions for $\delta/\delta Y_k$, $\delta/\delta \vec{P}_{ij}$ and $\delta/\delta \bar{P}_{ij}$ $(i,j \geqslant 2)$. Setting $K_{ij} = \delta H/\delta P_{ij}$ and applying both sides of the above operator identities to H, we then find

$$K_{1l} = Y_l\left(1 - \sum_{i=2}^{N} Y_i K_{i1}\right) .$$

and

$$K_{lk} = Y_k K_{l1} , \quad (l, k \geqslant 2), \quad l \neq k .$$

$$K_{l1x} = (a_1 - a_l)\lambda K_{l1} - P_{1l} + \sum_{i=2}^{N}(P_{1i}K_{li} - P_{il}K_{i1}) + P_{1l}\left(\sum_{i=2}^{N} Y_i K_{i1} + Y_l K_{l1}\right) \qquad (4.10) .$$

from which we obtain

$$(K_{lk})_x = (a_k - a_l)\lambda K_{lk} + P_{kl}(Y_l K_{l1} - Y_k K_{k1}) + \sum_{j=1}^{N}(P_{kj}K_{lj} - P_{jl}K_{jk}), \quad \left(\begin{matrix} l, k \geqslant 2 \\ l \neq k \end{matrix}\right) .$$

and

$$(K_{1l})_x = (a_l - a_1)\lambda K_{1l} + \sum_{i=2}^{N}(P_{li}K_{1i} - K_{il}P_{i1}) + P_{l1} - P_{l1}\sum_{i=2}^{N}(1 + \delta_{il}) Y_i K_{i1} .$$

It is easy to see that the above three equations are equivalent to the following single matrix equation:

$$(K^T)_x = \lambda[A, K^T] - [K^T, P]_F + [B, P] - [E_1, P] , \qquad (4.11)$$

where

$$B = \mathrm{diag}\left(\sum_{i=2}^{N} Y_i K_{i1}, -Y_2 K_{21}, -Y_3 K_{31}, \cdots, -Y_N K_{N1}\right) .$$

Using Eqs.(4.9) and (4.10) we can easily verify that

$$(Y_l K_{l1})_x = \sum_{k=1}^{N}(P_{lk}K_{lk} - P_{kl}K_{kl}) = ([P, K^T]_D)_{ll} , \quad (l \geqslant 2). \qquad (4.12)$$

Since $\mathrm{tr}[B,A] = 0$ for any pair of matrices A and B, we see that

$$([K^T, P]_D)_{11} = -\sum_{l=2}^{N}([K^T, P]_D)_{ll} = \sum_{l=2}^{N}(Y_l K_{l1})_x . \qquad (4.13)$$

which together with (4.10) imply that

$$B_x = [K^T, P]_D \qquad (4.14)$$

From Eqs.(4.9), (4.10) and the definition of L we finally obtain

$$(L - \lambda)K = [P^T, E_1]_A$$

as desired.

V. A NEW HIERARCHY OF HAMILTONIAN EQUATIONS

Consider the spectral problem $\psi_x = U\psi$ with

$$U = \begin{bmatrix} -\lambda - \lambda^{-1}\varepsilon v & u - v\lambda^{-1} \\ u + v\lambda^{-1} & \lambda + \lambda^{-1}\varepsilon v \end{bmatrix} , \quad \varepsilon = \pm 1 , \qquad (5.1)$$

which is a reduction of the spectral problem discussed in Ref.8. To derive the corresponding hierarchy of equations we introduce the auxiliary problem

$$\psi_t = V\psi , \quad V = \sum_{j=0}^{n} V_j \lambda^{n-j} , \quad n = 2m+1 ,$$

$$V_j = \begin{pmatrix} d_j & (e_j + f_j)/2 \\ (e_j - f_j)/2 & -d_j \end{pmatrix} \qquad (5.2)$$

Substituting U and V into the integrability condition (1.4) and comparing the coefficients of λ^j on both sides of the resulting equation, we obtain

$$v_t \varepsilon \delta_{j\,n+1} + d_{jx} + u f_j + v e_{j-1} = 0 .$$

$$2u_t \delta_{jn} - e_{jx} - 2f_{j+1} + 4 d_{j-1} v - 2\varepsilon v f_{j-1} = 0 ,$$

$$2v_t \delta_{j\,n+1} + f_{jx} + 4u d_j + 2e_{j+1} + 2\varepsilon v e_{j-1} = 0 , \qquad (5.3)$$

from which we can calculate successively e_j, f_j as follows:

$$e_0 = f_0 = 0$$

$$e_1 = 2u , \quad f_1 = 0 ; \quad e_2 = 0 , \quad f_2 = -(u_x + 2v) ,$$

$$e_{j+1} = -\tfrac{1}{2} f_{jx} + 2uI(uf_j + ve_{j-1}) - \varepsilon v e_{j-1} , \quad \left(I \equiv \tfrac{1}{2}\left(\textstyle\int_{-\infty}^{x} - \int_{x}^{+\infty}\right)dx'\right),$$

$$f_{j+1} = -\left(\tfrac{1}{2} e_{jx} + 2vI(uf_{j-1} + ve_{j-2}) + \varepsilon v f_{j-1}\right) ; \qquad (5.4)$$

and

$$u_t = -f_{n+1} ,$$

$$v_t = -\varepsilon v e_n . \qquad (5.5)$$

From (5.4) we deduce that

$$\begin{pmatrix} -f_{n+1} \\ -\varepsilon v e_n \end{pmatrix} = L^* \begin{pmatrix} -f_{n-1} \\ -\varepsilon v e_{n-2} \end{pmatrix}$$

where

$$L^* = \begin{bmatrix} \tfrac{1}{4}D^2 - (u^2 + \varepsilon v) - (u_x + 2v)Iu , & \tfrac{1}{2}D - \varepsilon u - \varepsilon(u_x + 2v)I \\ -\tfrac{1}{2}\varepsilon v D + 2\varepsilon u v I u , & 2uvI - \varepsilon v \end{bmatrix}$$

Therefore the corresponding hierarchy of equations reads

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -f_{2m+2} \\ -\varepsilon v e_{2m+1} \end{pmatrix} = L^{*m} \begin{pmatrix} -f_2 \\ -\varepsilon v e_1 \end{pmatrix}$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = L^{*m} \begin{pmatrix} u_x + 2v \\ -2\varepsilon u v \end{pmatrix}$$

(5.6)

In particular setting $m=1$ we obtain the first couple of equations (1.6) in this hierarchy.

Since this hierarchy of equations is derived from the isospectral problems (5.1) and (5.2), we can calculate the common conserved densities via the standard procedure [20]. To this end we set $Z = \psi_2/\psi_1$ then from (5.1) we have

$$Z_x = (u + v\lambda^{-1}) + (2\lambda + 2\lambda^{-1}\varepsilon v)Z - (u - v\lambda^{-1})Z^2 . \qquad (5.7)$$

Expanding Z into a series of λ^{-1} : $Z = \sum_{m=1}^{\infty} Z_n \lambda^{-n}$ and then comparing the coefficients of λ^{-n} we obtain

$$Z_1 = -u/2 , \qquad Z_2 = -(1/4)(u_x + 2v) ,$$

$$Z_3 = -(u_{xx} + 2v_x - 4\varepsilon u v - u^3)/8 ,$$

$$Z_{n+1} = \Big(Z_{nx} - 2\varepsilon v Z_{n-1} + u\sum_{j+k=n} Z_k Z_j - v \sum_{j+k=n-1} Z_k Z_j\Big)/2 , \quad n \geq 3.$$

The generating function H of the conserved densities $\{H_n\}$ is

$$H = \sum_{n=-1}^{\infty} H_n \lambda^{-n} = (-\lambda - \lambda^{-1}\varepsilon v) + (u - v\lambda^{-1})Z , \qquad (5.8a)$$

which gives

$$H_{-1} = -1 , \quad H_0 = 0 , \quad H_1 = -(\varepsilon v + u^2/2) , \quad H_2 = -(u^2)_x/8 .$$

$$H_n = uZ_n - vZ_{n-1} , \quad (n \geq 2) . \qquad (5.8b)$$

Note that only H_{2k-1} gives the non-trivial conserved densities for the whole hierarchy of equations. The variational derivatives of the first two non-trivial conserved densities are

$$\frac{\delta}{\delta q} H_1 = \begin{pmatrix} -u \\ -\varepsilon \end{pmatrix} , \quad \left(q = (u, v)^T , \ \frac{\delta}{\delta q} = \left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\right)^T\right)$$

$$\frac{\delta}{\delta q} H_3 = \begin{pmatrix} -u_{xx}/4 + u^3/2 + \varepsilon u v - v_x/2 \\ u_x/2 + \varepsilon u^2/2 + v \end{pmatrix}$$

It is easy to verify that

$$K\begin{pmatrix} u_x + 2v \\ -2\varepsilon v \end{pmatrix} = \begin{pmatrix} \delta H_3/\delta u \\ \delta H_3/\delta v \end{pmatrix} . \qquad (5.9)$$

where

$$K = \begin{bmatrix} uIu - D/4 & \varepsilon uI - 1/2 \\ \varepsilon Iu + 1/2 & I \end{bmatrix} .$$

Note that

$$K^* = -K , \quad KL^* = LK . \qquad (5.10)$$

It should be noted that the operator K is singular, in fact that

$$K\begin{pmatrix} f \\ -f_x/2 - \varepsilon u f \end{pmatrix} = 0$$

holds for any f.

Now we proceed to prove that

$$L\left(\delta H_{2k-1}/\delta q\right) = \delta H_{2k+1}/\delta q , \quad \delta H_{2k}/\delta q = 0 \qquad (5.11)$$

or equivalently

$$(L - \lambda^2)\, \delta H/\delta q = \lambda \begin{pmatrix} u \\ \varepsilon \end{pmatrix} \qquad (5.12)$$

To calculate $\delta H/\delta q$ we deduce first from (5.7) and (5.8a) that

$$1 = -\lambda^{-1}\varepsilon V_H(v) + Z\left(V_H(u) - \lambda^{-1} V_H(v)\right)$$

$$0 = V_H(u) + V_H(v)\lambda^{-1} + \lambda^{-1}\varepsilon Z V_H(v) - Z$$

from which it results that

$$V_H(u) = \varepsilon \qquad (5.13a)$$

$$V_H(v) = -\lambda\varepsilon(1 - \varepsilon Z)/(1 + \varepsilon Z) \qquad (5.13b)$$

In the same manner we deduce that

$$V_Z(u) = \frac{1}{(1+\varepsilon Z)} D + \frac{(H - \lambda - \varepsilon u)}{(1 + \varepsilon Z)} . \qquad (5.13c)$$

$$V_Z(v) = \frac{\varepsilon\lambda Z}{(1+\varepsilon Z)^2} D + \frac{\varepsilon\lambda}{(1+\varepsilon Z)^2}\left(HZ - \lambda Z - \lambda^{-1} v(1+\varepsilon Z) + u\right) . \qquad (5.13d)$$

Now by the chain rule (2.6) we have

$$\begin{pmatrix} \delta/\delta H \\ \delta/\delta Z \end{pmatrix} = \begin{pmatrix} V_H^*(u) & V_H^*(v) \\ V_Z^*(u) & V_Z^*(v) \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix} .$$

Applying the operators on both sides of the equation to H we find

$$1 = V_H^*(u) R + V_H^*(v) S ,$$

$$0 = V_Z^*(u) R + V_Z^*(v) S ,$$

where $R = \delta H/\delta u$, $S = \delta H/\delta v$. Using Eqs.(5.13) to (5.13d) we then obtain

$$R = \varepsilon + YS \qquad (5.14a)$$

$$S_x = 2\varepsilon u S - 2R , \qquad (5.14b)$$

where

$$Y = \lambda(1 - \varepsilon Z)/(1 + \varepsilon Z) .$$

Since Z satisfies the equation (5.7) we see that

$$Y_x = -(\lambda^2 + 2v\varepsilon) - 2\varepsilon v Y + Y^2 \qquad (5.14c)$$

the above three equations (5.14a) to (5.14c) can be used to calculate R_x, R_{xx} in terms of R, S and Y

$$R_x = -\left(2\varepsilon Y + (\lambda^2 + 2\varepsilon v) S + S Y^2\right)$$

$$R_{xx} = (8v + 4\varepsilon\lambda^2) + 4uY + S\left((-2\varepsilon v_x - 2\varepsilon\lambda^2 u - 4uv) + (4\lambda^2 + 8\varepsilon v) Y + (2\varepsilon u) Y^2\right).$$

Now it is straightforward to verify that

$$(\varepsilon u_x + 2v\varepsilon)R - 2uvS = (R_x/2 + \varepsilon uR + \varepsilon vS + \lambda^2 S)_x. \tag{5.15a}$$

$$R_{xx}/4 - (\varepsilon v + \lambda^2)R + \varepsilon uR_x/2 + (uv + \varepsilon\lambda^2 u + \varepsilon Dv/2)S = 0. \tag{5.15b}$$

In the following we suppose that u,v along with all their x-derivatives tend to zero when x goes to $\pm\infty$. Assuming this we have from (5.8b)

$$S = \delta H/\delta v \longrightarrow \delta H_1/\delta v\,\lambda^{-1} = -\varepsilon\lambda^{-1},$$

$$R \to 0 \quad , \qquad (x \to \pm\infty) .$$

Thus the integration of (5.15a) gives

$$I((\varepsilon u_x + 2v\varepsilon)R - 2uvS) = \lambda\varepsilon + R_x/2 + \varepsilon uR + \varepsilon vS + \lambda^2 S. \tag{5.16a}$$

which together with (5.15b) yield

$$R_{xx}/4 - (u^2 + \varepsilon v)R - \lambda^2 R + uI((u_x + 2v)R - 2uv\varepsilon S) = \lambda u. \tag{5.16b}$$

The combination of Eqs.(5.16a) with (5.16b) gives the desired equation (5.12). The proof of Eq.(5.11) is thus completed. From Eqs.(5.11), (5.9) and (5.10) we see that

$$K\binom{u}{v}_t = KL^{*m}\binom{u_x + 2v}{-2\varepsilon uv} = L^m K\binom{u_x + 2v}{-2\varepsilon uv} = L^m\frac{\delta H_3}{\delta q} = \frac{\delta H_{2m+3}}{\delta q}, \tag{5.17}$$

which shows that all equations in this new hierarchy take the form of generalized Hamiltonian equations (1.5b).

As a final remark, let us introduce the Poisson bracket {F,H} when $\delta F/\delta q$, $\delta H/\delta q$ lie in the image of K, $\delta F/\delta q = Kf$, $\delta H/\delta q = Kh$:

$$\{F, H\} = (Kf)^T h .$$

Set $\hat{H}_m = H_{2m}$ then from (5.17)

$$\frac{\delta \hat{H}_{m+1}}{\delta q} = KS_m, \quad \left(S_m \equiv L^{*m} f, \quad f = \binom{u_x + 2v}{-2\varepsilon uv}\right)$$

and accordingly

$$\{\hat{H}_{m+1}, \hat{H}_{n+1}\} = KS_m \cdot S_n = KL^{*m}f \cdot L^{*n}f =$$

$$= LKL^{*m-1}f \cdot L^{*n}f \overset{D}{\sim} KL^{*m-1}f \cdot L^{*n+1}f = KS_{m-1} \cdot S_{n+1} = \{\hat{H}_m, \hat{H}_{n+2}\}.$$

from which and the fact that $\{F,H\} \overset{P}{\sim} -\{H,F\}$ we deduce that

$$\{\hat{H}_m, \hat{H}_n\} \overset{D}{\sim} 0 .$$

It is not clear at present whether this new hierarchy of equations shares some other common properties possessed by many other completely integrable equations, such as exhibiting Backlund transformations and soliton solutions. We hope that these problems could be solved in the near future by the interested readers.

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REFERENCES

1) G.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, Phys. Rev. Lett. 19, 1095 (1967).

2) P.D. Lax, Comm. Pure Appl. Maths. 21, 467 (1968).

3) V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP, 62 (1972).

4) M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Maths. 53, 249 (1974).

5) F. Calogero and A. Degasperis, Nuovo Cimento B32, 201 (1976).

6) For a review see Ling-Lie Chau Wang, in Proceedings of Guangzhou Conf. on Theoretical Partial Physics, Guangzhou, China (1980), (Academica Sinica, Beijing, 1980), Talk given at the International School of Subnuclear Physics, Erice, Italy, 1980.

7) G.Z. Tu,"A simple approach to Hamiltonian structures of soliton equations"(I) Nuovo Cimento 73B, 15 (1983); (II) to appear in Sci. Exploration.

8) M. Boiti and G.Z. Tu, "A simple approach to Hamiltonian structures of soliton equations" (III) to appear in Nuovo Cimento.

9) M. Boiti, C. Laddomada, F. Pempinelli and G.Z. Tu,"On a new hierarchy of Hamiltonian soliton equations", to appear in J. Math. Phys.

10) M. Boiti, F. Pempinelli and G.Z. Tu,"Canonical structures of soliton equations via isospectral eigenvalue problems", submitted for publication.

11) F. Magri, J. Math. Phys. 19, 1156 (1978).

12) P.J. Olver, Math. Proc. Camb. Phil. Soc. 88, 71 (1980).

13) I.M. Gel'fand and I.Ya. Dorfman, Func. Anal. 13, 13 (1979).

14) G.Z. Tu, "On formal variational calculus of higher dimensions", J. Math. Anal. Appl. 93, (1983), in press.

15) M. Wadati, Stud. Appl. Math. 59, 153 (1978).

16) G.Z. Tu and M.Z. Qin, Kexue Tongbao 24, 913 (1979).

17) G.Z. Tu, J. Phys. A 15, 277 (1982).

18) F.J. Chinea and F. Guil, J. Phys. A 15, 2349 (1983).

19) A.C. Newell, Proc. R. Soc. A365, 283 (1979).

20) T.M. Alberty, T. Koikawa and R. Sasaki, Physica D5, 43 (1982).

21) M. Boiti and G.Z. Tu, Nuovo Cimento 71B, 253 (1982).