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A GENERATING FUNCTION FOR A CLASS OF EFFECTIVE CHEW-MANDELSTAM FUNCTIONS

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Abstract

We have obtained the generating function for a class of effective Chew-Mandelstam functions for arbitrary integral angular momentum. From this a closed formula for the Chew-Mandelstam functions themselves is derived in both the simple equal mass case and in the more complicated case of unequal masses.

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[Revised version]

1. INTRODUCTION

Phenomenologists in their attempts to fit experimental data to partial-wave amplitudes have used a variety of different approximations to the S matrix. A useful and commonly used ansatz is the K matrix parametrization for the multichannel scattering matrix $\mathcal{S}_\ell(s)$ in the ℓ^{th} partial wave:[1,2,3]

$$\mathcal{S}_\ell = 1 + 2i\rho_\ell^{1/2}\, T_\ell\, \rho_\ell^{1/2} \tag{1.1}$$

where

$$T_\ell = K(s)\big(1 - C_\ell(s)K(s)\big)^{-1} \tag{1.2}$$

and

$$\rho_\ell = \mathrm{Im}\, C_\ell \tag{1.3}$$

is a diagonal matrix of two-body phase-space factors. K(s) is a real, symmetric matrix whose elements are meromorphic functions of s, the invariant squared energy. Different choices for the two-body phase-space factors have been made,[1,2,3,4,5] but all must give the correct threshold behaviour, namely:

$$\rho_\ell \propto \frac{2k}{\sqrt{s}} \cdot k^{2\ell} \tag{1.4}$$

where k is the centre-of-mass three momentum, $k^2 = \frac{(s-a)(s-b)}{4s}$ and the convenient abbreviations $a = (m_1+m_2)^2$, $b = (m_1-m_2)^2$ have been introduced. In addition, it is common[1,2,4,5] to arrange for a damping factor at large s, so that ρ tends to a finite constant in this limit. This is done for two reasons:

(a) Resonance widths do not grow without bound as $s \to \infty$.

(b) The real part of C_ℓ may be defined in terms of the imaginary part via Eqs. (1.3) and (1.5) below

$$\text{Re } C_\ell(s) = \frac{s}{\pi}\int_a^\infty ds' \, \frac{\text{Im}C_\ell(s')}{s'(s'-s)} \; . \tag{1.5}$$

The left-hand cut structure of the partial-wave amplitudes may be approximated by choices of the functional form for ρ_ℓ as is done by Cutkosky *et al.*[4] In addition such structure may be simulated to some extent by the choice of terms and parameters in the K matrix itself.

For S waves there is no ambiguity; $\rho = \frac{2k}{\sqrt{s}}$ and Eqs. (1.3) and (1.5) yield the well-known Chew-Mandelstam function.[6,1,2,3]

In this paper, we wish to focus on a particular class of effective Chew-Mandelstam functions[7] for $\ell > 0$ singled out for extensive application by Edwards and Thomas.[2] Here one defines:

$$\rho_\ell = \left(\frac{2k}{\sqrt{s}}\right)\left(\frac{2k}{\sqrt{s}}\right)^{2\ell} \theta(s-a) \tag{1.6}$$

which satisfies both the threshold requirement, Eq. (1.4) and obeys the limit $\rho_\ell \to 1$ as $s \to \infty$ so that the real part of C_ℓ can be defined by a once-subtracted dispersion relation via Eqs. (1.3) and (1.5). Although Edwards and Thomas gave explicit forms for the C_ℓ for small ℓ, no general expression was written down. It might therefore be helpful for future work to find a general expression for $C_\ell(s)$. In this paper we exhibit a useful closed form for these functions. We proceed in two stages. First, we find the generating function $C(s,z)$ for the functions $C_\ell(s)$. This is defined by the formal power series expansion in z:

$$C(s,z) = \sum_{\ell=0}^{\infty} C_\ell(s) z^\ell \tag{1.7}$$

$C(s,z)$ having been found, we use it to obtain a closed form for the $C_\ell(s)$.

2. THE GENERATING FUNCTION

From Eqs. (1.5) and (1.7) we see that $C(s,z)$ satisfies the same once subtracted dispersion relation as the $C_\ell(s)$ themselves, namely:

$$\mathrm{Re}\ C(s,z) = \frac{s}{\pi}\int_a^\infty ds' \frac{\mathrm{Im}C(s',z)}{s'(s'-s)} \quad . \tag{2.1}$$

Further

$$\mathrm{Im}\ C(s,z) = \sum_{\ell=0}^{\infty}\left(\frac{2k}{s^{1/2}}\right)^{2\ell+1} z^\ell = \frac{2k}{s^{1/2}}\left(1-\frac{4k^2z}{s}\right)^{-1} \tag{2.2}$$

Hence

$$C(s,z) = \frac{s}{\pi}\int_a^\infty ds' \frac{(s'-a)(s'-b)}{(s'-s-i\varepsilon)} \frac{1}{(s'-a)^{1/2}(s'-b)^{1/2}P(s')} \tag{2.3}$$

where

$$P(s) = (1-z)s^2 + z(a+b)s - zab \quad . \tag{2.4}$$

At this point we remark that

$$\frac{(s'-a)(s'-b)}{(s'-s)} = s' + (s-a-b) + \frac{(s-a)(s-b)}{(s'-s)} \quad . \tag{2.5}$$

It has proven most convenient to split the integration in Eq. (2.3) into three parts in the fashion suggested by Eq. (2.5). Then the integrations can all be done in terms of elementary functions[8] (for details see Appendix A) and we obtain:

$$C(s,z) = I_1 + I_2 + I_3 \tag{2.6}$$

$$C_\ell(s) = I_{1,\ell} + I_{2,\ell} + I_{3,\ell} \tag{2.7}$$

$$I_1 = \frac{2k}{\pi s^{1/2}}\left(1-\frac{4k^2z}{s}\right)^{-1}\left[2\ell n\left(\frac{(s-a)^{1/2}+(s-b)^{1/2}}{2(m_1m_2)^{1/2}}\right) + i\pi\right] \tag{2.8}$$

$$I_2 = \frac{1}{\pi}\left(1-\frac{4k^2z}{s}\right)^{-1}\frac{1}{2z^{1/2}}\ \ell n\left|\frac{1+z^{1/2}}{1-z^{1/2}}\right| \tag{2.9}$$

$$I_3 = \frac{1}{\pi}\left(1-\frac{4k^2z}{s}\right)^{-1}\left(\frac{a+b}{2}-\frac{ab}{s}\right)\frac{Q}{(a+b)}\ \ell n\left|\frac{1+Q}{1-Q}\right| \tag{2.10}$$

where

$$Q = \frac{(a+b)}{2}\left(ab + \frac{z(a-b)^2}{4}\right)^{-1/2} . \qquad (2.11)$$

Note that $C(s,0)$ correctly reproduces the expression for the S wave Chew-Mandelstam function $C_0(s)$ given in Basdevant and Berger.[1] In general one has from Eq. (1.7)

$$C_\ell(s) = \frac{1}{\ell!}\frac{d^\ell C(s,z)}{dz^\ell} , \quad z = 0 . \qquad (2.12)$$

However, it is not necessarily trivial to find from Eqs. (2.8),(2.9), (2.10),(2.11) and (2.12) the explicit expressions for $C_\ell(s)$. The three parts I_1, I_2, I_3 differ very considerably in the ease with which they yield up the closed expressions. For example, we can write almost immediately:

$$I_{1,\ell} = \frac{2}{\pi}\left(\frac{2k}{s^{1/2}}\right)^{2\ell+1} \ln\left(\frac{(s-a)^{1/2} + (s-b)^{1/2}}{2(m_1m_2)^{1/2}}\right) + i\left(\frac{2k}{s^{1/2}}\right)^{2\ell+1} . \qquad (2.13)$$

On the other hand I_2 and I_3 require more discussion which we defer to the next section. We give separate treatment for the equal and unequal mass cases.

3. THE CLOSED EXPRESSIONS

A. The equal mass case; $m_1 = m_2 = m$.

In this case matters simplify remarkably because Q reduces to $z^{-1/2}$, whence $I_2 = I_3$ and

$$C(s,z) = I_1 + 2I_2 \qquad (3.1)$$

$$C(s,z) = \frac{2k}{\pi s^{1/2}}\left(1 - \frac{4k^2 z}{s}\right)^{-1}\left[2\ln\left(\frac{(s-4m^2)^{1/2} + s^{1/2}}{2m}\right) + i\pi\right] + \frac{2}{\pi}\left(1 - \frac{4k^2 z}{s}\right)^{-1}\frac{1}{2z^{1/2}}\ln\left|\frac{1+z^{1/2}}{1-z^{1/2}}\right| \qquad (3.2)$$

To simplify notation now and later let us introduce the abbreviations:

$$\lambda = \frac{4k^2}{s}; \quad \mu = \frac{(a-b)^2}{16ab}; \quad \nu = \frac{a+b}{2(ab)^{1/2}}; \quad \omega = \nu - \frac{(ab)^{1/2}}{s} \quad . \tag{3.3}$$

Now

$$\frac{1}{2z^{1/2}} \ln \left|\frac{1+z^{1/2}}{1-z^{1/2}}\right| = \sum_{r=0}^{\infty} \frac{z^r}{2r+1} \quad . \tag{3.4}$$

If Eq. (3.4) is multiplied by $(1-\lambda z)^{-1}$ and the result expanded in powers of z, one finds:

$$2I_{2,\ell} = \frac{2}{\pi} \sum_{r=0}^{\ell} \frac{\lambda^{\ell-r}}{2r+1} \quad . \tag{3.5}$$

Recalling Eq. (2.13), we get finally:

$$\mathrm{Im}\; C_\ell(s) = \lambda^{\ell+1/2} \tag{3.6}$$

$$\mathrm{Re}\; C_\ell(s) = \frac{2}{\pi}\left[\lambda^{\ell+1/2} \ln\left(\frac{(s-4m^2)^{1/2} + s^{1/2}}{2m}\right) + \sum_{r=0}^{\ell} \frac{\lambda^{\ell-r}}{2r+1}\right] \tag{3.7}$$

B. The general case; $m_1 \neq m_2$.

In this case we can still use Eqs. (2.13) and (3.5) to find $I_{1,\ell}$ and $I_{2,\ell}$, but further work is required to obtain $I_{3,\ell}$. I_3 can be written as follows:

$$I_3 = \frac{(ab)^{1/2}\omega}{2\pi} WXY \tag{3.8}$$

where

$$W = (1-\lambda z)^{-1} \tag{3.9}$$

$$X = \frac{2Q}{(a+b)} \tag{3.10}$$

$$Y = \ln \left|\frac{1+Q}{1-Q}\right| \tag{3.11}$$

and Q(z) is given by Eq. (2.11). Then the n^{th} derivatives of W,X,Y with respect to z at $z=0$ are:

$$\frac{d^nW}{dz^n} = n!\lambda^n; \quad \frac{d^nX}{dz^n} = \frac{(-1)^n 2^n (2n-1)!!\mu^n}{(ab)^{1/2}} ;$$

$$Y = 2\ln\left(\frac{m_1}{m_2}\right) \text{ and } \frac{d^nY}{dz^n} = \nu(n-1)! \sum_{p=0}^{n-1} \frac{(-1)^p 2^p (2p-1)!!\mu^p}{p!} \quad \text{for } n > 0 . \quad (3.12)$$

The notation $(2n+1)!! = 1.3.5.7...2n+1$, $(-1)!! = 1$ from Ref. 8 has been used here. At this point we introduce a further abbreviation:

$$\Omega(n) = \frac{(-1)^n \, 2^n (2n-1)!!}{n!} . \quad (3.13)$$

For the detailed proff of Eq. (3.12) see Appendix B. We now take the ℓ^{th} derivative of I_3 and using Eq. (3.12) in conjunction with Leibnitz's rule find at last the following closed expression for $C_\ell(s)$ in the general case:

$$\text{Im } C_\ell(s) = \lambda^{\ell+1/2} \quad (3.14)$$

$$\text{Re } C_\ell(s) = \frac{1}{\pi}\left[2\lambda^{\ell+1/2} \ln\left(\frac{(s-a)^{1/2} + (s-b)^{1/2}}{2(m_1 m_2)^{1/2}}\right) + \sum_{n=0}^{\ell} \frac{\lambda^{\ell-n}}{2n+1}\right]$$
$$+ \frac{\omega}{\pi} \ln\left(\frac{m_1}{m_2}\right) \sum_{n=0}^{\ell} \Omega(n) \, \lambda^{\ell-n} \, \mu^n$$
$$+ \frac{\omega\nu}{2\pi} \sum_{p=1}^{\ell} \frac{1}{p} \left(\sum_{q=0}^{\ell-p} \Omega(q) \, \lambda^{\ell-q-p} \mu^q\right)\left(\sum_{r=0}^{p-1} \Omega(r) \, \mu^r\right). \quad (3.15)$$

4. CONCLUSION

We have found the generating function $C(s,z)$ for the class of effective Chew-Mandelstam functions $C_\ell(s)$ used by Edwards and Thomas.[2] From $C(s,z)$ we have derived a closed form for the $C_\ell(s)$ in the general case of unequal masses. The advantages of proceeding in this seemingly indirect fashion are well known to mathematicians. As well as being elegant and giving greater power in dealing with refractory expressions, this method may afford insights denied to a more piecemeal approach. One example of this here is the discussion of the equal mass limit. At the level of the generating function [see Eqs. (2.8),(2.9),(2.10)] it is trivial to take the equal mass limit $a \to 4\ m^2$, $b \to 0$. But to do this at the level of the Chew-Mandelstam functions themselves requires a painstaking discussion of the cancellations between divergent terms [see Eq. (3.15)].

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APPENDIX A

$$\mathrm{Re}\ C(s,z) = \frac{s}{\pi}\int_a^\infty ds'\ J(s') \qquad (A.1)$$

where

$$J(s') = \frac{(s'-a)(s'-b)}{(s'-s)P(s')R(s')^{1/2}} \qquad (A.2)$$

and

$$R(s) = (s-a)(s-b) \qquad (A.3)$$

$$\begin{aligned} P(s) &= (1-z)s^2 + z(a+b)s - zab \\ &= s^2 - z(s-a)(s-b) \\ &= s(s - 4k^2 z)\ . \end{aligned} \qquad (A.4)$$

Now use

$$\frac{(s'-a)(s'-b)}{(s'-s)} = s' + (s-a-b) + \frac{(s-a)(s-b)}{(s'-s)} \qquad (A.5)$$

to write

$$\begin{aligned} J(s') &= J_1 + J_2 \\ &= \frac{E}{(s'-s)R(s')^{1/2}} + \frac{F + Gs'}{P(s')R(s')^{1/2}} \end{aligned} \qquad (A.6)$$

where

$$E = \frac{(s-a)(s-b)}{P(s)} \qquad (A.7)$$

$$F = s-a-b - \frac{(s-a)(s-b)}{s}\left(\frac{zab}{P(s)} + 1\right) \qquad (A.8)$$

$$G = 1 - \frac{(s-a)(s-b)(1-z)}{P(s)} \quad . \tag{A.9}$$

Let

$$I_1 = \frac{s}{\pi}\int_a^\infty ds'\ J_1(s') \quad . \tag{A.10}$$

Make the substitution $t = (s'-s)^{-1}$ and use Eq.(2.261) on page 81 of Ref. 8 to get:

$$I_1 = \frac{2k}{\pi s^{1/2}}\left(1 - \frac{4k^2 z}{s}\right)^{-1}\left[2\,\ln\left(\frac{(s-a)^{1/2} + (s-b)^{1/2}}{2(m_1 m_2)^{1/2}}\right) + i\pi\right] . \tag{A.11}$$

Hence Eq. (2.8) is proved.

Now look at

$$L = \frac{s}{\pi}\int_a^\infty ds'\ J_2(s') \tag{A.12}$$

Make the substitution

$$s' = \beta t(1+t)^{-1}$$

where

$$\beta = \frac{2ab}{a+b} \quad . \tag{A.13}$$

Then L can be brought to the form:

$$L = \frac{s\beta}{\pi(1-z)}\int_{\frac{a}{\beta-a}}^{-1} dt\ \frac{A + Bt}{(t^2+x)(t^2+y)^{1/2}} \tag{A.14}$$

where

$$A = -i\ \frac{(a+b)^3\ (ab)^{-3/2}}{(a-b)}\ [4ab + z(a-b)^2]^{-1}\ F \tag{A.15}$$

$$B = \frac{GA}{F} \tag{A.16}$$

$$x = \frac{-z(a+b)^2}{(a-b)^2 z+4ab} \tag{A.17}$$

$$y = \frac{-(a+b)^2}{(a-b)^2} \quad . \tag{A.18}$$

Break L into two parts:

$$L = I_2 + I_3 \tag{A.19}$$

$$I_2 = \frac{s\beta}{\pi(1-z)} \int_{\frac{a}{\beta-a}}^{-1} dt \, \frac{A}{(t^2+x)(t^2+y)^{1/2}} \tag{A.20}$$

$$I_3 = \frac{s\beta}{\pi(1-z)} \int_{\frac{a}{\beta-a}}^{-1} dt \, \frac{Bt}{(t^2+x)(t^2+y)^{1/2}} \; . \tag{A.21}$$

In I_2 make the substitution $t^2 = -v^2(t^2+y)$ as suggested in Sect. 2.25 of Ref. 8. After some elementary steps, one gets finally:

$$I_2 = \frac{1}{\pi}\left[1 - \frac{4k^2 z}{s}\right]^{-1} \frac{1}{2z^{1/2}} \, \ell n \left|\frac{1+z^{1/2}}{1-z^{1/2}}\right| \tag{A.22}$$

and Eq. (2.9) is proved. In I_3 make the substitution $t^2+y = -u^2$. Again after some work, one finds:

$$I_3 = \frac{1}{\pi}\left[1 - \frac{4k^2 z}{s}\right]^{-1} \left[\frac{a+b}{2} - \frac{ab}{s}\right] \frac{Q}{(a+b)} \, \ell n \left|\frac{1+Q}{1-Q}\right| \tag{A.23}$$

where

$$Q = \frac{(a+b)}{2}\left[ab + \frac{z(a-b)^2}{4}\right]^{-1/2} \tag{A.24}$$

and Eq. (2.10) is proved.

One might worry that $C(s,z)$ could have singularities in the z plane which would invalidate the expansion around $z=0$ in Eq. (1.7). Fortunately this is not so. Look at Eqs (2.8),(2.9),(2.10). I_1, I_2, I_3 all have the simple pole at $z = \frac{s}{4k^2} = \frac{(1-a)^{-1}}{s} \frac{(1-b)^{-1}}{s}$. But $s \geq a > b$, hence the pole is always at $z \geq 1$. I_1 has no other singularities in z. I_2, I_3 both have a branch point at $z=1$. Despite appearances, I_2 does *not* have a branch point at $z=0$, nor does I_3 have one at $z = -\frac{4ab}{(a-b)^2} = -\frac{(m_1{}^2-m_2{}^2)^2}{4m_1{}^2m_2{}^2}$. These are the only singularities for finite z. Therefore the expansion 1.7 is valid.

APPENDIX B

Given

$$W = (1-\lambda z)^{-1} \tag{B.1}$$

$$X = 2Q(a+b)^{-1} \tag{B.2}$$

$$Y = \ln \left|\frac{1+Q}{1-Q}\right| \tag{B.3}$$

with

$$Q = \frac{(a+b)}{2}\left[ab + \frac{z(a-b)^2}{4}\right]^{1/2} \tag{B.4}$$

and

$$\Omega(n) = \frac{(-1)^n 2^n (2n-1)!!}{n!} \tag{B.5}$$

one must show that:

$$\frac{1}{n!}\frac{d^n W}{dz^n} = \lambda^n \tag{B.6}$$

$$\frac{1}{n!}\frac{d^n X}{dz^n} = \frac{\Omega(n)\ \mu^n}{(ab)^{1/2}} \tag{B.7}$$

$$\frac{1}{n!}\frac{d^n Y}{dz^n} = \frac{\nu}{n}\sum_{p=0}^{n-1}\Omega(p)\ \mu^p \quad \text{for} \quad n>0 \tag{B.8}$$

$$Y = 2\ \ln\left[\frac{m_1}{m_2}\right]; \quad \text{all at } z=0\ . \tag{B.9}$$

(B.6) is trivial and follows at once from (B.1). From (B.2) one has:

$$X = 2Q(a+b)^{-1} = (ab)^{-1/2}\ [1+4\mu z]^{-1/2} \tag{B.10}$$

[recall the abbreviations μ,λ,ν from Eq. (3.3)].

Expanding (B.10) in powers of z yields

$$X = (ab)^{-1/2}\sum_{n=0}^{\infty} 4^n z^n \mu^n \binom{-1/2}{n} \tag{B.11}$$

where $\binom{n}{r}$ is the binomial coefficient. The coefficient of z^n is

$$\frac{(-1)^n 4^n \mu^n\ (2n-1)!!(ab)^{-1/2}}{2^n n!} = \frac{\Omega(n)\ \mu^n}{(ab)^{1/2}}$$

and so (B.7) is proved. Introduce the useful abbreviations:

$$\alpha = 4\mu; \quad \beta = \frac{a+b}{2}; \quad \gamma = ab; \quad \delta = \frac{(a-b)^2}{4}; \quad \nu = \frac{a+b}{2(ab)^{1/2}}; \quad Y^n = \frac{d^n Y}{dz^n}. \tag{B.12}$$

Then from (B.3):

$$Y^0 = \ln\left[\frac{\beta + (\gamma+\delta z)^{1/2}}{\beta - (\gamma+\delta z)^{1/2}}\right]. \tag{B.13}$$

Further it will be shown that for $n > 0$;

$$Y^n = \frac{\nu\, S_n(z)}{2^{n-1}(1-z)^n(1+\alpha z)^{n-1/2}} \tag{B.14}$$

where $S_n(z)$ satisfies the recursion rule:

$$S_{n+1}(z) = \frac{2dS_n(z)}{dz}\,[1+(\alpha-1)z-\alpha z^2] + S_n(z)\,[2n(1-\alpha)+\alpha+\alpha z(4n-1)] \tag{B.15}$$

with the initial conditions

$$S_1 = 1,\ \frac{dS_1}{dz} = 0 . \tag{B.16}$$

Take the logarithmic derivative of Eq. (B.14) to find:

$$\frac{d}{dz}(\ln Y^n) = \frac{1}{S_n}\frac{dS_n}{dz} + \frac{n}{1-z} + \frac{\alpha(1/2-n)}{1+\alpha z} . \tag{B.17}$$

Multiply (B.17) by Y^n to get:

$$Y^{n+1} = \frac{\nu\left[2(1-z)(1+\alpha z)\frac{dS_n}{dz} + 2S_n[n(1+\alpha z) + \alpha(1/2-n)(1-z)]\right]}{2^n(1-z)^{n+1}(1+\alpha z)^{n+1/2}} . \tag{B.18}$$

Comparison of (B.14) and (B.18) then proves (B.15) by induction from Y^1. In particular, (B.16) is easily established by differentiating (B.13) once. Introduce the notation for the coefficients of the polynomial

$$S_n(z) = \sum_{i=0}^{n-1} s_n^i z^i . \tag{B.19}$$

Then Eq. (B.15) translates into the following recursion rule for the S_n^i:

$$S_{n+1}^k = 2(k+1)S_n^{k+1} + [2(n-k) - \alpha(2(n-k) - 1)]S_n^k + \alpha[2(2n-k) + 1]S_n^{k-1} \quad (B.20)$$

where S_b^a is always zero for $a \gtrless b$. The solution to (B.20) which satisfies the initial conditions (B.16) is :

$$S_n^k = \binom{n-1}{n-k-1} \frac{(2n-1)!!\ \alpha^k 2^{n-k-1}(n-k-1)!}{(2n-2k-1)!!} \sum_{p=0}^{n-k-1} \frac{(-1)^p (2p-1)!!\ \alpha^p}{2^p\ p!} . \quad (B.21)$$

Here $\binom{n}{r}$ denotes the binomial coefficient and the double factorial notation is explained in Sect. 3. Equation (B.21) can then be established from (B.20) by induction on n. This lengthy algebra will not be reproduced here. From (B.19) and (B.21) one has:

$$S_n(0) = S_n^0 = 2^{n-1}(n-1)! \sum_{p=0}^{n-1} \frac{\Omega(p)\,\alpha^p}{4^p} . \quad (B.22)$$

Combining this result with (B.14) one finds:

$$\frac{1}{n!} Y^n(z=0) = \frac{\nu}{n} \sum_{p=0}^{n-1} \Omega(p)\,\mu^p \quad \text{for} \quad n > 0 ,$$

proving Eq. (B.8). Equation (B.9) follows at once from putting $z=0$ in (B.3). Thus finally Eq. (B.6) to (B.9) are proved.

To establish (3.15), recall (3.8):

$$I_3 = \frac{(ab)^{1/2}\,\omega WXY}{2\pi} . \quad (B.23)$$

By Leibniz's rule one has:

$$\frac{d^\ell I_3}{dz^\ell} = \frac{(ab)^{1/2}\omega}{2\pi} \sum_{p=0}^{\ell} \binom{\ell}{p} Y^p \sum_{q=0}^{\ell-p} \binom{\ell-p}{q} W^{\ell-p-q}\, X^q \quad (B.24)$$

where the obvious notation $Y^n = \frac{d^nY}{dz^n}$, $Y^0 = 0$, *etc.* has been used.

Hence

$$\frac{d^{\ell}I_3}{dz^{\ell}} = \frac{(ab)^{1/2}\omega}{2\pi}\left[\binom{\ell}{0}Y\sum_{q=0}^{\ell}\binom{\ell}{q}W^{\ell-q}\;X^{q} + \sum_{p=1}^{\ell}\binom{\ell}{p}Y^{p}\sum_{q=0}^{\ell-p}\binom{\ell-p}{q}W^{\ell-p-q}\;X^{q}\right] \tag{B.25}$$

and

$$\frac{d^{\ell}I_3}{dz^{\ell}} = \frac{(ab)^{1/2}\omega}{2\pi}\left[2\ell n\left(\frac{m_1}{m_2}\right)\sum_{q=0}^{\ell}\binom{\ell}{q}\frac{(\ell-q)!\lambda^{\ell-q}q!\Omega(q)\mu^{q}}{(ab)^{1/2}} + \sum_{p=1}^{\ell}\binom{\ell}{p}\frac{p!\nu}{p}\times \times\sum_{r=0}^{p-1}\Omega(r)\mu^{r}\sum_{q=0}^{\ell-p}\binom{\ell-p}{q}\frac{(\ell-p-q)!\lambda^{\ell-p-q}q!\Omega(q)\mu^{q}}{(ab)^{1/2}}\right] \tag{B.26}$$

So

$$\frac{1}{\ell!}\frac{d^{\ell}I_3}{dz^{\ell}} = \frac{\omega}{\pi}\,\ell n\left(\frac{m_1}{m_2}\right)\sum_{n=0}^{\ell}\Omega(n)\lambda^{\ell-n}\;\mu^{n} + \frac{\omega\nu}{2\pi}\sum_{p=1}^{\ell}\frac{1}{p}\left[\sum_{q=0}^{\ell-p}\Omega(q)\lambda^{\ell-p-q}\;\mu^{q}\right]\left[\sum_{r=0}^{p-1}\Omega(r)\;\mu^{r}\right]\;. \tag{B.27}$$

Combining this result with those for $I_{1,\ell}$ and $I_{2,\ell}$ one gets at long last Eq. (3.15).