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SOME PROBLEMS ON NON-LINEAR SEMIGROUPS AND THE BLOW-UP OF INTEGRAL SOLUTIONS *

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I. INTRODUCTION

It is well-known that the semigroup S on $\overline{D(A)}$ generated by a non-linear (possible multivalued) operator A (via (2.8) or (2.13) may not be differentiable (see Crandall and Liggett [4]). However, in continuous case this semigroup is differentiable everywhere on $[0,+\infty)$ (Theorem 2.3 here, of R.H. Martin Jr.). The most general condition under which an operator of dissipative type generates a semigroup is the "tangential condition" (2.9) introduced by Kobayashi [8] and extended by the author (see (2.19)) to non-autonomous case. This condition is strictly more general than the standard "range condition" (2.10). In applications to PDE the range condition is most useful. However, the tangential condition is still important in PDE, since e.g. in continuous case it is equivalent to Nagumo condition (2.14) (which can be applied in PDE). Moreover, this condition (2.9) has a unifying effect in the theory of non-linear semigroups. Indeed, first of all from (2.9) (Theorem 2.1) we can easily derive the exponential formula of Crandall and Liggett ((2.13) in Theorem 2.2 here). This is a known fact. Another important fact is that Theorem 2.3 (i.e. the differentiability of S in continuous case) can also be easily derived from Theorem 2.1. This is important since the original proof of Theorem 2.3 is more difficult than the proof of Theorem 2.1 itself. Such aspects are presented in Sec.II. Finally we note that condition (2.9) can also be used in the theory of m-dissipativity of the sum of two m-dissipative operators [8,13].

The theory of semigroups on closed subsets of Banach manifolds is started in [9], as an extension of Nagumo-Brezis theorem (see Theorem 2.6 of this paper).

The main result in Sec.III is given by Theorem 3.5 (and Corollary 3.1), which asserts that classical results on the behaviour of the maximal solution u as $t \uparrow t_{max}$, holds also for integral solutions. The proof of this fact is not obvious and uses essentially some inequalities of Benilan.

In the first part of Sec.III we recall some results on blowing-ups from semilinear case. This is to give the reader (not familiarized with the subject) the possibility to see some examples in such a case.

Some problems for further study are pointed out in Remark 1 and 2.

II. A UNIFYING APPROACH IN THE THEORY OF NON-LINEAR SEMIGROUPS

Let X be a real Banach space of norm $\|\cdot\|$ and let X^* be its dual.

Recall some useful notions. A multivalued operator $A : D(A) \subset X \to 2^X$ is identified with its graph (denoted again by A) $A \subset X \times X$, where $D(A)$ is the domain of A.

Definition 1.1

The operator (set) $A \subset X \times X$ is said to be dissipative if for all $[x_j, y_j] \in A$ (i.e. $x_j \in D(A)$ and $y_j \in Ax_j$, $j=1,2$) we have

$$\|x_1 - x_2\| \le \|x_1 - x_2 - \lambda(y_1 - y_2)\|, \ \forall \lambda > 0 \tag{2.1}$$

It is known (see e.g. [13],[4]) that (2.1) is equivalent to

$$\langle y_1 - y_2, x_1 - x_2 \rangle_i \le 0, \ \forall [x_j, y_j] \in A, \ j=1,2$$

where

$$\langle y, x\rangle_i = \inf\{x^*(y), x^* \in J(x)\} = \lim_{t \uparrow 0} \frac{\|x+ty\|^2 - \|x\|^2}{2t} \tag{2.2}$$

As usual, $J : X \to 2^{X^*}$ denotes the duality mapping. The set A is said to be ω-dissipative ($\omega \in \mathbb{R}$) if $A - \omega I$ is dissipative (I - the identity on X) A is said to be quasi-ω-dissipative if [8]

$$\langle y_1, x_1 - x_2\rangle_i + \langle y_2, x_2 - x_1\rangle_i \le \omega\|x_1 - x_2\|^2, \ \forall [x_j, y_j] \in A, \ j=1,2 \tag{2.3}$$

Note that in the applications to PDE, we do not know quasi-dissipative operators which are not dissipative. However, theoretically, the generality from dissipativity to quasi-dissipativity costs nothing in terms of analytical difficulty (in some cases).

Definition 1.2.

A semigroup on a subset $D \subset X$ is a function $S : [0,+\infty) \times D \to D$ (i.e. $S(t) : D \to D$, $\forall\, t \geqslant 0$) satisfying also

$$S(t+\tau) = S(t), S(\tau), \quad t, \tau \quad 0 \tag{1}$$

$$S(0) = I \tag{2}$$

$$\lim_{t \downarrow 0} S(t)x = x, \quad \forall\, x \in D. \tag{3}$$

If in addition

$$\|S(t)x - S(t)y\| \le e^{\omega t}\|x - y\|, \ \forall\, x, y \in D, \ t \geqslant 0 \tag{4}$$

then S is said to be a semigroup of type ω on D.

The theory of the semigroups S generated by dissipative (possible multivalued) operators A, is important in the study of the Cauchy problem

$$u'(t) \in Au(t),\ u(0) = x_0 \in \overline{D(A)},\ t \geq 0 \tag{2.4}$$

This because under natural conditions on A (see theorems 2.1 and 2.3 below), the solution $u = u(t;x)$ of 2.4) is given by $u(t;x_0) = S(t)x_0$. For what we have to point out in this section, we first need to give briefly the definition of DS-approximate solutions [8]. To this goal, let us fix an arbitrary $T > 0$. For each positive integer n, suppose that there are:

a partition $\Delta_n = \{0 = t^n_0 < t^n_1 < \ldots < t^n_{N_n-1} < T \leq t^n_{N_n}\}$ of $[0,T]$ with $\max_{1\leq j\leq N_n} (t^n_j - t^n_{j-1}) \to 0$ as $n \to \infty$ and some elements $x^n_j \in D(A)$, $P^n_j \in X$ such that

$$\frac{x^n_j - x^n_{j-1}}{t^n_j - t^n_{j-1}} - P^n_j \in Ax^n_j,\ j = 1,2,\ldots,N_n \tag{2.5}$$

$$x^n_0 \to x \in \overline{D(A)},\ P_n = \sum_{j=1}^{\infty} \|P^n_j\|(t^n_j - t^n_{j-1}) \to 0 \text{ as } n \to \infty \tag{2.6}$$

The step function

$$u_n(t) = \begin{cases} x^n_0, & \text{if } t = 0 \\ x^n_i, & \text{if } t \in (t^n_{i-1}, t^n_i] \cap (0,T] \end{cases} \tag{2.7}$$

is said to be a DS-approximate solution to (2.4). The operator $S(t)$ defined by the limit of $u_n(t)$ (provided it exists), i.e.

$$S(t)x_0 = \lim_{n\to\infty} u_n(t),\ t \in [0,T],\ T > 0 \tag{2.8}$$

does not depend on the DS-approximate solution u_n (and $S(t)$ is said to be the semigroup generated by A).

Very general conditions under which both u_n and $\lim_{n\to\infty} u_n(t)$ exist, are given by the following result of Kobayashi [8].

Theorem 2.1

Assume that the set $A \subset X \times X$ is quasi-ω-dissipative and satisfies the "tangential condition"

$$\lim_{h \downarrow 0} h^{-1} d[x; R(I-hA)] = 0,\ \forall x \in \overline{D(A)} \tag{2.9}$$

Then for each $T > 0$ and for each $x_0 \in \overline{D(A)}$, the problem (2.4) has a DS-approximate solution u_n (given by (2.7) and the limit (2.8) holds uniformly on $[0,T]$. Moreover, the semigroup S generated by A (via (2.8)) is of type ω.

Remark 2.1

By $d[x,D]$, one denotes the distance from the point $x \in X$ to the set $D \subset X$. In the applications to PDE, the following "range condition" is useful

$$\bigcap_{0<\lambda\leq\lambda_0} R(I-\lambda A) \supset \overline{D(A)} \tag{2.10}$$

where $\lambda_0 > 0$ is sufficiently small. In the case, $R(I - \lambda A) = X$, $\forall \lambda > 0$, A is said to be m-dissipative. Clearly, (2.10) implies (2.9), but the converse implication is not necessarily true.

First of all, from Theorem 2.1 we can easily derive the exponential formula of Crandell and Liggett [4]. This is a well-known fact [8,13]. Indeed, let us assume that (2.10) holds (and that A is dissipative). Let $t \in (0,T)$ be arbitrarily fixed. Set $t^n_j = j\frac{t}{n}$, $j = 0,1,\ldots,N_n$ where N_n is the first positive integer such that $\frac{t}{n} \cdot N_n \geq T$. Let $x^n_0 \in D(A)$ with the property that $x^n_0 \to x$ $\overline{D(A)}$. Define $x^n_j = (I - \frac{t}{n}A)^{-1}x^n_{j-1}$, $j = 1,2,\ldots,N_n$. In this case (2.5) holds with $P^n_j = 0$, and clearly

$$x^n_j = (I - \tfrac{t}{n}A)^{-j} x^n_0,\ j = 0,1,\ldots,N_n \tag{2.11}$$

Consequently, $u_n(s) = x^n_j$ for $(j-1)\frac{t}{n} < s \leq j\frac{t}{n}$. Since $(n-1)\frac{t}{n} < t \leq n.\frac{t}{n}$, it follows that

$$u_n(t) = x^n_n = (I - \tfrac{t}{n}A)^{-n} x^n_0 \in D(A) \tag{2.12}$$

Accordingly, from Theorem 2.1 one obtains the following basic result of Crandall and Liggett [4].

Theorem 2.2

Let $A \subset X \times X$ be ω-dissipative such that (2.10) holds. Then the limit

$$\lim_{n\to\infty} (I - \frac{t}{n} A)^{-n} x_0 = S(t)x_0, \quad t \geqslant 0 \tag{2.13}$$

exists uniformly on compact subsets of $[0,+\infty)$ and S is a semigroup of type ω on $\overline{D(A)}$.

In general, the semigroup S generated by the A via the exponential formula (2.13) may not be differentiable. Recall that S is said to be differentiable at $t_0 > 0$, if the function $t \to S(t)x_0$ is differentiable at $t = t_0$, in the strong sense, $\forall\, x_0 \in \overline{D(A)}$. An important case in which S is differentiable on $[0,+\infty)$ is the following one. "$D(A) = \overline{D(A)} \equiv D$ and $A:D \to X$ is continuous." Precisely, the following result holds (Martin, R.H. Jr. [13]).

Theorem 2.3

Let D be a (non-empty) closed subset of X. If $A:D \to X$ is a continuous and dissipative operator, satisfying the tangential condition

$$\lim_{h \downarrow 0} h^{-1} d[x + hAx; D] = 0 \quad , \ \forall\, x \in D \tag{2.14}$$

then the semigroup S on D generated by A (via (2.8)) is everywhere differentiable on $[0,+\infty)$ and (consequently) the function $u(t) = S(t)x_0$ is the unique solution the Cauchy problem

$$u'(t) = Au(t), \ u(0) = x_0 \in D, \ t \geqslant 0 \tag{2.15}$$

This result is an extension of Nagumo and Brezis result, from A-Lipschitz continuous, to A-dissipative (or only ω-dissipative) and continuous. The original proof of Theorem 2.3 is more difficult than the proof of Theorem 2.1. Our main remark in this section is just to show that Theorem 2.3 can be easily derived from Theorem 2.1, too. Let us briefly show this fact. By standard arguments [10,13,14], under the hypotheses of Theorem 2.3 we can construct a polygonal approximate solution of the form

$$y_n(t) = x_i^n + (t - t_i^n)(Ax_i^n + p_i^n) \ , \ t_i^n \le t \le t_{i+1}^n \ , \ \| p_i^n \| \le \frac{1}{n} \tag{2.16}$$

with
$$x_{i+1}^n = x_i^n + (t_{i+1}^n - t_i^n)(Ax_i^n + p_i^n) \ , \quad t_i^n \le T, \ i = 0,1,\dots$$

for some $T > 0$. Moreover $x_i \in D$ and

$$\| Ax_{i+1}^n - Ax_i^n \| \le \frac{1}{n} \ , \ i = 0,1,\dots \quad ; \ n = 1,2,\dots$$

The main remark is that the elements x_i^n satisfy

$$\frac{x_{i+1}^n - x_i^n}{t_{i+1}^n - t_i^n} - \varepsilon_i^n = Ax_{i+1}^n$$

with
$$\varepsilon_i^n = p_i^n + Ax_i^n - Ax_{i+1}^n \quad , \quad \text{so} \ \| \varepsilon_i^n \| \le \frac{2}{n}$$

Consequently, the step function

$$\tilde{u}_n(t) = \begin{cases} x_0 \ , & t = 0 \\ x_{i+1}^n \ , & t_i^n < t \le t_{i+1}^n \ , \ i = 0,1,\dots \end{cases} \tag{2.17}$$

is a DS-approximate solution to the problem (2.15). Moreover, it is an elementary fact to prove that in this case (2.9) is equivalent to (2.14) [13]. By Theorem 2.1 it follows that

$$\lim_{n\to\infty} \tilde{u}_n(t) = S(t)x_0 \tag{2.18}$$

exists uniformly on $[0,T]$, and defines the semigroup S on D (for $t \in [0,T[$). On the other hand it is easy to see that

$$\| y_n(t) - u_n(t) \| \le \frac{2M}{n} \ , \ \forall\, t \in [0,T]$$

where $M > 0$ is a constant independent of t and n. Accordingly, we also have $\lim_{n\to\infty} y_n(t) = S(t)x_0 \in D$, uniformly on $[0,T]$. Without Theorem 2.1, the proof of the convergence of $y_n(t)$ as $n \to \infty$, is a very delicate problem, since $y_n(t) \notin D$ for $t \neq t_i^n$ (so $Ay_n(t)$ has no sense for $t \neq t_i^n$). But it is standard to show that (with $y(t) = \lim_{n\to\infty} y(t)$)

$$y(t) = x_0 + \int_0^t Ay(\tau)\, d\tau \quad , \ t \in [0,T]$$

which shows that $y(t) = S(t)x_0$ is a solution to (2.15) on $[0,T]$. The uniqueness and the extendability of y on $[0,+\infty)$ are also standard, and thus Theorem 2.3 follows from Theorem 2.1, too.

Remark 2.2

In time dependent case (i.e. $A = A(t)$) the condition (2.9) was extended by the author in [11], under the form

$$\lim_{h\downarrow 0} h^{-1} d[x; R(I-hA(t+h))] = 0 \ , \ \forall\, x \in \overline{D(A(t))} \ , \ t \geq 0 \tag{2.19}$$

while (2.14) was extended by the author and Vrabie [15] as follows

$$\lim_{h\downarrow 0} h^{-1} d[x + hA(t)x; D(A(t+h))] = 0 \ , \ \forall\, x \in D(A(t)),\ t \geq 0 \tag{2.20}$$

where $D(A(t))$ is supposed to be closed (in (2.20)).

We end this section with the problem of the one parameter semigroup on a closed subset D of a Banach manifold M [9].

Let $A: M \to TM$ (the tangent bundle of M) be a vector field on M. Then A is said to be quasi-tangent to $D \subset M$, if for each $x \in D$, there is a chart (U, φ) of M at x such that

$$\lim_{h\downarrow 0} h^{-1} d[\varphi(x) + h D(\varphi)_x(Ax); \varphi(U \cap D)] = 0 \tag{2.21}$$

Firstly one proves that the property (2.21) is independent of the chart (U, φ). Secondly, in the case in which D is a submanifold of M, one proves that (2.21) is equivalent to the tangency to D in classical sense (i.e. with the fact that $A: D \to TD$). As usual, D is said to be a flow-invariant set with respect to the vector field A if any integral curve γ of A starting from D, remains in D as long as it exists (to the right of the initial date $t = t_0 \geq 0$).

Recall the following result from [9], which is an extension of Nagumo and Brezis result from Banach spaces to Banach manifolds.

Theorem 2.6

Let M be a C^k-manifold ($k \geq 1$) and let D be a closed subset of M. Then D is a flow-invariant set with respect to the locally Lipschitz vector field $A: M \to TM$, if A is quasi-tangent to D (in the sense of (2.21)). If in addition D is compact, then A generates a semigroup $S(t)$ on D, given by $S(t)x_0 = \gamma(t)$, $t \geq 0$, where γ is the integral curve of A, starting from $x_0 \in D$, at $t = 0$ (i.e. $\gamma(0) = x_0 \in D$).

Remark 2.3

In the case of $X = R = (-\infty, +\infty)$, any non-increasing function $A: R \to R$ generates a (non-linear) semigroup. Indeed, in this case A, can be extended to a maximal dissipative set $\tilde{A}$, as follows

$$\tilde{A}x = \begin{cases} [A(x+), A(x-)], & \text{if } A \text{ is not continuous at } x \\ Ax, & \text{if } A \text{ is continuous at } x, \end{cases}$$

where

$$A(x-) = \lim_{y\uparrow x} Ay,\quad A(x+) = \lim_{y\downarrow x} Ax.$$

Or, it is known that the problem

$$\tilde{x}'(t) \in \tilde{A}\tilde{x}(t) \ , \ \text{a.e. on } [0, +\infty) \ , \ \tilde{x}(0) = x_0$$

has a unique solution $\tilde{x}(t) = S(t)x_0$, $t \geq 0$. It is not difficult to check that actually we have

$$\tilde{x}'(t) = A\tilde{x}(t) \ , \ \text{a.e. on } [0, +\infty) \ , \ \tilde{x}(0) = x_0$$

(which is left to the reader). By a different method, this result can be also derived from some techniques of Filippov (Trans. Am. Math. Soc. 42 (1964)) 9-277).

III. ON THE BLOWING-UP OF MILD AND INTEGRAL SOLUTIONS

3.1 A brief survey on blow-up of mild solutions

Let A be a linear (densely defined) operator generating the linear semigroup S of class C_0. Let also $f: R_+ \times D \to X$ be a continuous function, where D is a closed subset of X.

For applications in PDE, the following abstract problem is important

$$u'(t) = Au(t) + f(t, u(t)),\ u(t_0) = x_0 \in D,\ u(t) \in D,\ t \geq t_0 \tag{3.1}$$

In general this problem may have no strong solutions but under additional conditions either on A or on f, it has mild solutions. Let us recall this notion.

If J_u is a subinterval of $[t_0,+\infty)$ with $t_0 \in J_u$, then by a mild solution to (3.1) we mean a continuous function $u:J_u \to X$ satisfying Volterra integral equation

$$u(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)\, f(s,u(s))\, ds \tag{3.2}$$

and the constraint

$$u(t) \in D, \quad t \in J_u \tag{3.3}$$

A necessary condition for the existence of a solution to (3.2) and (3.3) is the following one [10]

$$\lim_{h\downarrow 0} h^{-1} d[S(h)x + h f(t,x); D] = 0, \quad \forall x \in D,\ t \geq 0 \tag{3.4}$$

If for each $t_0 \geq 0$ and $x_0 \in D$, the problem (3.2) and (3.3) has a local solution u, then by Zorn's lemma there is a maximal solution

$$u : [t_0, t_{max}) \to D \quad \text{to } (3.2)+(3.3).$$

The following result of classical nature is well-known (see e.g. [13], [17]).

<u>Theorem 3.1</u>

Let $f:R_+ \times D \to X$ be a continuous function which maps bounded subsets into bounded subsets. Moreover, suppose that for each $t_0 \geq 0$ and $x_0 \in D$ (with D-closed) the problem (3.2) and (3.3) has a local solution. Let $u:[t_0,t_{max}) \to D$ be a maximal solution to (3.2) corresponding to initial data (t_0,x_0). Then

a) Either $t_{max} = +\infty$, or

b) $t_{max} < +\infty$ and $\lim_{t\uparrow t_{max}} ||u(t)|| = +\infty$.

In the situation b) we say that the solution u blows-up in finite time. The first results of this type were given with "$\limsup_{t\uparrow t_{max}}$" in place of "$\lim_{t\uparrow t_{max}}$". For concrete PDE it is important to know which one of the situations a) or b) holds. We shall mention here such cases. Generally, a simple but restrictive condition in which we have the situation a) is given by

$$||f(t,x)|| \leq a(t)||x||, \quad x \in D,\ t \geq 0 \tag{3.5}$$

Indeed, in view of (3.5), if we assume that $t_{max} < +\infty$, then by (3.2) and Gronwall's lemma it follows that $||u(t)||$ is bounded on $[t_0,t_{max})$ (which contradicts b)). Consequently, we have (under (3.5)) $t_{max} = +\infty$.

If we assume that $S(t)$ is compact for $t > 0$ (Pazy [17]) we have local existence of the solution to (3.2) and (3.3) [10]. If $S(t)$ is not (necessarily) compact (for $t > 0$), but $x \to f(t,x)$ is dissipative, then we have global existence (i.e. the situation a)) [13]. Finally, if the mild solution u is differentiable at a point $t > t_0$, then it satisfies (3.1) at t.

Before the presentation of the non-linear version of Theorem 3.1 (with $D = X$) we want to recall the problem of the blowing-up of some concrete PDE. To this goal, we first assume that A generates a holomorphic semigroup $S(t)$ of class C_0 with $||S(t)|| \leq M$, $\forall\, t \geq 0$, and $A^{-1} \in L(X)$. Then for $0 < \alpha < 1$ the fractional power $(-A)^{\alpha}$ can be defined by

$$(-A)^{\alpha}x = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} t^{\alpha-1} A(t+A)^{-1}x\, dt \tag{3.6}$$

With respect to the norm $||u||_{(\alpha)} = ||(-A)^{\alpha}u||$, the space $X_\alpha = D(-A^{\alpha})$ is a Banach space. Moreover, if $f:X_\alpha \to X$ is locally Lipschitz (i.e. $||f(u) - f(v)|| \leq C_B||u - v||_\alpha$, $\forall\, u,v \in B$ (B - a bounded subset of X_α, $C_B > 0$) then there is a unique function

$$u \in C([0,t_{max});X_\alpha) \cap C^1([0,t_{max});X) \quad \text{with}$$

$$u(t) \in D(A) \text{ on } [0,t_{max}) \quad \text{, satisfying}$$

$$u'(t) = Au(t) + f(u(t)), \quad 0 \leq t < t_{max}, \quad u(0) = x_0 \in D(A) \tag{3.7}$$

and

$$\lim_{t\uparrow t_{max}} ||u(t)||_\alpha = +\infty . \tag{3.8}$$

For the proof see Ball [1].

Let us consider the following PDE <u>of parabolic type</u>

$$u_t = \Delta u + u|u|^{\gamma-1}, \quad x \in \Omega, \quad \gamma > 1 \tag{3.9}$$

$$u(0,x) = u_0(x) \ \text{ in } \Omega, \quad u|_\Gamma = 0 \tag{3.10}$$

where Ω is a bounded domain of R^n with smooth boundary Γ, and $u_0 \in H_0^1(\Omega)$. The behaviour of the solution u to the problem (3.9) and (3.10) depends on the initial data u_0. To see this aspect, we need the energy functional

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{\gamma+1}|u|^{\gamma+1}\right) dx \tag{3.11}$$

where $E:H^1(\Omega)\cap L^{\gamma+1}(\Omega)\to R$, Δ and ∇ are the Laplace (gradient) operators, respectively. The following result holds [1].

Theorem 3.2

Let $1<\gamma<n(n-2)$ for $n\geq 3$ and $\gamma>1$ if $n=1$ or $n=2$. If $u_0\in H_0^1(\Omega)$ there is a unique solution u to (3.9) and (3.10), $u\in C([0,t_{max}); H^1_0(\Omega))\cap C^1([0,t_{max}); L^2(\Omega))$, $u(t)\in H^1_0(\Omega)\cap H^2(\Omega)$, $t\in[0,t_{max})$. If in addition we choose u_0 such that $E(u_0)\leq 0$ (which it is possible since $\gamma>1$), then $t_{max}<+\infty$ and $\lim_{t\uparrow t_{max}}\int_\Omega |u|^{\gamma+1}dx=+\infty$.

Note that for the proof one takes $X=L^2(\Omega)$, $A=\Delta$, $D(A)=H_0^1(\Omega)\cap H^2(\Omega)$, $X_{1/2}=H_0^1(\Omega)$ and $f(u)=u|u|^{\gamma-1}$. To have $f:X_{1/2}\to X$ we need $|u|^\gamma\in L^2$ (i.e. $u\in L^{2\gamma}$). Or, for the embedding of $H^1(\Omega)\subset L^{2\gamma}(\Omega)$ we need $\frac{1}{2\gamma}>\frac{1}{2}-\frac{1}{n}$ for $n\geq 3$ (i.e. the condition appearing in the theorem).

Thus the existence of the solution u as in the theorem, is a consequence of the above result on (3.7). For the last part of the conclusion of the theorem one uses the function

$$F(t)=\int_\Omega (u(t,x))^2dx \quad , \quad 0\leq t<t_{max} \tag{3.12}$$

Using Green's formula and (3.10) we have

$$\int_\Omega u\,\Delta u\,dx=-\int_\Omega|\nabla u|^2dx \tag{3.13}$$

hence (by (3.9) and (3.13))

$$\dot F(t)=2\int_\Omega u\,u_t\,dx=-2\int_\Omega|\nabla u|^2dx+2\int_\Omega|u|^{\gamma+1}dx \tag{3.14}$$

A simple combination of the energy inequality $E(u(t,\cdot))\leq E(u_0)$ (with $E(u_0)\leq 0$) and (3.14) yield

$$\dot F(t)\geq 2\,\frac{\gamma-1}{\gamma+1}\int_\Omega|u|^{\gamma+1}dx\geq k\,F(t)^{\frac{\gamma+1}{2}} \quad , \quad 0\leq t<t_{max} \tag{3.15}$$

with a constant $k>0$. Integrating (3.15) it follows $t_{max}<+\infty$.

We shall give two more examples here, both of them based on a result of Segal [20]. We start with a result of Brezis and Gallouet [3] on the non-linear Schrödinger equation

$$-i\,u_t-\Delta u+k\,u\,|u|^2=0 \quad , \quad x\in\Omega\ ,\ t\geq 0 \tag{3.16}$$

for the case $n=2$. One of their main theorems is the following one.

Theorem 3.3

Let $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$. If one of the conditions below holds

1) Either $k\geq 0$, or

2) $k<0$ and $|k|\int_\Omega|u_0(x)|^2dx<4$,

then there is a unique solution u to (3.16) and (3.10) such that

$$u\in C([0,+\infty); H^2(\Omega))\cap C^1(R_+; L^2(\Omega))$$

(i.e. $t_{max}=+\infty$).

For the existence of u one applies the result of Segal (mentioned above) while for the proof of $t_{max}=+\infty$ one proves that

$$\|u(t)\|_{H^2(\Omega)} \text{ is bounded on } [0,t_{max}).$$

Finally, the last example is the case of the hyperbolic equation

$$u_{tt}=\Delta u+u\,|u|^{\gamma-1} \quad , \quad x\in\Omega\ ,\ t\geq 0 \tag{3.17}$$

with the conditions

$$u|_\Gamma=0\ ,\ u(0,x)=u_0(x)\ ,\ u_t(0,x)=u_1(x)\ ,\ x\in\Omega \tag{3.18}$$

In this case one uses the energy first integral

$$E(u(t,\cdot),u_t(t,\cdot))=E(u_0,u_1) \quad , \quad t\geq 0 \tag{3.19}$$

where

$$E(u_0,u_1)=\int_\Omega\Big[\tfrac{1}{2}|u_1|^2+\tfrac{1}{2}|\nabla u_0|^2-\tfrac{1}{\gamma+1}|u_0|^{\gamma+1}\Big]dx \tag{3.20}$$

As in the previous results, the choice of the initial conditions u_0 and u_1 plays a crucial role in the behaviour of the solution near t_{max}. Precisely, one can prove for example [1].

Theorem 3.4

Let γ be as in Theorem 3.2. If $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$, then there is a unique maximal solution u to the problem (3.17) and (3.18), such that $u, u_t \in C([0,t_{max});X)$. If in addition we choose u_0 and u_1 with the property $E(u_0,u_1) \leq 0$, then $t_{max} < +\infty$, *and*

$$\lim_{t \uparrow t_{max}} \|u\|_{L^{\gamma+1}} = \lim_{t \uparrow t_{max}} \left(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \right) = +\infty$$

For the proof of the existence one takes

$$X = H_0^1(\Omega) \times L^2(\Omega) \quad \text{and} \quad A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \quad \text{with}$$

$$D(A) = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega), \quad f\binom{u}{v} = \binom{0}{u|u|^{\gamma+1}}$$

and thus the problems (3.17) and (3.18) are reduced to one of the form (3.1). To prove that $E(u_0,u_1) \leq 0$ implies $t_{max} < +\infty$, we consider again the function (3.12) and the energy integral (3.19). Similarly to (3.15), we have

$$F''(t) = \int_\Omega \left(2|u_t|^2 - 2|\nabla u|^2 + 2|u|^{\gamma+1}\right) dx \geq$$

$$2\frac{\gamma-1}{\gamma+1}\int_\Omega |u|^{\gamma+1} dx - 4E(u_0,u_1) \geq \tilde{K}\, F(t)^{\frac{\gamma+1}{2}}, \quad 0 \leq t < t_{max} \tag{3.21}$$

which by integration yields $t_{max} < +\infty$.

3.2 Global behaviour of solutions to non-linear evolution equations

In this section we shall prove that under reasonable hypotheses the conclusion of Theorem 3.1 remains also valid for the non-linear equation (differential inclusion)

$$u'(t) \in Au(t) + F(t,u(t)) + f(t) \tag{3.22}$$

with the initial condition

$$u(t_0) = x_0 \in \overline{D(A)} \cap U, \tag{3.23}$$

where U is an open subset of X such that $U \cap \overline{D(A)}$ in non-empty.

Recall that by an integral solution to (3.22) and (3.23) on the sub-interval $J \subset [0,+\infty)$ with $0 \in J$, we mean a continuous function $u: J \to \overline{D(A)} \cap U$,

satisfying

$$\|u(t)-x\|^2 \leq \|u(t_0+h)-x\|^2 + 2\int_{t_0+h}^{t} \langle F(\tau,u(\tau)) + f(\tau) + y, u(\tau)-x \rangle_s \, d\tau \tag{3.24}$$

for all $[x,y] \in A$, $0 \leq h$, $t_0+h \leq t$, where $\langle \cdot,\cdot \rangle_s$ is defined by (2.2) with "sup" and "$t \downarrow 0$" in place of "inf" and "$t \uparrow 0$", respectively.

Our basic hypotheses are the following.

(I1) $F:[0,+\infty) \times U \to X$ is a continuous (or satisfies Carathéodory conditions) and maps bounded subsets into bounded subsets.

(I2) $A:D(A) \subset X \to 2^X$ is a m-dissipative (possible multivalued) operator whcih generates the semigroup S on $\overline{D(A)}$ (via (2.13)).

(I3) There is $a > 0$ such that the functions $\{t \to S(t)x,\ x \in \overline{D(A)}\}$ are equi-continuous at $t = a$ on bounded subsets of $\overline{D(A)}$.

(I4) $f \in L^1_{loc}([0,+\infty); X)$.

(I5) For each $t_0 \geq 0$ and $x_0 \in \overline{D(A)} \cap U$, the problem (3.22) and (3.23) has a local solution.

Remark 3.1

If we assume that $S(t)$ is compact for $t > 0$, then (I3) and (I5) are fulfilled [22]. ($S(t)$ is compact for $t > 0$ iff (*) $(I-\lambda A)^{-1}$ is compact for $\lambda > 0$ (**) $\{t \to S(t)x,\ x \in B\}$ are equicontinuous on bounded subsets $B \subset \overline{D(A)}$) (by Brezis's result [2]). A possible generalization of Brezis' result would be the characterization of the compactness of $S(t)$ generated by A via Theorem 2.1. In this direction one condition is also (**). But what is the analogous of (*)? An answer to this question would be (theoretically) interesting.

Remark 3.2

Let $Y \subset X$ be a continuously included subspace of X and suppose that the restriction of $S(t)$ to Y is also a semigroup on Y. We can get existence for (3.22) by assuming the compactness of $S(t)$ only in X. This is useful in PDE, since for instance is easier to verify the compactness of $S(t)$ in $L^2(\Omega)$ than in $L^\infty(\Omega) \subset L^2(\Omega)$ (see [15]). The main result of this section is given by the following theorem.

Theorem 3.5

Suppose that (I1), (I2), (I4) and (I5) are fulfilled. Then the problem (3.22) and (2.23) has at least a maximal (integral) solution $u:[t_0,t_{max}) \to U \cap \overline{D(A)}$. Moreover, if in addition (I3) also holds, then the following situations may occur

1) either $t_{max} = +\infty$, or

2) $t_{max} < +\infty$ and $\lim_{t \uparrow t_{max}} ||u(t)|| = +\infty$ (if u is unbounded on $[t_0,t_{max})$), or

3) $t_{max} < +\infty$ and $\lim_{t \uparrow t_{max}} u(t) \equiv u^* \in \partial U \cap \overline{D(A)}$ (if u is bounded on $[t_0,+\infty)$ and U is strictly included in X).

First note that ∂U is the boundary of U. The existence of a maximal solution to the problem (3.22) and (2.23) follows from (I5), Zorn's lemma and

Lemma 3.1

Let u and v be integral solutions to (3.22) on $[t_0,T_1]$ and $[T_1,T_2]$, respectively, with $t_0 < T_1 < T_2$ and $u(T_1) = v(T_1)$. Define $w:[t_0,T_2] \to X$ by $w(t) = u(t)$ on $[t_0,T_1]$ and $w(t) = v(t)$ on $[T_1,T_2]$. Then w is an integral solution to (3.22) and (3.23) on $[t_0,T_2]$.

The proof of the lemma is standard and it is left to the reader.

Proof of Theorem 3.5

We first prove 2), i.e. assume that

$$t_{max} < +\infty \quad \text{and} \quad \limsup_{t \uparrow t_{max}} ||u(t)|| = +\infty \tag{3.25}$$

and prove that (3.25) implies $\liminf_{t \uparrow t_{max}} ||u(t)|| = +\infty$. Indeed, if $\liminf_{t \uparrow t_{max}} ||u(t)|| < \infty$, then there is $r_0 > 0$ and a sequence $t_k \uparrow t_{max}$ with the properties

$$||u(t_k)|| \leq r_0, \quad t_0 < t_k < t_{k+1}, \quad k = 1,2,\ldots \tag{3.26}$$

Set $B = \{u(t_k), k = 1,2,\ldots\}$. Since B is bounded subset of $\overline{D(A)}$, there is $b > 0$ such that (by (I3))

$$||S(h+a)u(t_k) - S(a)u(t_k)|| < 1, \ \forall h \in [0,b], \ k = 1,2,\ldots \tag{3.27}$$

Moreover, since $S(a)$ is non-expansive, $S(a)B$ is bounded (i.e. $||S(a)u(t_k)|| \leq r$, $r > 0$, $\forall k = 1,2,\ldots$ Say $r \geq r_0$).

Set

$$M = \sup\{||F(t,x)||, t \in [0,t_{max}], x \in U, ||x|| \leq 2 + 3r\}$$

According to (I1), we have $M < +\infty$. We may assume that b is sufficiently small so that

$$Mb + \int_{t_m-b}^{t_m} ||f(\tau)|| d\tau < 1 \tag{3.27'}$$

($t_m = t_{max}$). Now let k_0 be sufficiently large such that $t_{max} - b < t_{k_0}$. Then we can prove that

$$||u(t)|| \leq 2 + 3r, \ \forall t \in [t_{k_0}, t_{max}) \tag{3.28}$$

Indeed, if (3.28) were false, then would exist $\bar{h} > 0$ satisfying

$$0 \leq \bar{h} \leq b, \ ||u(t_{k_0}+s)|| < 2 + 3s, \ \forall s \in [0,\bar{h}), \ t_{k_0} + \bar{h} < t_{max} \quad ||u(t_{k_0}+\bar{h})|| = 2 + 3r \tag{3.29}$$

Set

$$v(t) = S(t - t_{k_0})S(a)u(t_{k_0}), \quad t \geq t_{k_0}$$

Then v is the integral solution to the problem

$$v'(t) \in Av(t), \ v(t_{k_0}) = S(a)u(t_{k_0}), \ t \geq t_{k_0}$$

By the well-known inequality of Benilan (which gives an estimate of the difference of two integral solutions) (see e.g. [13] p.256, or [22]) we have

$$||u(t_{k_0}+\bar{h}) - v(t_{k_0}+\bar{h})|| \leq ||u(t_{k_0}) - v(t_{k_0})|| + \int_{t_{k_0}}^{t_{k_0}+\bar{h}} ||F(s,u(s))|| ds$$
$$+ \int_{t_{k_0}}^{t_{k_0}+\bar{h}} ||f(s)|| ds \leq 2r + Mb + \int_{t_m-b}^{t_m} ||f(s)|| ds < 2r + 1 \tag{3.30}$$

(by (3.27)). An obvious combination of (3.27), (3.29) and (3.30) yields the absurdity

$$2 + 3r = ||u(t_{k_0}+\bar{h})|| \leq ||u(t_{k_0}+\bar{h}) - v(t_{k_0}+\bar{h})|| + ||v(t_{k_0}+\bar{h}) - v(t_{k_0})|| + ||S(a)u(t_{k_0})||$$
$$\leq 1 + 2r + ||S(\bar{h}+a)u(t_{k_0}) - S(a)u(t_{k_0})|| + r < 2 + 3r.$$

and thus (3.28) is proved. But (3.28) contradicts the unboundedness of u (see (3.25)) on $[t_0,t_{max})$, hence $\liminf_{t\uparrow t_{max}} ||u(t)||$ has to be $+\infty$. It remains to prove 3). Therefore, let us assume that $||u(t)|| \leq M$, $M > 0$, for all $t \in [t_0,t_{max})$ with $t_{max} < +\infty$. Given an arbitrary $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) > 0$ such that

$$\delta M_1 + \int_{t_m-\delta}^{\delta} ||f(\tau)||\, d\tau < \frac{\varepsilon}{3} \tag{3.31}$$

where $M_1 \geq ||F(t,u(t))||$, for all $t \in [t_0,t_m)$, $t_m \equiv t_{max}$. Denote by v the integral solution to the problem

$$v'(t) \in Av(t)\,,\quad v(t_m-\delta) = u(t_m-\delta)\quad,\quad t_m-\delta \leq t$$

Then (similarly to (3.30))

$$||u(t)-v(t)|| \leq \int_{t_m-\delta}^{t_m} ||F(\tau,u(\tau))||\, d\tau + \int_{t_m-\delta}^{t_m} ||f(\tau)||\, d\tau \leq$$

$$\delta M_1 + \int_{t_m-\delta}^{t_m} ||f(\tau)||\, d\tau < \frac{\varepsilon}{3}\,,\ \forall\ t \in [t_m-\delta, t_m) \tag{3.32}$$

Let $\eta = \eta(\varepsilon) > 0$ be such that $\eta(\varepsilon) < \delta$ and $||v(t) - v(\tau)|| < \frac{\varepsilon}{3}$, for all $t,\tau \in [t_m - \eta, t_m]$. Then we have obviously

$$||u(t) - u(\tau)|| \leq ||u(t)-v(t)|| + ||v(t)-v(\tau)|| + ||v(\tau)-u(\tau)|| < \varepsilon$$

for all $t,\tau \in [t_m - \eta, t_m)$, and therefore $\lim_{t\uparrow t_m} u(t) \equiv \tilde{u}$ exists, and $\tilde{u} \in \overline{D(A)}$. The non-continuability of u and Lemma 3.1 imply necessarily $\tilde{u} \in \partial U$ (since otherwise, u would be extended to an interval containing $[t_0,t_{max}]$). This completes the proof of Theorem 3.5.

In the case $U = X$, the situation 3) cannot occur and hence Theorem 3.5 becomes

<u>Corollary 3.1</u>

Assume that (I1) to (I5) (with $U = X$) are satisfied. Then for the maximal solution $u:[t_0,t_{max}) \to \overline{D(A)}$ we have either $t_{max} = +\infty$, or : $t_{max} < +\infty$ and $\lim_{t\uparrow t_{max}} ||u(t)|| = +\infty$.

<u>Remark 3.3</u>

If F satisfies an estimate of the form (3.5), then in the conditions of Corollary 3.1 we have $t_{max} = +\infty$ (like in semilinear case). Indeed, if v is the solution to $v'(t) \in Av(t)$, $v(t_0) = u(t_0)$, $t \geq t_0$, then we have

$$||u(t) - v(t)|| \leq \int_{t_0}^{t} ||F(\tau,u(\tau))||\, d\tau + \int_{t_0}^{t} ||f(\tau)||\, d\tau$$

hence

$$||u(t)|| \leq \int_{t_0}^{t} a(\tau)||u(\tau)|| + \int_{t_0}^{t} ||f(\tau)||\, d\tau + ||v(t)||,\ t_0 \leq t < t_{max} \tag{3.33}$$

If $t_{max} < +\infty$, then v is bounded on $[t_0,t_{max})$ and (3.33) would imply (by Gronwall's lemma) the boundedness of u. Hence $t_{max} = +\infty$.

<u>Remark 3.4</u>

If $x \to F(t,x)$ is dissipative then it is proved in [13] (Chap.5) that $t_{max} = +\infty$. Finally, the boundary value problem for $u''(t) \in -Au(t)$ with A dissipative has also a global character [see the references in Pavel [13], [21]). A better version of Theorem 3.5 as well as some concrete applications will be given elsewhere.

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