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UNIVERSALITY OF THE TOPOLOGY OF PERIOD DOUBLING DYNAMICAL SYSTEMS

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Universality of the Topology of Period

Doubling Dynamical Systems

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ABSTRACT

The evolution of the topology of the invariant manifolds of the attractors of 3-D autonomous dynamical systems during period doubling is shown to be universal. The overall topology of the nth attractor is shown to depend only on the topology of the first attractor at birth.

Recently, the geometrical aspects of period doubling 3-D continuous dynamical systems have been investigated [1-3]. Numerical results have revealed the twisted nature of the invariant manifolds of the periodic attractors, and it has been pointed out that if period doubling is to occur, it is crucial that the invariant manifolds form nonorientable surfaces with an odd number of twists. Here a twist is defined as a rotation by π of the tangent planes of the invariant manifolds about the orbit over one period. It was found that at birth of a new periodicity the invariant manifolds of the attractor always form orientable surfaces, and that some mechanism takes effect which winds or rewinds the manifolds so as to change an orientable to a nonorientable surface. Invoking similarities with 1-D systems and using a "paper sheet" model, Uezu [2] argued that this winding and rewinding mechanism may be thought of as being caused by the passage of the periodic orbit through a "fold." The number of folds through which the periodic orbit passes in this model was thought to correspond to the number of twists deleted or added. Further numerical results for two particular systems [3] revealed a relationship between the winding/rewinding mechanism and the change of the eigenvalues from positive real to negative real.

In the following we use mathematical properties of 3-D autonomous period doubling systems to exhibit the nature of this winding/rewinding mechanism. In particular we show that, as the bifurcation sequence cascades, exactly one twist is alternately deleted or added, and we prove that the topological character of the invariant manifolds of the attractor of any 3-D autonomous dynamical flow is universal and depends only on initial conditions, i.e., the topology of the first attractor at birth.

Consider a 3-D autonomous dynamical system depending on a parameter μ

$$\dot{\vec{x}} = \vec{F}_\mu(\vec{x}). \tag{1}$$

First we like to list some general properties of the system. Let

$$\dot{\vec{q}} = \overleftrightarrow{J}\,\vec{q} \tag{2}$$

be the associated vector variational equation. Here $\overleftrightarrow{J}$ is the 3 × 3 Jacobian matrix associated with $\vec{F}_\mu$ evaluated along the nth attractor with period T_n. Let $\overleftrightarrow{Q}(t)$ be a 3 × 3 matrix which satisfies the matrix variational equation

$$\dot{\overleftrightarrow{Q}} = \overleftrightarrow{J}\,\overleftrightarrow{Q} \tag{3}$$

integrated along the attractor with $\overleftrightarrow{Q}(0) = \overleftrightarrow{I}$, $\overleftrightarrow{I}$ being the identity matrix. Integrating (3) over T_n we get $\overleftrightarrow{Q}(T_n)$. Then (2) has the solution

$$\vec{q}\,(T_n) = \overleftrightarrow{Q}\,(T_n)\,\vec{q}_o, \tag{4}$$

where $\vec{q}_o = \vec{q}(0)$.

The matrix $\overleftrightarrow{Q}(T_n)$ has the property that its determinant D_n has the value

$$D_n = e^{-2k_n T_n}, \tag{5}$$

where we have defined

$$k_n = -\frac{1}{2T_n} \int_0^{T_n} \mathrm{Tr}\,(\vec{J})\, dt. \tag{6}$$

Let $\gamma_i^{(n)}$ be the eigenvalues and $\vec{v}_i^{(n)}$ the eigenvectors of $\vec{Q}(T_n)$, with $i = 1,2,3$. One of the eigenvectors, say $\vec{v}_3^{(n)}$, is the vector field $\vec{F}_\mu$ itself. The corresponding eigenvalue $\gamma_3^{(n)}$ equals unity independently of the value of μ. Then we find

$$\gamma_1^{(n)} = \alpha_n + \left(\alpha_n^{\,2} - D_n\right)^{1/2} \tag{7}$$

and

$$\gamma_2^{(n)} = \alpha_n - \left(\alpha_n^{\,2} - D_n\right)^{1/2}, \tag{8}$$

where

$$\alpha_n = \frac{\mathrm{Tr}\,\left(\vec{Q}(T_n)\right) - 1}{2}. \tag{9}$$

The evolution of the eigenvalues $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ of $\vec{Q}(T_n)$ as μ is monotonically varied is shown in Fig. 1. For a given periodicity Eq. (5) requires that the product $\left(\gamma_1^{(n)}\ \gamma_2^{(n)}\right)$ is a constant. In addition, $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ must obey Eqs. (7) and (8). Hence $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ start out real positive as the nth attractor is born, say $\gamma_1^{(n)} = 1$, $\gamma_2^{(n)} = D_n$. Then, as μ is varied they move closer, meeting at $D_n^{\,1/2}$. They become complex conjugates and move on the circle of radius $D_n^{\,1/2}$. They meet again at $-D_n^{\,1/2}$, separate, and move along the negative real axis. Since now α_n must be negative, $\gamma_1^{(n)} \rightarrow -D_n$ and $\gamma_2^{(n)} \rightarrow -1$. At $\gamma_2^{(n)} = -1$ a period doubling bifurcation occurs, and the entire scenario repeats for the new periodicity.

Next we like to find an expression that enables us to determine the number of twists of the invariant manifold of the attractor.

Since the twisted nature of the invariant manifold is determined by the components of the eigenvectors perpendicular to $\vec{v}_3^{(n)}$, we use a similarity transformation to diagonalize $\vec{\vec{Q}}(T_n)$. Under the transformation the eigenvalues remain the same, while the eigenvectors are mutually orthogonal such that the vectors $\vec{v}_i^{(n)}$ lie along the x_i-coordinate axes $(i = 1,2,3)$. Hence, we are interested only in the 1,2-components of $\vec{q}$ in (4). Using Floquet's theory the solution to (4) is given by

$$\vec{q}\,(mT_n) = e^{-k_n T_n m} \begin{bmatrix} ae^{+\eta_n T_n m} \\ be^{-\eta_n T_n m} \end{bmatrix} \tag{10}$$

for $\gamma_1^{(n)}$, $\gamma_2^{(n)}$ real, and by

$$\vec{q}\,(mT_n) = ae^{-k_n T_n m} \begin{bmatrix} \cos(-i\eta_n T_n m + b) \\ \sin(-i\eta_n T_n m + b) \end{bmatrix} \tag{11}$$

for $\gamma_1^{(n)}$, $\gamma_2^{(n)}$ complex [4]. Here a and b are real constants relating to the initial condition $\vec{q}_o$; m is the number of iterations; and η_n is defined as

$$\eta_n = \frac{1}{2T_n} \ln \left(\gamma_1^{(n)}/\gamma_2^{(n)}\right) \tag{12}$$

so that

$$\gamma_1^{(n)} = e^{(-k_n + \eta_n)T_n} \tag{13}$$

and

$$\gamma_2^{(n)} = e^{(-k_n - \eta_n)T_n} . \tag{14}$$

Note that η_n is purely imaginary if $\gamma_1^{(n)}$, $\gamma_2^{(n)}$ are complex conjugates.

Hence, knowing η_n and noting that k_n must be real, the evolution of the eigenvectors, and thus of the invariant manifolds of the system as μ is varied, can be found from (10) and (11). The evolution of η_n, however, can be determined easily from the evolution of the $\gamma_i(n)$ and using (13) and (14).

The evolution of η_n is as follows. For the case depicted in Fig. 1 η_n starts (apart from an initial "phase factor") equal to + k_n. As $\gamma_1^{(n)}$ decreases to $D_n^{1/2} = e^{-k_n T_n}$, η_n drops to 0. Then $\gamma_1^{(n)}$ ($\gamma_2^{(n)}$) marches into the upper (lower) complex plane, and η_n is purely imaginary changing from 0 to $i\pi/T_n$. After $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ have met again and $\gamma_1^{(n)} \to - e^{-2k_n T_n}$, the real part of η_n increases from 0 to $-k_n$, while the imaginary part (an odd multiple of π/T_n) stays fixed and gives rise to the minus signs of $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$. At $\gamma_2^{(n)} = -1$ the system undergoes period doubling, i.e, $T_n \to T_{n+1} = 2T_n$. Thus the imaginary part of the exponent in (10) becomes an even multiple of π, and $\gamma_1^{(n+1)}$ and $\gamma_2^{(n+1)}$ both are positive real. The evolution of η_{n+1} for the new periodicity is identical to that of η_n. (See Table 1.)

At the period doubling transition one also finds that $D_{n+1} = [D_n]^2$, $\gamma_1^{(n+1)} = [\gamma_2^{(n)}]^2$, and $\gamma_2^{(n+1)} = [\gamma_1^{(n)}]^2$. Hence $\gamma_1^{(n)}$ matches onto $\gamma_2^{(n+1)}$, and $\gamma_2^{(n)}$ matches onto $\gamma_1^{(n+1)}$, i.e., the eigenvalue associated with the positive (negative) root of the old periodicity matches onto the eigenvalue associated with the negative (positive) root of the new periodicity. Similarly $\vec{v}_1^{(n+1)}$ ($\vec{v}_2^{(n+1)}$) matches onto $\vec{v}_2^{(n)}$ ($\vec{v}_1^{(n)}$). Hence period doubling corresponds to the coordinate transformation (a rotation and inversion)

$$\begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix} \tag{15}$$

where $x_i^{(n)}$ ($x_i^{(n+1)}$) are the coordinates of the old (new) periodicity.* This means that a clockwise (counter-clockwise) twist of the invariant manifolds in the old coordinate frame appears as a counter-clockwise (clockwise) twist in the new frame after period doubling. (Here the sense of rotation is defined with respect to the x_1,x_2-axes.)

Combining the above results we finally find the following evolution of the invariant manifolds as μ varies. For values of μ for which the eigenvalues are positive (negative) real, the vectors tangent to the invariant manifolds are imaged parallel (antiparallel) after one iteration corresponding to exactly an even (odd) number of twists. Hence no new twists can be added in this case. New twists are introduced into the invariant manifolds thus only (i) if the periodicity doubles so that the attractor is twice as long as before resulting in twice as many twists, or (ii) if μ traverses a range of values for which the eigenvalues are complex conjugates. In the latter case η_n always changes by a positive increment of $i\pi/T_n$. Thus, invoking

*An alternate way of accounting for the interchange $\gamma_i^{(n+1)} \to \gamma_j^{(n)}$ $i \neq j$ during period doubling is to let $\eta_{n+1} \to -\eta_{n+1}$ in (10) through (14) and to use the old coordinate frame.

continuity, a full counter-clockwise twist has been added to the invariant manifold by the time μ is such that the $\gamma_i^{(n)}$ are real again. Counting twists when the $\gamma_i^{(n)}$ reemerge real, one therefore finds a twist, either deleted or added to the total number of twists depending on whether the manifolds are twisted clockwise (deletion) or counter-clockwise (addition) at a given periodicity.

Consequently, the number of twists N_n of the nth attractor with two eigenvalues negative real is described by the difference equation

$$N_n = 2\, N_{n-1} + (-1)^{(n-1)}, \tag{16}$$

which has the solution

$$N_n = [(3N_o + 2)\; 2^{n-1} + (-1)^{n-1}]/3, \tag{17}$$

where N_o is the number of twists of the first attractor at birth. This formula was found by computational methods in [3] for the parametric pendulum and Duffing's equation; similar results were found in [2] for the forced Brusselator and the Lorenz model. The above analysis shows that Eq. (17) applies to any 3-D autonomous period doubling system, and that the evolution of the topology of the invariant manifolds of the attractors is universal. The topological character of the manifold of a particular attractor thus depends only upon the overall topology of the first attractor at birth, i.e., on N_o. As a result, only periodic solutions which have the same value of N_o can be bifurcating solutions of each other. Furthermore, N_n is always odd, and the topological character of the manifolds just before period doubling is that of a Moebius strip [5].

It is an interesting and open question as to whether there may be a general relationship that determines N_o. It may be possible to predict the evolution of N_o as attractors are destroyed and created as U is varied, or to find a relationship between the N_o's when several attractors exist simultaneously. Knowledge of such general relationships determining N_o would certainly be very useful in classifying and finding the possible attractors of a given system.

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Tables

Table 1. Evolution of η_n as μ is varied during a given periodicity T_n

$\gamma_1^{(n)}$	$Re(\eta_n)$	$Im(\eta_n T_n)$
$1 \rightarrow e^{-k_n T_n}$	$k_n \rightarrow 0$	unchanged, even multiple of π
complex, upper half plane	0	change by $\pm \pi$
$-e^{-k_n T_n} \rightarrow -e^{-2k_n T_n}$	$0 \rightarrow -k_n$	unchanged, odd multiple of π

Figure Captions

FIG. 1. Evolution of the eigenvalues $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ as a parameter is monotonically varied.

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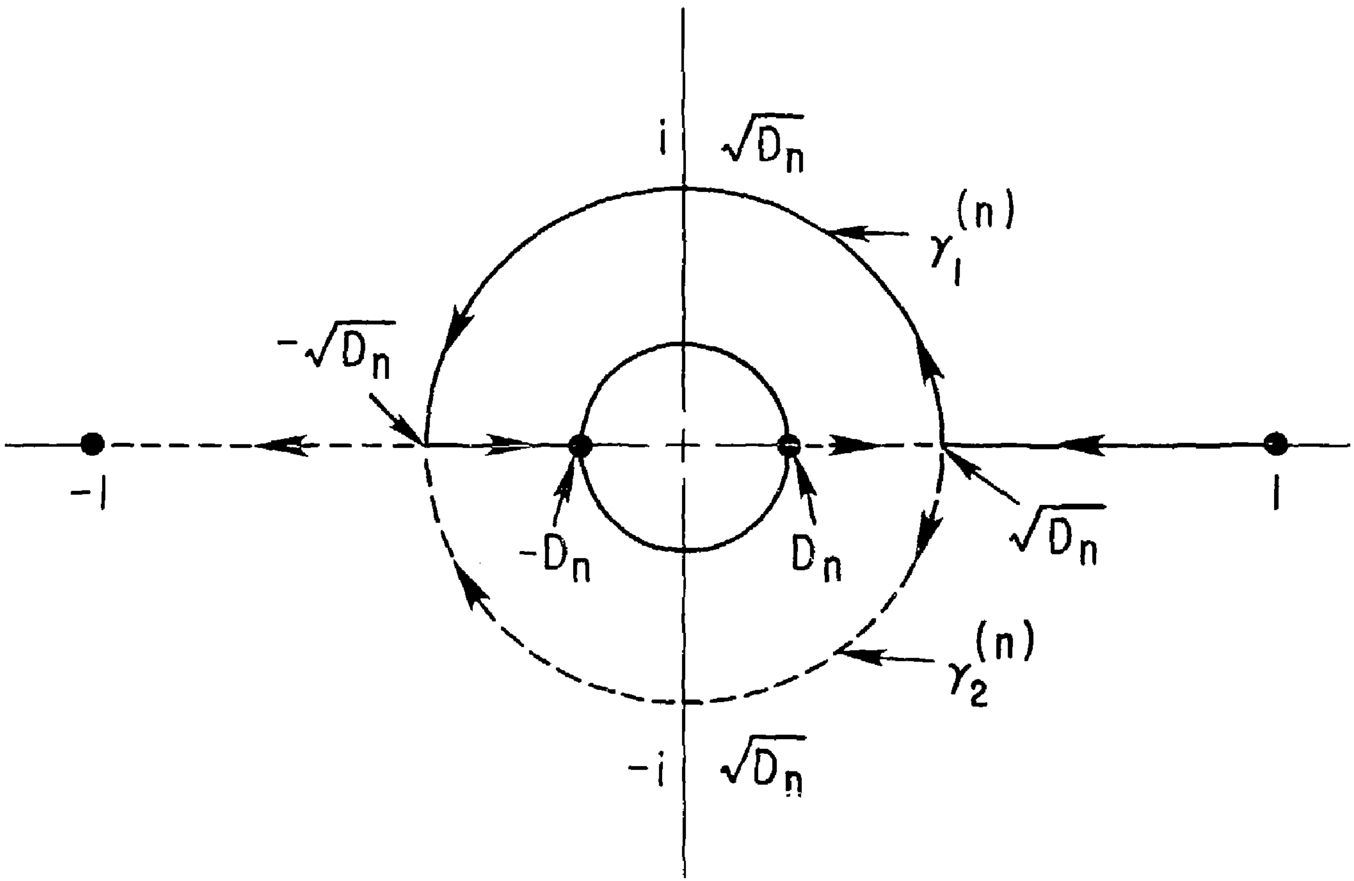

Fig. 1

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