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DISCLAIMER

The Ideal of the Perfect Magnet – Superconducting Systems *

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The Ideal is a measure of the best.
But who speaks for the Ideal,
And when is it ever timely or safe?

I. Introduction

In this report, we study an *iron-free*, superconducting, elliptical coil quadrupole which has been proposed by General Atomics for use in the SLC final focus system. Beth(1) has shown that such coils might provide a pure quadrupole field ignoring 3-D effects. Similarly, recent studies of rare earth *permanent magnets have shown that, at least in principle, these magnets can also be made arbitrarily* pure. Since similar claims can be made for conventional iron-core electromagnets either by demanding pure hyperbolic pole contours or using tricks(2), it is interesting to consider just how wide the gulf between principle and practice really is for each type of magnet and what it takes to bridge it (and where one is most likely to fall off). Here we consider only the superconducting option because its greater strength, variability and linearity make it potentially useful for the SLC and the low-beta insertions of high energy storage rings such as PEP.

II. Description

In principle, it is possible to design the 2-D coils of an iron-free magnet to provide only a single, pure multipole field whose strength varies linearly with current. For instance, Brechna(3) has shown how two intersecting circular or elliptical coils carrying a uniform current density can produce pure dipole fields. He also shows how to get a pure quadrupole by superimposing two elliptical coils. By a similar analysis one can show that it is not even necessary that the two coils have identical shapes but only the same aspect ratio ($\epsilon \equiv a/b = a'/b' > 1$) to get a pure quadrupole. It isn't possible to get pure dipoles in this way but one can make other systems such as constant gradient magnets. Figure 1 shows two configurations with elliptical coil contours ($\epsilon = \epsilon' \approx 2$) which produce pure dipole and pure quadrupole fields. For such pure fields, one finds:

Dipole Configuration:

$$B = -\mu_o \lambda J\left(\frac{2x_1 b}{a+b}\right) \xrightarrow{x_1=a-b} \mu_o \lambda J\left(\frac{\epsilon-1}{\epsilon+1}\right)2b \qquad (1)$$

Quad. Configuration:

$$G = -\mu_o \lambda J\left(\frac{b}{a+b} - \frac{a'}{a'+b'}\right) \xrightarrow{\epsilon=\epsilon'} \mu_o \lambda J\left(\frac{\epsilon-1}{\epsilon+1}\right)$$

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where x_1 is the arbitrary separation between the centers of each ellipse and the center of the magnet and ΔJ is the average current density in the z-direction. The strength limitation of such magnets will be determined by the achievable critical current densities and packing fractions as one lets $\epsilon \to \infty$. For the most commonly used NbTi alloy (4) at 4.2° K and $B_{max} = 1.5(3.0)((5.0))T$, one has $J_c = 4.2(3.0)((1.9))$ kA/mm^2. Given a beam stay-clear area, the required fields and a conservative value for the average current density such as $<J> = 2 \times 10^8 A/m^2$ one can compute the required coil dimensions. Thus, for a gradient of $100T/m$ at a 'radius' $b = 1.5$ cm one finds $a = 3.5$ cm. All prototype calculations done here are based on these numbers.

III. Quantitative Discussion

In practice, the theoretically perfect 2-D coil shape is seldom achievable and even if it were, it might prove academic because of higher order, optical "non-linearities" indigenous to the multipole or its 3-D realization(5). Exact optical calculations are the final arbiter of such questions. Here we are only interested in specific questions concerning the fields and whether they are actually achievable or not. Thus, since people now believe they can wind such coils, some questions to be asked are how do they do it, how accurately must they do it, and what are the various sensitivities. Brechna, without elucidating, commented that "in practice it is impossible to wind dipole (or quadrupole) coils which have the shapes ideally obtained by superposition of circles or ellipses." He then went on to approximate each elliptical coil cross section by 12 rectangular current strips whose field harmonics were corrected by an additional set of independent windings which were used to correct the first symmetry allowed harmonic and other possible errors.

The theoretical coil shape is based on the assumption of a uniformly distributed current density which is never strictly possible in practice because the coil is composed of a matrix of wires and the current within each wire is composed of elementary filaments. Replacing a uniform current distribution by a uniform matrix of current filaments and voids will generally produce field errors quite apart from those due to unavoidable errors such as the actual placement of the wires, variations in wire size or variations along z — the 3-rd dimension. This is clear when one goes to the limit i.e. uses only one wire or mesh point to simulate the problem. For this discussion we will always use the much studied and comparatively ductile type II superconductor NbTi as described in ref. 4. In practice, each wire has many filaments of NbTi (typically 1-100 μm) set in a Cu matrix which provides a low resistivity alternate circuit. If the ratio of Cu to superconductor is high, the heating of the Cu is small and can be accomodated by the He cooling system. Then, when the superconductor goes locally normal, it can recover without propagating and driving the whole system normal or possibly damaging it. The maximum ratio of NbTi to Cu which has been used in magnet wire is $\alpha \approx 1$. This allows quenches and also the possibility of damage. For fully stabilized operation, i.e. no quenches, the ratio needs to be much smaller e.g. more like 1/30 but this depends on the application e.g. the current required in a given wire.

The theoretically best packing fraction for circular wires in a bulk coil is $\pi/4$ — ignoring boundary effects. The actual packing fraction is NA_w/A_c where N is the number of turns and A is the actual area of the wire or coil. In our case, the area of each of the four 'elliptical' coils is:

$$A_c = \frac{ab}{2}[\pi - 4\,\mathrm{Tan}^{-1}(b/a)] - \sum \Delta_i \tag{2}$$

where Δ_i is the perturbation due to the winding mandrel, synchrotron radiation port or errors. Ideally then, one would like the number of turns to approach:

$$N \to \frac{\pi}{4}(A_c/A_w) = A_c/D_w^2. \tag{3}$$

For the ideal elliptical shape with $a = 3.5, b = 1.5$ cm and $D_w = 20$ mils one has $A_c = 4.7$ cm^2 and $N \gtrsim 1500$ ignoring coatings and the like. Using the value $<J> = 20$ kA/cm^2 then gives a total of

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$I_t = 80$ kA-t, a power supply current of $I_w \approx 53$ A and $<J_w> \approx 26$ kA/cm^2. For a Cu:Sc ratio of $\alpha = 4:1$, this gives $J_{Sc} \approx 130$ kA/cm^2, a value well below the critical current density at 4.2°K and 1.5*T*. A prototype coil wound with the proposed conductor would determine just how good the packing fraction could be made and thus the gradient limit in this case. However it is clear that even with rather conservative assumptions, a gradient of 2 T/cm in this type of 3 cm bore quad is reasonable. This is far beyond the capabilities of good conventional or permanent magnet quads.

What are the effects of a reduced packing fraction (PF)? If the actual wire radius is less than its maximal value so that $R_w \equiv R_m - \delta$, one has

$$\delta/R_m = 1 - 2\sqrt{PF/\pi}. \tag{4}$$

For $PF = 0.60$, $\delta \approx 0.15\ R_w$. If $\vec{s}_i$ is the i-th positional wire error and there is no net coil motion i.e. $\sum \vec{s}_i \equiv 0$, it is clearly possible to have rms positional wire errors $\sigma_s = 2\sqrt{2}\delta = 0.42\ R_w$. One can translate this into a field effect by using the Biot-Savart law.

For a straight conductor with current I_w, parallel to the magnet's axis, which is displaced a distance s from its ideal or symmetric position (determined by the corresponding turn on the opposite side of the mandrel), the field error is essentially that of a dipole element. The resulting field at any point (r, θ) due to such an error at (r_s, θ_s) will be

$$B_\theta = \frac{\mu_0 I_w}{2\pi}\sum n\left(\frac{r^{n-1}}{r_s^{n+1}}\right)[s_r \cos n(\theta-\theta_s) - s_\theta \sin n(\theta-\theta_s)]$$

$$B_r = \frac{\mu_0 I_w}{2\pi}\sum n\left(\frac{r^{n-1}}{r_s^{n+1}}\right)[s_r \sin n(\theta-\theta_s) + s_\theta \cos n(\theta-\theta_s)] \tag{5}$$

where $s_{r,\theta}$ are the components of the positional error, $\vec{s}$. One need not Fourier analyze the current to compute the field error at x, y due to a single dipole element at x_s, y_s. This is

$$\delta B = \frac{\mu_0 I_w}{2\pi S^2}\left(s_r e^{j\phi} + s_\theta e^{j(\phi-\pi/2)}\right) \quad \text{where} \quad S^2 = (x-x_s)^2 + (y-y_s)^2 \tag{6}$$

Looking only at one component, one then has

$$\delta B_x = \frac{\mu_0 I_w}{2\pi}\sum \frac{s_{ri}\cos\psi_i + s_{\theta i}\sin\psi_i}{S_i^2}. \tag{7}$$

Assuming a normal distribution for the positional deviations with probable errors $\sigma_{sr} = \sigma_{s\theta} \equiv \sigma_s$, gives probable field errors of

$$\delta B_x = \frac{\mu_0 I_w}{\pi}\frac{\sigma_s\sqrt{N}}{<S^2>} = \delta B_y. \tag{8}$$

For a quadrupole coil with any random but consistent mesh of wires possessing perfect 45° mechanical symmetry around the aperture (Fig. 1 shows the generic coil cell), the first symmetry allowed harmonic is the duodecapole ($n = 6$ or $2n$-pole). Allowing random wire errors between coil segments as described above then implies a fractional error of:

$$\frac{\delta B_x}{B_x} = \frac{I_w \sigma_s \sqrt{N}}{<J>\pi <S^2>\, y}\left(\frac{\epsilon+1}{\epsilon-1}\right) = \frac{2}{\pi}\frac{\sigma_s R_w}{<S^2>}\frac{\sqrt{A_c}}{y}\left(\frac{a+b}{a-b}\right). \tag{9}$$

All of these quantities have sizes which are characteristic of either the coil or wire. At the half-value radius, $y = b/2$, one has

$$\frac{\delta B_x}{B_x} = \frac{4}{\pi}\frac{\sigma_s R_w}{<S^2>}\sqrt{\frac{A_c}{b^2}}\left(\frac{a+b}{a-b}\right) \equiv k\frac{\sigma_s R_w}{<S^2>} \tag{10}$$

where k is determined strictly by the physical dimensions of the coil and has a value $k = 4.2$ for GA's example. An upper and lower limit can be established by using upper and lower limits on $<S^2>, \sigma_s$ etc. Using $b \lesssim S \lesssim a$ and $\delta \lesssim \sigma_s \lesssim 2\sqrt{2}\delta$, where δ corresponds to $PF = 0.60$, gives

$$1.3 \times 10^{-3} \lesssim \frac{\delta B_x}{B_x} \lesssim 0.020. \tag{11}$$

This result is consistent with more detailed calculations given below. Clearly, the smaller the wire radius and the higher the packing fraction, the better the field quality should be!

There are several reasons why one might argue against this. Arbitrarily increasing the number of turns increases both the inductance and the production costs of a magnet. It also raises questions on how sensitive the resulting system will be to beam losses or synchrotron radiation i.e. it is conceivable that an otherwise cryostable coil could be unstable in an actual operating environment. Ignoring end-effects, the external, self-inductance per unit length of the magnet studied here is:

$$L = \frac{N^2}{I_t^2}\mu_o \int H^2 dA \;\rightarrow\; 2.87 \times N^2[\mu H/m]. \tag{12}$$

This expression can be calculated analytically, ignoring end-effects, in terms of the known physical dimensions of a magnet because H/I is a constant which depends only on the specific geometry. A simple example is given directly below. Such inductance could prove to be expensive for large strings of magnets but should be acceptable for a few. The alternative of a few turns and very high currents is also expensive and hard to deal with. However, because typical cell magnets don't need to be of very high quality, it appears that IR and cell quads should be treated quite differently i.e. somewhat like conventional magnets presently.

Temperature variations, for whatever reason, can cause flux changes and increased heating which may result in quenches. Simple adiabatic calculations can establish guidelines for choice of wire and filament size. For instance, instability of the "fluxoids" or local, circulating supercurrents which oppose the penetration of the external field can provide a considerable variation in stored energy. Because the energy density is size dependent ($\approx \frac{1}{8}\mu_o J^2 R^2$), decreasing the wire or filament size and embedding the system into a material with high thermal capacity increases stability. The limiting condition for stability depends on where the operating line intersects the critical current, field and temperature surface. Leaving λ unspecified in Eq. 1 gives:

$$R_f \lesssim \frac{4}{J_c}\sqrt{\frac{\rho C_p}{\mu_o}(T_c - T_o)} \;\xrightarrow[2.0T/cm]{1.0T/cm}\; \begin{matrix}150\mu m\\ 235\mu m\end{matrix} \tag{13}$$

where T (= 4.2° K) is the operating temperature, ρ (=6.2 × 10^3 kg/m^3) is the density and C_p(=0.87 J/kg) is the thermal capacity of NbTi at 4.2° K. This is not a problem for most high energy physics applications because the limiting gradient ($G_{max}(J_c)$) is usually pushed high enough that the magnet becomes marginal for other reasons ?!?

Stray electrons passing through the coil provide local temperature variations via ionization which sets a lower limit on filament size i.e.

$$D_f \gtrsim \sqrt{\frac{6}{\pi}\frac{dE}{ds}/[\rho C_p(T_c - T_o)]} \rightarrow 0.1\mu m \tag{14}$$

for $dE/ds \approx 1.44 \times 10^{-10} J/m$. Technical limitations on production methods currently limit filaments to $D_f \gtrsim 1\mu m$ for long wire lengths (e.g. > 10 km). However, dumping the beam into a coil or even systematic steering or misalignment errors can clearly quench the magnet rather easily but this will depend on both the coil and wire design as well as the beam profiles and masks etc. Experimental study in this area would be useful since superconducting magnets and kickers in injection lines would be most useful. Furthermore, since superconducting wires and coils are rather 'good' particle detectors they should be studied in this context. Systematic effects such as synchrotron radiation from a perfectly aligned beam also require consideration and provide input for the coil design in magnets for use in electron machines (see references 6 and 7). Besides quenching the magnets one also wants to avoid sinking such systematic heat sources at He temperatures.

IV. Some Results of Calculations

Figures 2-5 show a sequence of mesh and field plots for the first four calculations in Table I. Although we would have liked to increase the boundary radius somewhat further to insure convergence, we believe the results would remain essentially the same. First, one sees a consistent variation of the $n = 6$ strength with perturbation of the coil shape adjacent to the winding mandrel i.e. as one proceeds from the coil configuration of Fig. 2b to that of Fig. 3a. Similarly, the results are relatively insensitive to the detailed shape of the coil at the vertex of the ellipse — the point furthest from where the field is calculated. Removing coil along the axes, as shown in Fig. 3b effects changes in the sign of the leading order error ($n = 6$) with very little effect on $n = 10$. These results suggest that one might systematically perturb the coil shape in such a way as to both simplify construction and not hurt the leading order, symmetry allowed, error harmonics i.e. $n = 6$ and 10. Gaps along the x-y axes of the magnet will occur in the winding process but could be used to pass synchrotron radiation or even the primary beam and might also be used to improve coil cooling. A less obvious advantage of increasing the mandrel cutout is that this reduces the peak field in the coil somewhat. For an ideal ellipse, this peak field (B_p) is:

$$B_p = Gab\sqrt{2/(a^2+b^2)} \xrightarrow{a \to \infty} \sqrt{2}Gb. \tag{15}$$

The limit of the coil distortion procedure is to approximate the shape by rectangular coil segments such as attempted in ref. 6. The number of segments or free parameters then determine the number of symmetry allowed harmonics that can be made zero i.e. two rectangular segments would imply that the $n = 6$ and 10 terms could be made zero in a perfectly symmetric magnet. On the other hand, one also expects such a system to be more sensitive to coil alignment errors. This will be considered in detail in a subsequent report which also considers the effect of shape on the turn efficiency i.e. $\delta G/\delta i_j$.

Figures 4 and 5 show the results of midplane symmetric calculations. An attempt was made to keep the mesh as uniform as possible. The coil shapes and areas are quite uniform e.g. $\sigma_{A_e} = 1.1 \times 10^{-5} < A_e >$. Adding an iron shield at 6.0 cm such as shown in Fig. 5 perturbs the harmonics only slightly. Its primary function is to contain the field and improve efficiency so it must not be allowed to saturate. A thickness of 1 cm in this particular case (G = 1 T/cm) keeps the maximum field in the iron below 12 kG and results in no decrease of the gradient compared to $\mu = \infty$. The improvement in the gradient is $\delta G/G = 5.2\%$. This number can be increased by decreasing the iron radius but this increases the amount of iron required. Beyond the cost of the steel and support system, the optimal radius of the steel or even whether one uses steel depends on the magnitude and range of gradients required. This is the usual tradeoff between strength and field quality.

Finally, we have looked at the sensitivity to various systematic errors i.e. when $\sum \vec{s}_i \neq 0$ such as misalignment of a single coil (Line 7) and correlated misalignment of a pair of opposing coils (Line 8). An excitation current error in a single coil such as might result from a short in one of the coils (Line 9) relates to the positional error (Line 7). Lastly, a single wire locational error is simulated (Line 10) by placing a small dipole current pair at $r = 1.425$ cm. Although this would be the largest possible effect for a single turn, it is clear that the effect could easily become significant when taken over all turns as discussed above.

V. Conclusions

To obtain gradients of more than 2T/cm for radii greater than one cm or so quickly becomes impossible with any means other than superconducting elements. For any chosen gradient, there will be a limiting radius which depends on the basic quad design i.e. coil geometry, the coil material and the overall packing fraction, λ. Much of the above dealt with the proportionality constant, λ, in Eq. 1, *which was derived for 'perfect' dipoles and quadrupoles*, because we restricted ourselves to a specific design (1,3) and coil material (4). In this case, the relationship between the various parameters is:

$$\lambda b = \left(\frac{B_{max}(T)}{\sqrt{2}\mu_o}\right)\left(\frac{1}{J_c(A/m^2)}\right)\frac{\epsilon+1}{\epsilon-1}\sqrt{1+\frac{1}{\epsilon^2}} \xrightarrow{\epsilon\to\infty} k\frac{B_{max}(T)}{J_c(kA/mm^2)} \quad (cm) \tag{16}$$

where $k = 1/17.8$. *As shown in Fig. 6, the function* B/J_c generally increases steadily with B and reaches ≈ 1.0 at $B \approx 3.0T$. For the high field magnets required for high energy physics, one sees that full cryostable operation is a long way off. Because B_{max} needs to be large, the coils need to be large $(\epsilon \to \infty)$ as does the packing fraction $(\lambda \to 1)$. Thus, stable operation $(\lambda \lesssim 1/30)$ is seldom possible e.g. for $\epsilon \approx 2$ and a 1 cm radius, $B_{max} \approx 0.8T$ and $G_{max} = 0.63T/cm$ *which is rather poor by either* conventional or permanent magnet standards.

Thus, although there are problems with this design, there appears to be no reason why it couldn't be built to meet the present specifications *using standard NbTi wire. To make* best use of the design, the packing fraction needs to be maximized i.e. it may be necessary to use a computer controlled wire winding system to monitor and correct wire position to *optimize resolution*. In this same area it does seem sensible to build some prototype coils with the proposed materials (filament diameter $D_f \approx 5-10\mu m$ and large α). This would provide some necessary education and some valuable input on the number of turns possible for a given wire type as well as its critical current properties, mechanical rigidity etc. In the meantime, since there are alternative designs which are simpler and less expensive (6,7), we will attempt a similar *error* analysis on them for comparison since it is clearly possible to design a simpler quadrupole coil with synchrotron radiation ports and any number of symmetry allowed harmonics equal to zero. However, one also expects such systems to be *more* sensitive to errors so the important questions to be answered are what are the allowable tolerances on the error harmonics of these *quadrupoles for their intended* use and what are such harmonics sensitive to in the coil design?

Epilogue

The word 'perfect' comes from the Latin 'perfectus' meaning done or finished so one can easily appreciate the goal. However, to use this word in conjunction with magnets may well seem a contradiction since every magnet, regardless of its generic affiliation or the intent of the builder, always seems to turn out unique — and virtually never in a positive sense. This is one reason why so many magnet companies go broke and why most laboratories try to build their own.

Judged strictly by 2D field quality, the elliptic coil quad provides a nearly perfect magnet in that it nearly conforms to the definition of its type. In fact, according to some *optics codes*, a *perfect* quad would require only the absence of a dipole component. Unfortunately, there are some practical aspects which have to be considered. For instance, the magnet should provide reproducible results and elude any tendency to destroy itself. Words such as 'training' and 'normalization' may help some accomodate to such foibles more easily. For others, the potential *for self destruction is too human* to ever be palatable in an element which should be 'transparent' to the user. Although the present application is not wholly immune from such objections, there is a considerable safety margin or energy range available.

Acknowledgements

The authors wish to thank J. Alcorn, K. Brown, D. Leith, J. Purcell, J. Voss and especially Steve St. Lorant for discussions on the magnet and the subject of superconductivity.

1) R. A. Beth, *I.E.E.E. Trans. Nucl. Sci.* NS 14 (1967) 386-388. Beth also points out that "It is further possible to eliminate the external or stray field completely while producing the prescribed inner field by adding a second confocal elliptic current sheet." Ignoring end-effects i.e. considering only the *transverse* field, one can appreciate this possibility using symmetry and Ampere's circuital law.

2) H. D. Ferguson, J. E. Spencer and Klaus Halbach, *Nucl. Inst. and Meth.* 134 (1976) 409-420.

3) H. Brechna, *Superconducting Magnet Systems*, Springer-Verlag, New York (1973).

4) D. C. Larbalestier, *Adv. Cryo. Eng. Mat.* 26 (1979) 10; also D. G. Hawksworth and D. C. Larbalestier, *Ibid.* 479. All calculations assume use of Nb - 46.5 wt. % Ti (62.8 At. %) which has had success in magnets used for high energy physics in this country. The critical temperature is $T_c \approx 9°$ K. At 4.2° K this alloy has a critical induction of $B_{c_2} \approx 11$ T.

5) J. E. Spencer, BNL-31761 (1982) 126-162.

6) H. Shoaee, J. E. Spencer and S. J. St.Lorant, SLC CN-194.

7) W. R. Nelson and J. E. Spencer, SLC CN-196.

Table I: Some calculations for superconducting quadrupoles with coil shapes formed from sections of ellipses. All field harmonics are normalized to the quadrupole component and defined as $Q_n \equiv A_n/A_2(\%)$ where A_n is the amplitude of the n-th harmonic. They are calculated at 'half-radius' i.e. $R = b/2$ where a, b are the lengths of the semimajor and semiminor axes. Figures 2 – 5 show some of the configurations and results of calculations done with a gradient of $G = 1.0$ T/cm.

	Description	n=1	2	3	4	5	6	10	Comments
1)	Pure Ellipse		100				-0.040	0.008	See Figure 2
2)	Mandrel Cutout		100				0.018	0.003	See Figure 3
3)	Mandrel&SR Cutouts		100				-0.281	-0.007	See Figure 3
4)	Mesh/Coil Variations	-0.001	100	0.002	-0.002	-0.000	0.036	-0.002	See Figure 4
5)	Iron Shield ($\mu = \infty$)	-0.002	100	-0.001	-0.005	0.002	0.034	-0.002	Shield at $r = 6.0$ cm
6)	Iron Shield ($\mu \neq \infty$)	-0.000	100	-0.001	-0.003	0.002	0.035	-0.002	1 cm Thick Shield
7)	$\partial Q_n/\partial x$(mm)	-4.929		-0.689	-0.149	0.007	0.014	0.001	$\delta A_2 = 150$ G/mm/coil
8)	$\partial Q_n/\partial y$(mm)				0.303		0.026	0.000	$\delta A_2 = 150$ G/mm/coil
9)	$\partial Q_n/\partial I$(%)	0.770		0.069	0.016	0.003	-0.000	-0.000	Single coil error
10)	$\partial Q_n/\partial s$(mm)	0.028		0.002	-0.008	-0.012	-0.014	-0.006	Single wire error ($r = 1.425$ cm)

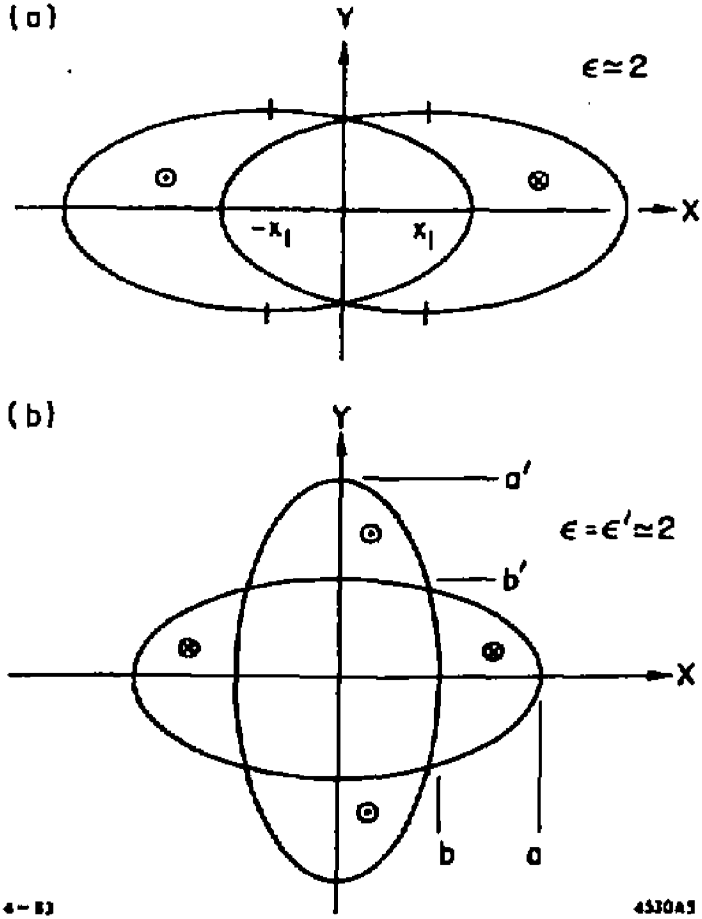


Fig. 1

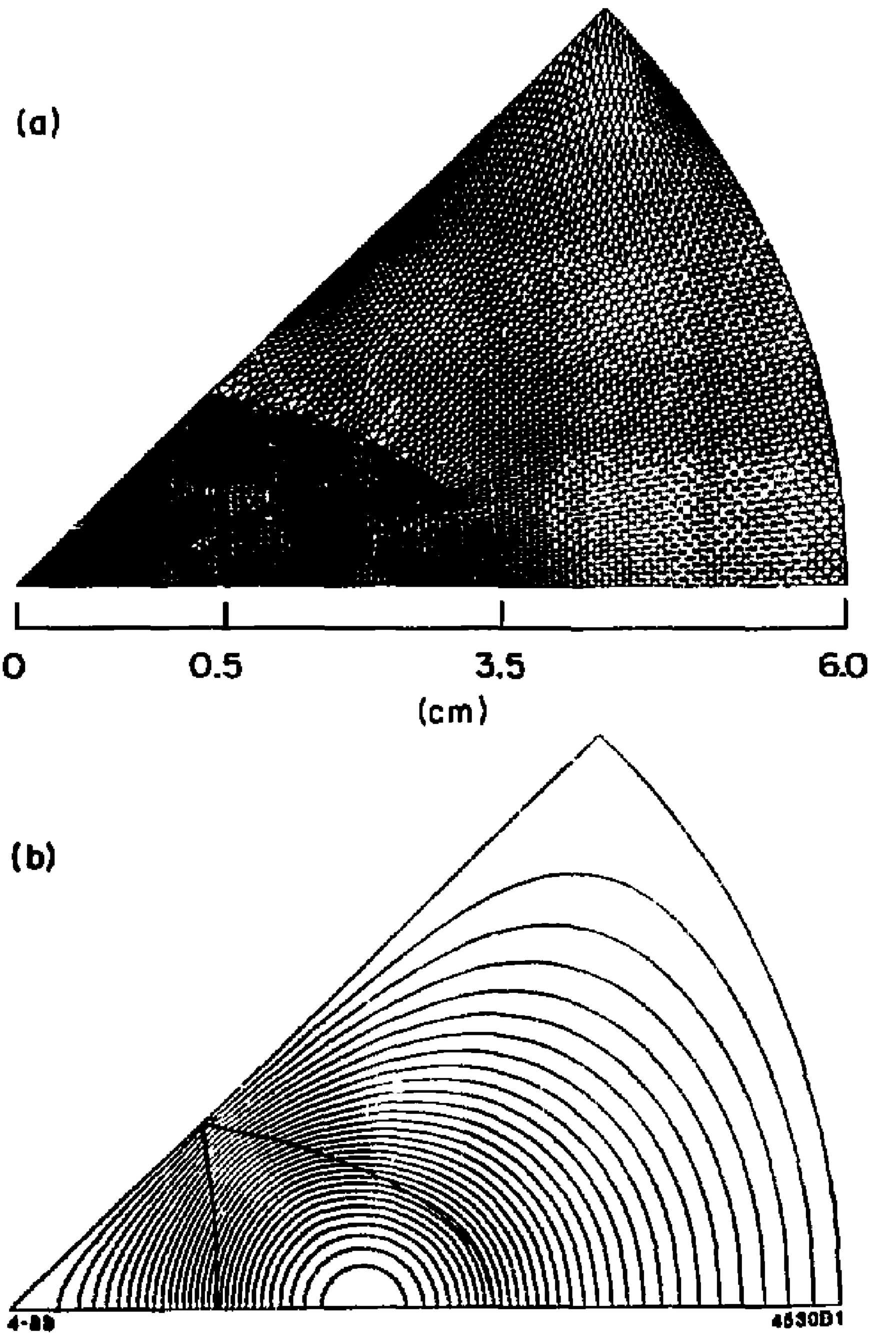


Fig. 2

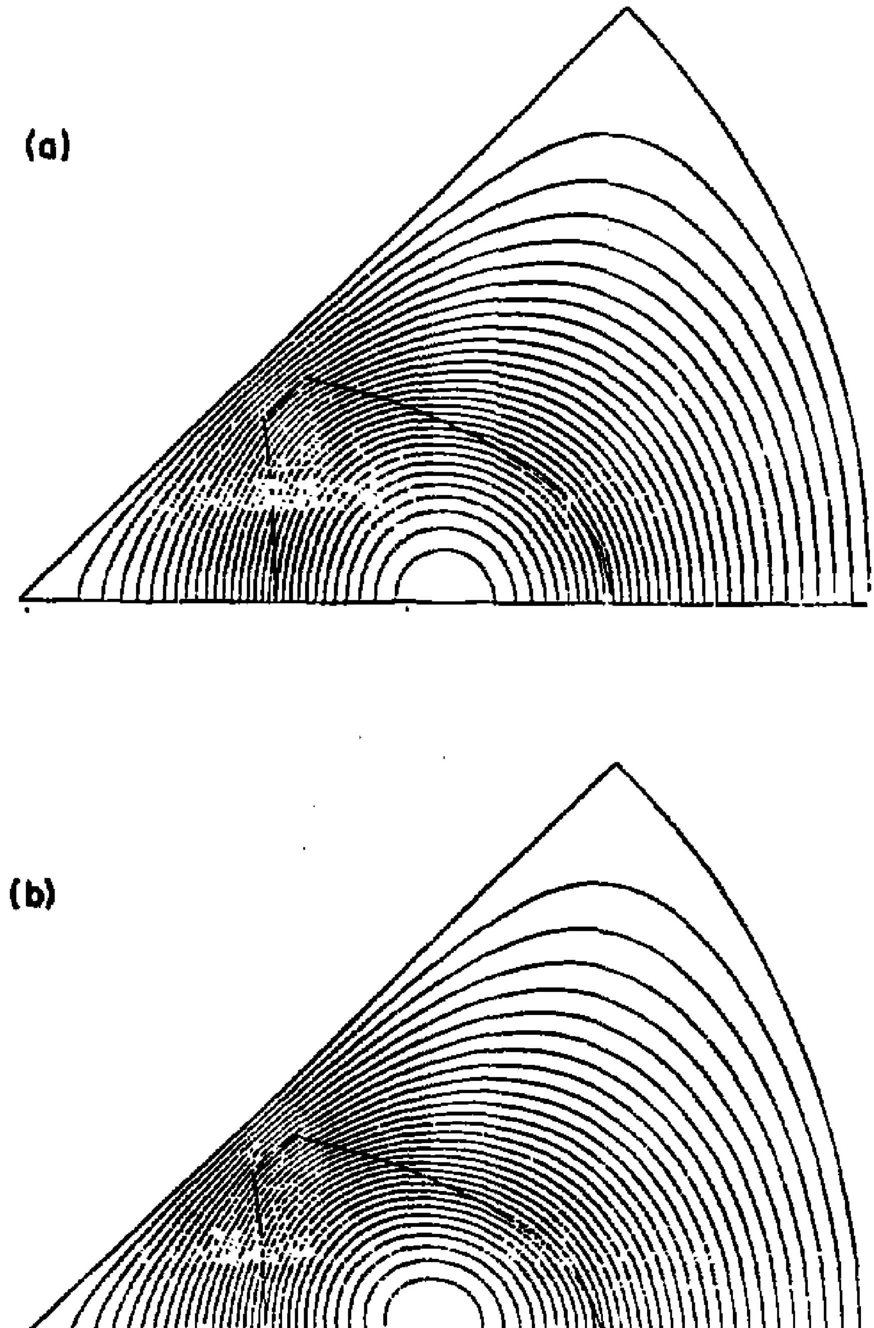


Fig. 3

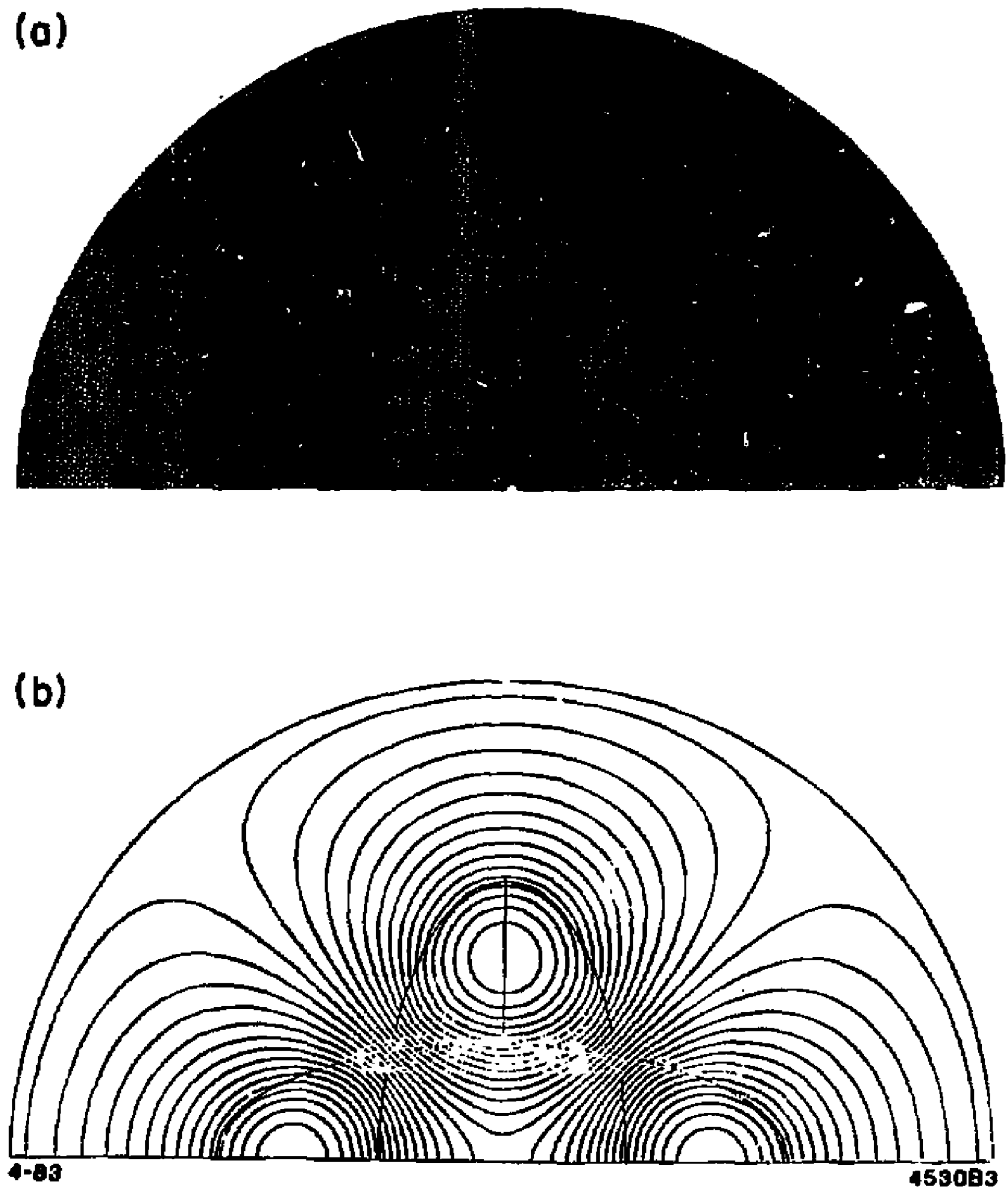


Fig. 4

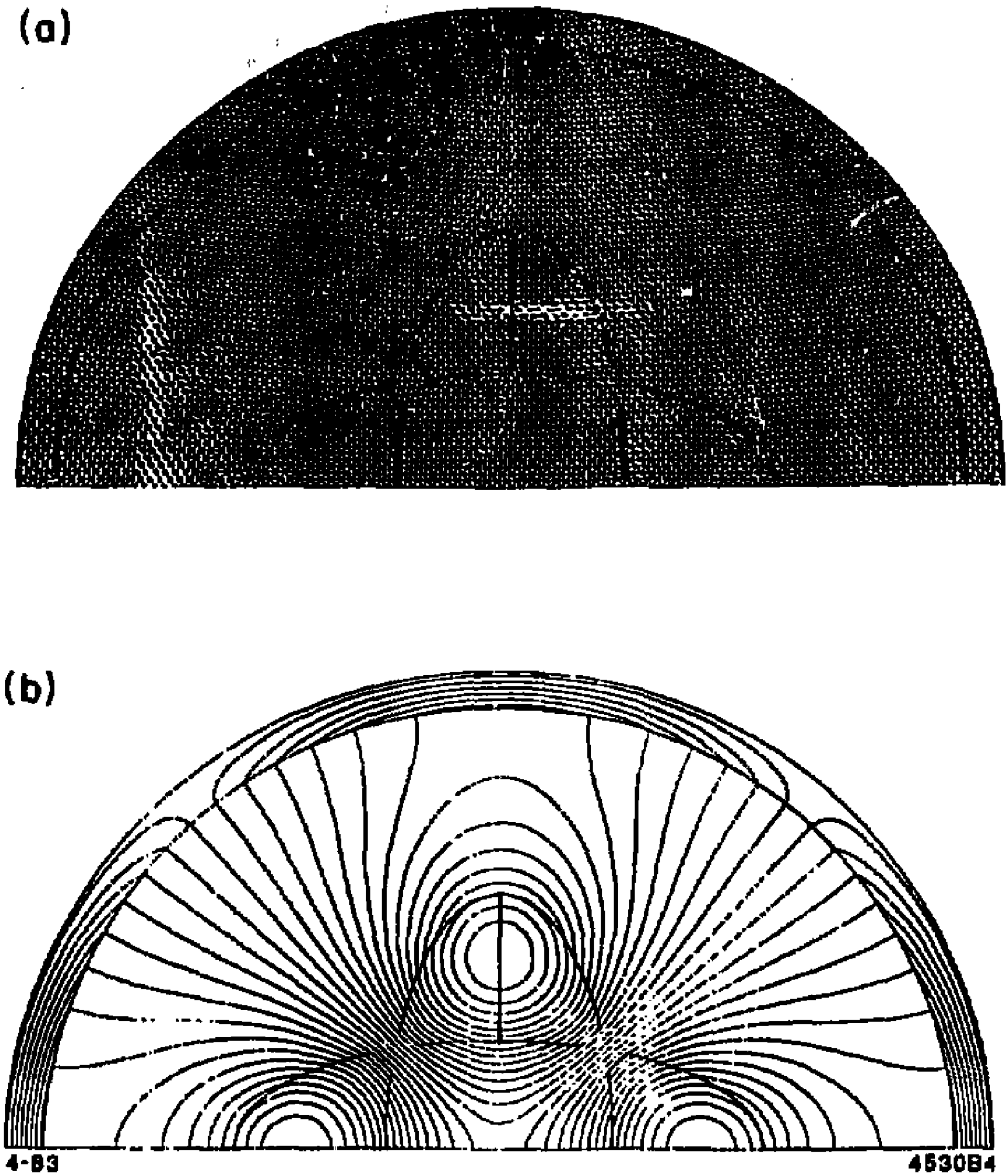
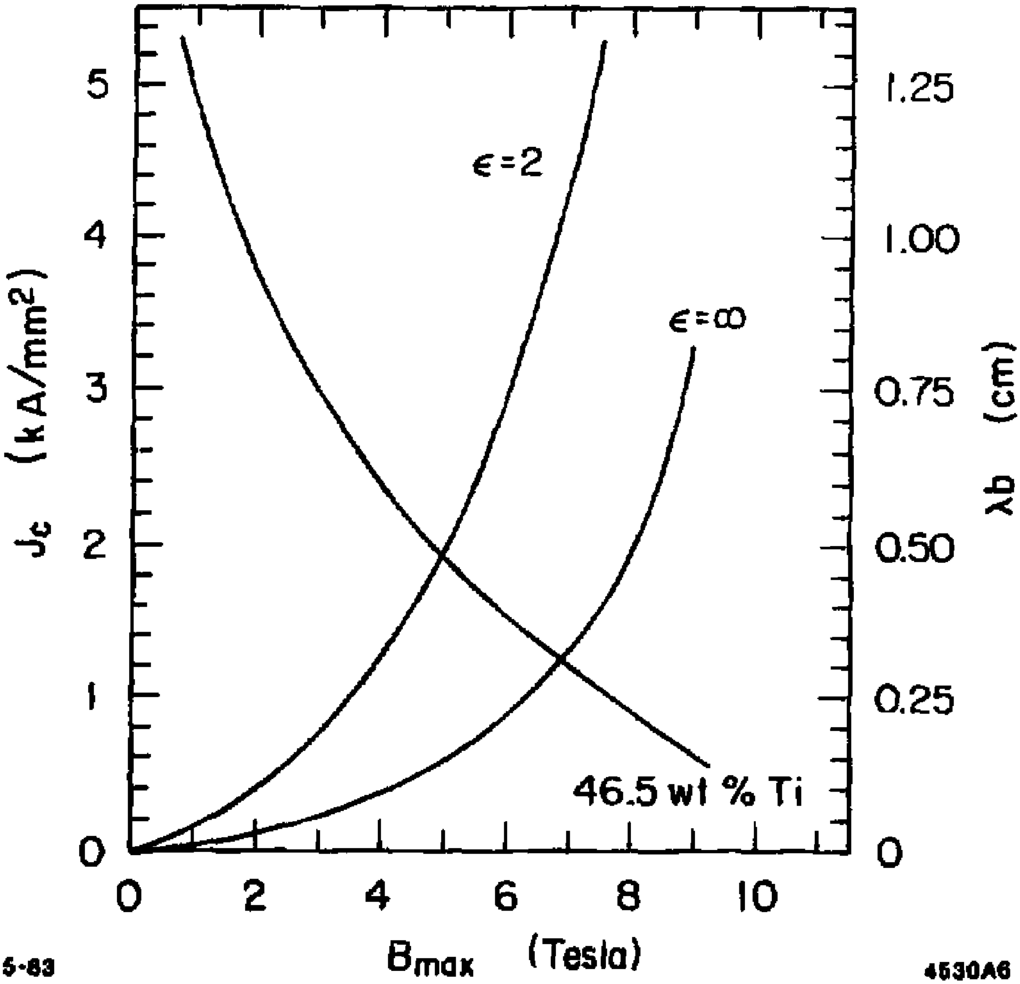


Fig. 5



5
4
3
2
1
0
J_c (kA/mm^2)
1.25
1.00
0.75
0.50
0.25
0
λb (cm)
ε=2
ε=∞
46.5 wt % Ti
0
2
4
6
8
10
B_{max} (Tesla)
5-83
4530A6

Fig. 6