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Stability of the Coherent Quadrupole Oscillations Excited by the Beam-Beam Interaction*

Y. KAMIYA† AND A. W. CHAO
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

1. Introduction

The coherent beam motions in which all beam bunches execute rigid dipole oscillations under the influence of the beam-beam force have been investigated before.[1,2,3] The corresponding stability limit was found to be a sawtooth function in the (ξ, ν) space, for which the maximum tolerable value for ξ shrinks to zero when ν is close to an integer (ξ is the beam-beam strength parameter and ν is the tune). It has also been shown that additional stopbands appear near half integral tunes in case there are errors in the betatron phase advances between interaction points.

In this note we will study the coherent quadrupole motion in the presence of beam-beam interaction, using a linear approximation to the beam beam force. The corresponding beam-beam limit is determined by evaluating the eigenvalues of a system of linear equations describing the coherent quadrupole motion. We will then find that the stability of the quadrupole motions imposes severe limits on the beam current, as is the case for the dipole instability. Preliminary results of this study have appeared elsewhere.[4] We will also find that

(1) the coherent quadrupole beam-beam limit is a sawtooth function with the maximum tolerable ξ shrinks to zero at all half integral tunes, and

(2) in case of errors in betatron phase advances, stopbands appear when ν is close to a quarter of an integer.

Depending on the value of the tune, the quadrupole instability may be a more severe limit than the dipole instability. This would then result in beam size oscillations and consequently loss in luminosity.

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† Visitor from KEK, Japan.

2. Mathematical Model

The linear motion of a single particle between the interaction points is described by

$$X_2 = M X_1 \quad , \tag{1}$$

where M is the 4×4 transport matrix. The Σ matrices associated with the beam distribution are given by averaging over all particles, i.e.,

$$\Sigma_2 = \langle X_2 \tilde{X}_2 \rangle = \langle M X_1 \tilde{X}_1 \tilde{M} \rangle = M \Sigma_1 \tilde{M} \quad , \tag{2}$$

where the brackets mean averages over beam distribution.

In general, the Σ matrix has ten independent elements, but if we restrict ourselves to the case in which the beam distribution is upright in $x-y$ space, all skew components such as $\langle xy \rangle$, $\langle xy' \rangle$, et cetera, vanish and the number of elements is reduced to six. We then define a six-dimensional vector as

$$\begin{aligned} \Sigma &= (\ \sigma_{11}\ \sigma_{12}\ \sigma_{22}\ \sigma_{33}\ \sigma_{34}\ \sigma_{44}\ ) \\ &= (\ \langle x^2 \rangle\ \langle xx' \rangle\ \langle x'^2 \rangle\ \langle y^2 \rangle\ \langle yy' \rangle\ \langle y'^2 \rangle\ ) \end{aligned} \quad . \tag{3}$$

In normalized phase space, the 4×4 transformation M from one interaction point to another is a rotation matrix

$$M = \begin{bmatrix} C_x & S_x & 0 & 0 \\ -S_x & C_x & 0 & 0 \\ 0 & 0 & C_y & S_y \\ 0 & 0 & -S_y & C_y \end{bmatrix} \quad , \tag{4}$$

where $C_{x,y} = \cos \mu_{x,y}$, $S_{x,y} = \sin \mu_{x,y}$, with $\mu_{x,y}$ the horizontal and vertical phase advances between the interaction points.

The corresponding 6×6 transformation matrix for Σ can be written as

$$M^R = \begin{bmatrix} A_x & 0 \\ 0 & A_y \end{bmatrix}$$

where

$$A_x = \begin{bmatrix} C_x^2 & 2S_xC_x & S_x^2 \\ -C_xS_x & C_x^2 - S_x^2 & C_xS_x \\ S_x^2 & -2S_xC_x & C_x^2 \end{bmatrix} \tag{5}$$

$$A_y = (x \rightarrow y \quad \text{in} \quad A_x) \quad .$$

Similarly, the 4×4 transformation matrix for the linearized beam-beam kick is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -k_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k_y & 1 \end{bmatrix} \qquad \text{with} \qquad k_{x,y} = 4\pi\xi_{x,y} \quad , \tag{6}$$

where ξ_x and ξ_y are the beam-beam tune shift parameters defined as

$$\xi_x = \frac{N\, r_e\, \beta_x}{2\pi\,\gamma\, a(a+b)} \quad , \qquad \xi_y = \frac{N\, r_e\, \beta_y}{2\pi\,\gamma\, b(a+b)} \quad , \tag{7}$$

with N the number of particle in the bunch; r_e the classical radius of electron; and a and b the unperturbed rms beam sizes in the horizontal and vertical dimensions, respectively. The unperturbed beam-beam kick matrix for Σ is then given by

$$M^K = \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix}$$

where

$$K_{x,y} = \begin{bmatrix} 1 & 0 & 0 \\ -k_{x,y} & 1 & 0 \\ k_{x,y}^2 & -2k_{x,y} & 1 \end{bmatrix} \quad . \tag{8}$$

We next consider the perturbed beam-beam kicks, taking into account of the fact that the beam size at interaction point is different from the ideal value and assuming that the difference is small so that the linear terms dominate. The coefficients that describe the effects of the beam size variation of one beam on the on-coming beam are given by the partial derivatives of k_x and k_y:

$$\begin{aligned} \sigma_{11} \frac{\partial k_x}{\partial \sigma_{11}} &= -k_x \frac{2+\frac{b}{a}}{1+\frac{b}{a}} \frac{1}{2} \\ \sigma_{11} \frac{\partial k_x}{\partial \sigma_{33}} &= -k_x \frac{1}{1+\frac{b}{a}} \frac{\beta_y}{2\beta_x \frac{b}{a}} \\ \sigma_{33} \frac{\partial k_y}{\partial \sigma_{11}} &= -k_y \frac{1}{1+\frac{b}{a}} \frac{1}{2} \left(\frac{b}{a}\right)^2 \frac{\beta_x}{\beta_y} \\ \sigma_{33} \frac{\partial k_y}{\partial \sigma_{33}} &= -k_y \frac{1+\frac{2b}{a}}{1+\frac{b}{a}} \frac{1}{2} \quad . \end{aligned} \tag{9}$$

We will now make the simplifying assumption that the horizontal tune shift is equal to the vertical tune shift, i.e., $\xi_x = \xi_y = \xi$. Then we have

$$\frac{b}{a} = \frac{\beta_y}{\beta_x} \quad , \tag{10}$$

and the second and third of Eq. (9) become

$$\begin{aligned} \sigma_{11} \frac{\partial k_x}{\partial \sigma_{33}} &= -k_x \frac{1}{1+\frac{b}{a}} \frac{1}{2} \\ \sigma_{33} \frac{\partial k_y}{\partial \sigma_{11}} &= -k_y \frac{1}{1+\frac{b}{a}} \frac{1}{2} \left(\frac{b}{a}\right) \quad . \end{aligned} \tag{11}$$

It can be seen that under this assumption, all coefficients are independent of β-functions and only depend on ξ and the aspect ratio b/a.

To describe the beam-beam effect on the quadrupole motion of two colliding beams, we first form a twelve-dimensional vector (Σ^1, Σ^2) whose components are the Σ matrix elements of the two beams. The transformation matrix for the beam-beam kick is then given by

$$\begin{bmatrix} K & 0 & P & Q \\ 0 & K & R & T \\ P & Q & K & 0 \\ R & T & 0 & K \end{bmatrix} , \tag{12}$$

where

$$P = \begin{bmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ r & 0 & 0 \end{bmatrix} , \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ t & 0 & 0 \end{bmatrix} ,$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ w & 0 & 0 \\ y & 0 & 0 \end{bmatrix} , \quad T = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & 0 & 0 \end{bmatrix} ,$$

describe the coupling between the quadrupole motions of the two colliding beams and

$$\begin{aligned} p &= -\sigma_{11} \frac{\partial k}{\partial \sigma_{11}} , \quad & r &= 2k\sigma_{11} \frac{\partial k}{\partial \sigma_{11}} - 2\sigma_{12} \frac{\partial k}{\partial \sigma_{11}} = -2kp + 2\frac{\sigma_{12}}{\sigma_{11}} p \\ q &= -\sigma_{11} \frac{\partial k}{\partial \sigma_{33}} , \quad & t &= -2kq + 2\frac{\sigma_{12}}{\sigma_{11}} q \\ w &= -\sigma_{33} \frac{\partial k}{\partial \sigma_{11}} , \quad & y &= -2kw + 2\frac{\sigma_{34}}{\sigma_{33}} w \\ x &= -\sigma_{33} \frac{\partial k}{\partial \sigma_{33}} , \quad & z &= -2kx + 2\frac{\sigma_{34}}{\sigma_{33}} x \quad . \end{aligned} \tag{13}$$

We now assume that at the midpoint of collision the unperturbed beams have upright distributions in the $x-x'$ and the $y-y'$ phase spaces. Then just before the interaction we have the condition

$$\frac{\sigma_{12}}{\sigma_{11}} = \frac{k}{2} \quad . \tag{14}$$

We will ignore the dynamic-β effect in which the β-function changes according to the beam-beam strength. This effect, when included, does not cause significantly different results. Under these assumptions, we have the following simplifications

$$r = -kp\,, \qquad y = -kw\,, \qquad t = -kq\,, \qquad z = -kx \quad . \tag{15}$$

In order to obtain the total transformation matrix around the whole machine and for multiple bunches, we need to arrange the Σ matrix elements into the vector

$$\Sigma = \left(\sigma_{11}^1,\ \sigma_{12}^1,\ \sigma_{22}^1,\ \sigma_{33}^1,\ \sigma_{34}^1,\ \sigma_{44}^1,\ \ldots,\ \sigma_{11}^k,\ \sigma_{12}^k,\ \sigma_{22}^k,\ \sigma_{33}^k,\ \sigma_{34}^k,\ \sigma_{44}^k \right) \quad . \tag{16}$$

Matrices transforming this vector between interaction points and through the beam-beam collisions are then established following the procedure described above. A program has been written for this purpose. Multiplying these matrices sequentially to obtain a total transformation matrix, we then numerically evaluate the eigenvalues of this total matrix. Instability occurs when any one of these eigenvalues has an absolute value bigger than one. The analytical calculation in the case of a single bunch per beam and without errors in the betatron phase advance between interaction points is presented in the Appendix.

3. Results

The analytic results derived in the appendix are used to prepare Figs. 1 and 2. These results apply to the case of one bunch per beam. Fig. 1 shows the stable region in the (ν_x, ν_y) space for a round beam ($b/a = 1$) for two values of the unperturbed beam-beam parameter $\xi = 0.02$ and 0.06. The shaded area indicate instability of at least one of the beam-beam quadrupole modes. Instabilities occur near the resonances $\nu_x = 0.5$, $\nu_y = 0.5$ and the sum resonance $\nu_x + \nu_y = 1$.

Figure 2 shows the results for a flat beam with $b/a = 0.1$. The stability region is not very different from that for the round beam case. In the following, we will consider round beams only.

Figure 3 is what happens along the diagonal line $\nu_x = \nu_y$ versus the beam-beam parameter ξ. The shaded regions are unstable against quadrupole motions. For reference, the unstable region against the single particle motion in the presence of linearized beam-beam force is also shown as double-shaded area. The figure repeats with period $\nu = 1$, thus representing a sawtooth behavior.

When there are more than one bunch per beam, we perform the calculations numerically. The sawtooth diagrams for two, three and four bunches per beam are shown in Figs. 4, 5 and 6, respectively.

The program can also include betatron phase errors between interaction points. Some of these results are given as Figs. 7, 8 and 9. In these results the tune errors between interaction points have been set to ± 0.05. Fig. 7 shows the sawtooth diagram for one bunch per beam and

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} + & - \\ - & + \end{pmatrix} \qquad . \tag{17}$$

Note that the main effect of phase errors is the excitation of stopbands around quarter integral tunes. Fig. 8 is for two bunches per beam and

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} + & + & - & - \\ - & - & + & + \end{pmatrix} \qquad . \tag{18}$$

Figure 9 is for three bunches per beam and

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \end{pmatrix} \qquad . \tag{19}$$

It can be mentioned that if the phase errors in the case of two bunches per beam are such that

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad , \tag{20}$$

the effect of phase errors on quadrupole mode stability is negligible. In particular, the 1/4–integer stopbands are not excited.

Finally, Fig. 10 shows the mode spectrum for the case of three bunches per beam and $\nu_x = 0.24$, $\nu_y = 0.19$ and no phase errors. At $\xi = 0$, the mode frequencies degenerate into $2\nu_x$ and $2\nu_y$ as they should.

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APPENDIX

In this apppendix we will derive analytically the formula for eigenvalues in the case of a single bunch per beam and without betatron phase errors. Two cases will be considered:

Case (i) $\nu_x = \nu_y, \quad k_x = k_y \quad$ and $\quad b/a \neq 1.$

Case (ii) $\nu_x \neq \nu_y, \quad k_x \neq k_y \quad$ and $\quad b/a \neq 1.$

Case (i). The 12×12 total transformation matrix M from one interaction point to another can be written as

$$M^{tot} = M^R M^K = \begin{array}{|c|c|c|c|} \hline AK & 0 & AP & AQ \\ \hline 0 & AK & AR & AT \\ \hline AP & AQ & AK & 0 \\ \hline AR & AT & 0 & AK \\ \hline \end{array} \quad , \qquad (A1)$$

where M^R and M^K are the rotation matrix and the beam-beam matrix given by

$$M^R = \begin{array}{|c|c|c|c|} \hline A & 0 & 0 & 0 \\ \hline 0 & A & 0 & 0 \\ \hline 0 & 0 & A & 0 \\ \hline 0 & 0 & 0 & A \\ \hline \end{array} \quad , \qquad M^K = \begin{array}{|c|c|c|c|} \hline K & 0 & P & Q \\ \hline 0 & K & R & T \\ \hline P & Q & K & 0 \\ \hline R & T & 0 & K \\ \hline \end{array} \quad , \qquad (A2)$$

where

$$A = \begin{bmatrix} C^2 & 2SC & S^2 \\ -CS & C^2 - S^2 & CS \\ S^2 & -2SC & C^2 \end{bmatrix} , \qquad K = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ k^2 & -2k & 1 \end{bmatrix} ,$$

$$P = \begin{bmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ r & 0 & 0 \end{bmatrix} , \qquad Q = \begin{bmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ t & 0 & 0 \end{bmatrix} ,$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ w & 0 & 0 \\ y & 0 & 0 \end{bmatrix} , \qquad T = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & 0 & 0 \end{bmatrix} ,$$

where $C = \cos\mu$, $S = \sin\mu$ with μ the phase advance between the interaction points.

The eigenvalues for a half machine can be found by solving the eigenvalue problem of the matrix M^{tot}

$$\det\left[M^{tot} - \lambda I \right] = 0 \quad . \tag{A3}$$

The direct calculation of the above determinant of matrix would be tedious but noticing that the matrix $M^{tot} - \lambda I$ has form

$$\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \quad , \tag{A4}$$

with X and Y 6×6 matrices, we can reduce Eq.(A3) to two equivalent equations

$$\det\left[X + Y\right] = 0 \qquad \text{and} \qquad \det\left[X - Y\right] = 0 \quad . \tag{A5}$$

We will first concentrate on the first equation of (A5). The matrix $X + Y$ is expressed as

$$X + Y = \begin{bmatrix} AK + AP - \lambda I & AQ \\ AR & AK + AT - \lambda I \end{bmatrix} \quad ,$$

where

$$AK = \begin{bmatrix} C^2 - 2SCk + S^2k^2 & 2SC - 2kS^2 & S^2 \\ -CS - kC^2 + kS^2 + k^2CS & C^2 - S^2 - 2kSC & CS \\ S^2 + 2SCk + C^2k^2 & -2SC - 2kC^2 & C^2 \end{bmatrix} \quad ,$$

$$AP = \begin{bmatrix} p(2SC - kS^2) & 0 & 0 \\ p(C^2 - S^2 - kCS) & 0 & 0 \\ p(-2SC - kC^2) & 0 & 0 \end{bmatrix} \quad .$$

$$AQ,\ AR,\ AT = (AP \text{ with } p \text{ replaced by } q,\ w,\ x) \quad . \tag{A6}$$

After some algebraic manipulations, the first of Eqs. (A5) can be shown to be equivalent to the equation

$$\det \begin{bmatrix} 1-\lambda & 2SC & S^2 & 0 & 0 & 0 \\ \lambda(p-k) & C^2-S^2-\lambda & CS & \lambda q & 0 & 0 \\ 2(1-\lambda) & -k(C^2-S^2+\lambda) & 1-CSk-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & 2SC & S^2 \\ \lambda w & 0 & 0 & \lambda(x-k) & C^2-S^2-\lambda & CS \\ 0 & 0 & 0 & 2(1-\lambda) & -k(C^2-S^2+\lambda) & 1-CSk-\lambda \end{bmatrix} = 0 \ , \quad (A7)$$

which after some calculations yields the eigenvalues

$$\lambda = 1 \quad \text{for two degenerate modes} \quad ,$$

and the rest of the eigenvalues satisfy

$$\Lambda = 2\cos 2\mu + S(p+x)\cos\mu \pm S\cos\mu\ \sqrt{(p-x)^2+4qw} \quad , \qquad (A8)$$

where $\Lambda = \lambda + (1/\lambda)$ and $\cos\mu = C - (kS/2)$.

Similarly, from the second equation in (A5), we obtain

$$\lambda = 1 \quad \text{for two modes} \quad ,$$

and the rest of the eigenvalues satisfy

$$\Lambda = 2\cos 2\mu - S(p+x)\cos\mu \pm S\cos\mu\ \sqrt{(p-x)^2+4qw} \quad . \qquad (A9)$$

Equations (A8) and (A9) then determines the stability of the quadrupole motions under the linearized beam-beam force. Note that using the expressions for p, q, w and x, it can be shown that $p+x$ and $(p-x)^2+4qw$ are independent of b/a, which means all eigenvalues in Case (i) are independent of the aspect ratio b/a.

Case (ii). The calculation of the determinant for Case (ii) is very similar to Case (i) except for being more complicated, so we will present only the outline and the results of the calculation.

The rotation matrix and beam-beam kick matrix in this case are given by

$$\begin{array}{|c|c|c|c|} \hline A_x & 0 & 0 & 0 \\ \hline 0 & A_y & 0 & 0 \\ \hline 0 & 0 & A_x & 0 \\ \hline 0 & 0 & 0 & A_y \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|} \hline K_x & 0 & P & Q \\ \hline 0 & K_y & R & T \\ \hline P & Q & K_x & 0 \\ \hline R & T & 0 & K_y \\ \hline \end{array} \quad . \qquad (A10)$$

Multiplying these two matrices, we have the total matrix M_{tot} for a half machine. We next reduce the determinant equation associated with the twelve-dimensional matrix to two equations involving six-dimensional matrices as in Case (i). The eigenvalues are then calculated, yielding the results

$$\lambda = 1 \quad \text{for four modes} \quad ,$$

and the remaining eight eigenvalues satisfy

$$\Lambda = \cos 2\bar{\mu}_1 + \cos 2\bar{\mu}_2 \pm \sqrt{(\cos 2\bar{\mu}_1 - \cos 2\bar{\mu}_2)^2 + 4\,q\,w\,S_1\,S_2 \cos\mu_1\ \cos\mu_2} \quad , \qquad (A11)$$

where

$$\cos 2\bar{\mu}_1 = \cos 2\mu_1 \pm p\,S_1 \cos\mu_1$$

$$\cos 2\bar{\mu}_2 = \cos 2\mu_2 \pm x\,S_2 \cos\mu_2$$

$$\cos\mu_1 = C_1 - \frac{k_x S_1}{2}$$

$$\cos\mu_2 = C_2 - \frac{k_y S_2}{2} \quad .$$

Stability of the quadrupole motions of the two colliding beams is determined by whether all eigenvalues have absolute value equal to unity.

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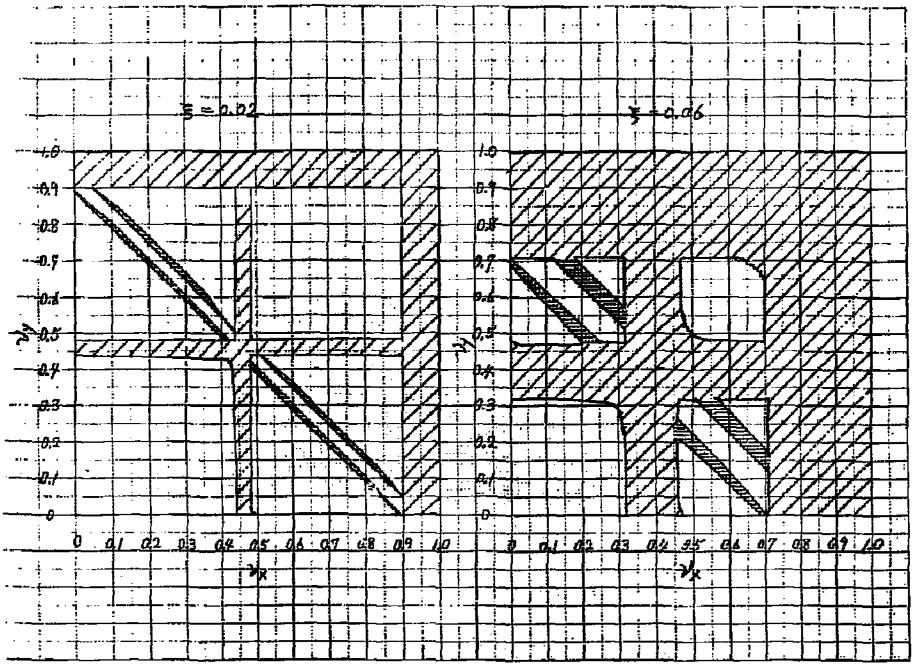

Fig. 1

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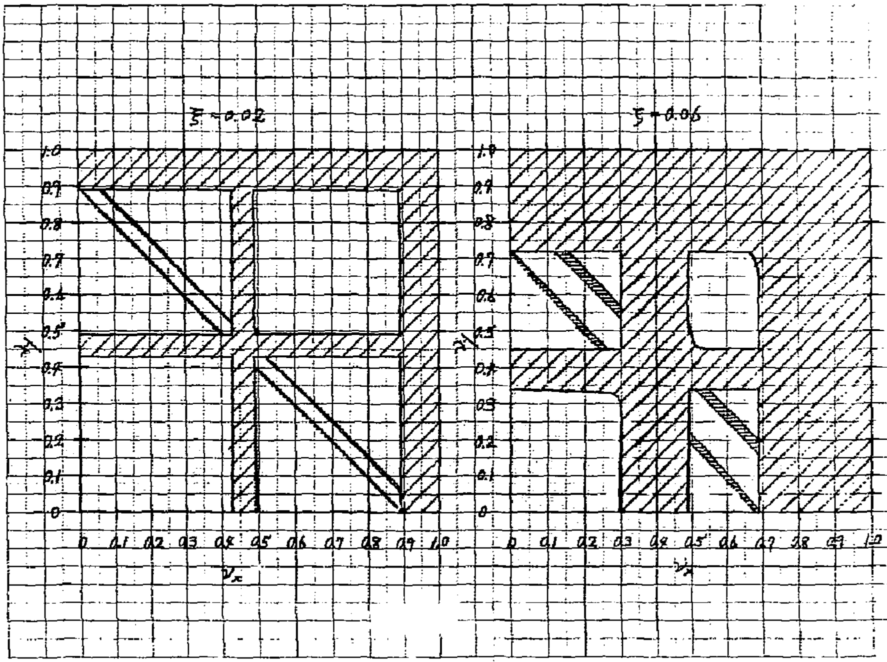

Fig. 2

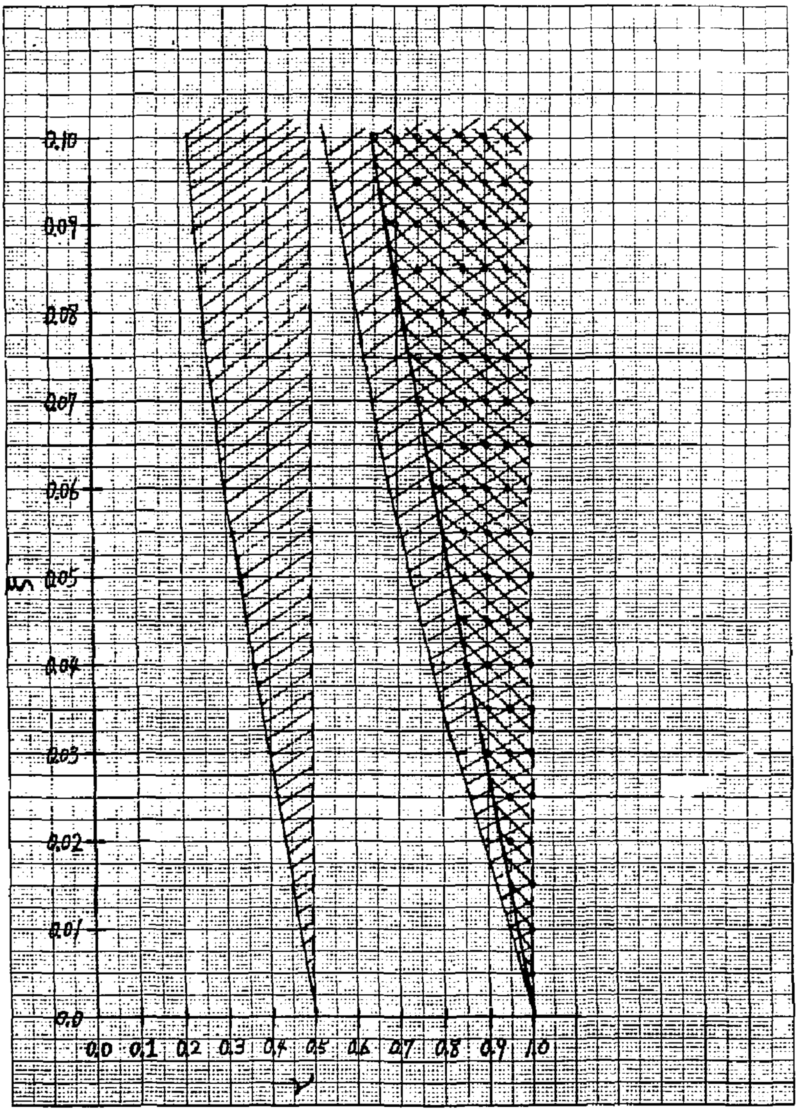
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Fig. 3

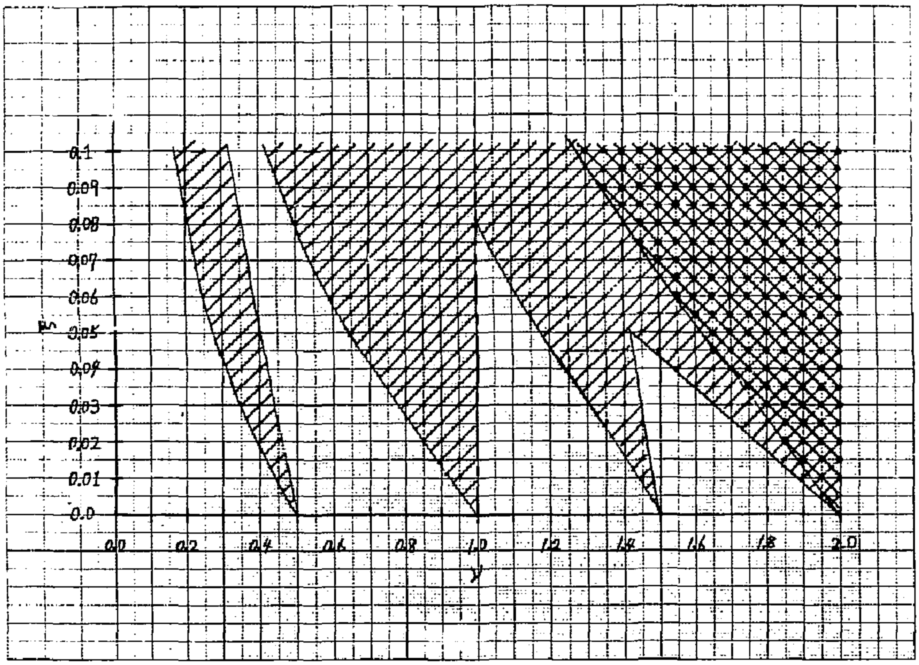
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Fig. 4

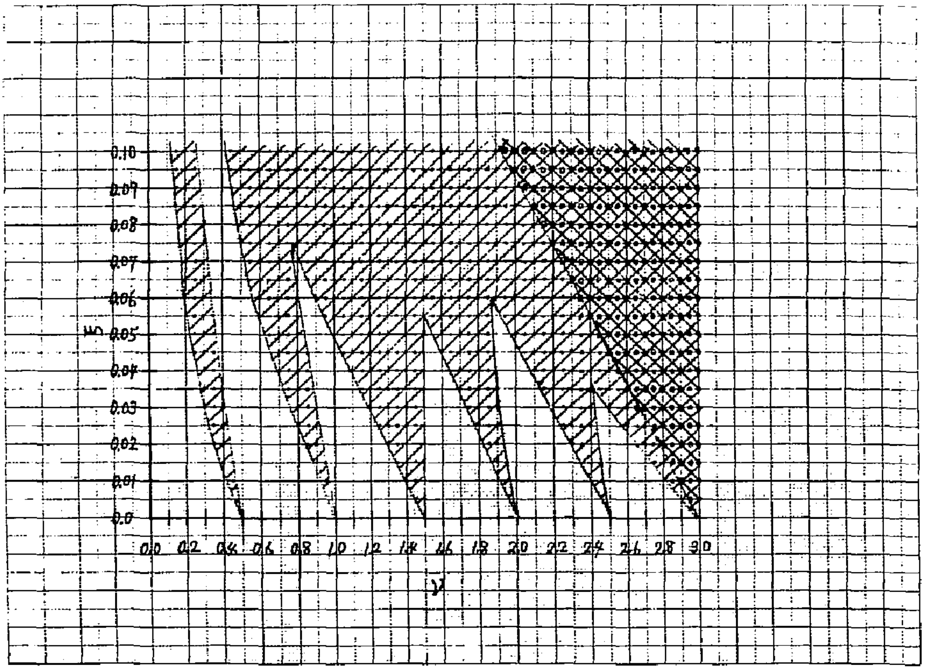
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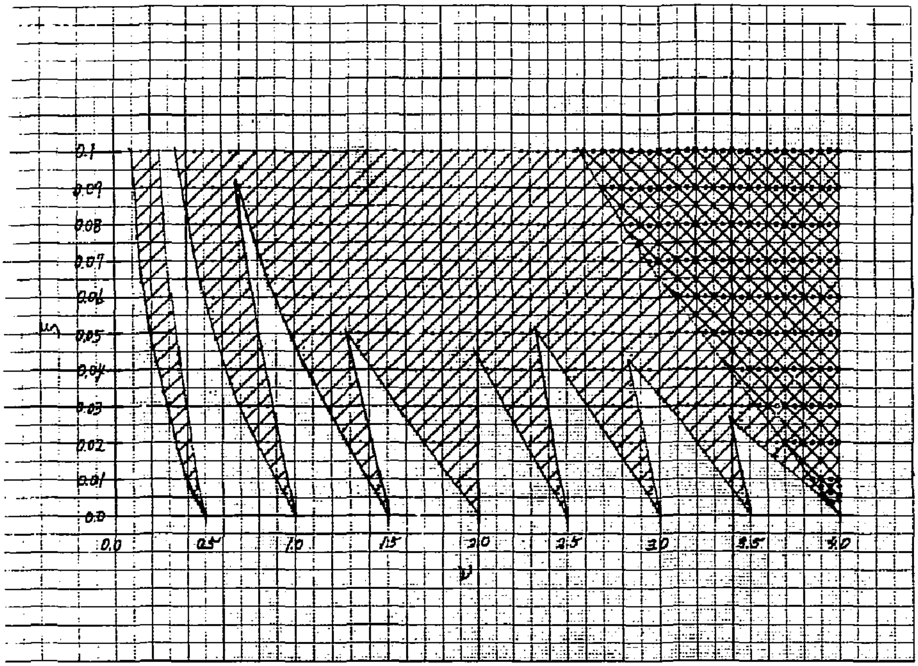
Fig. 5



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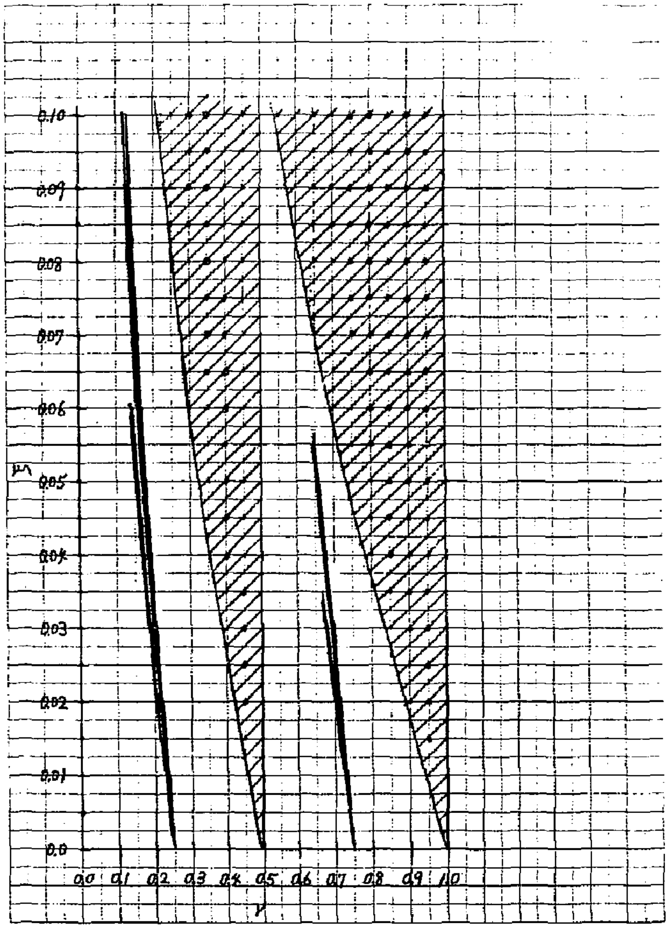
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Fig. 7

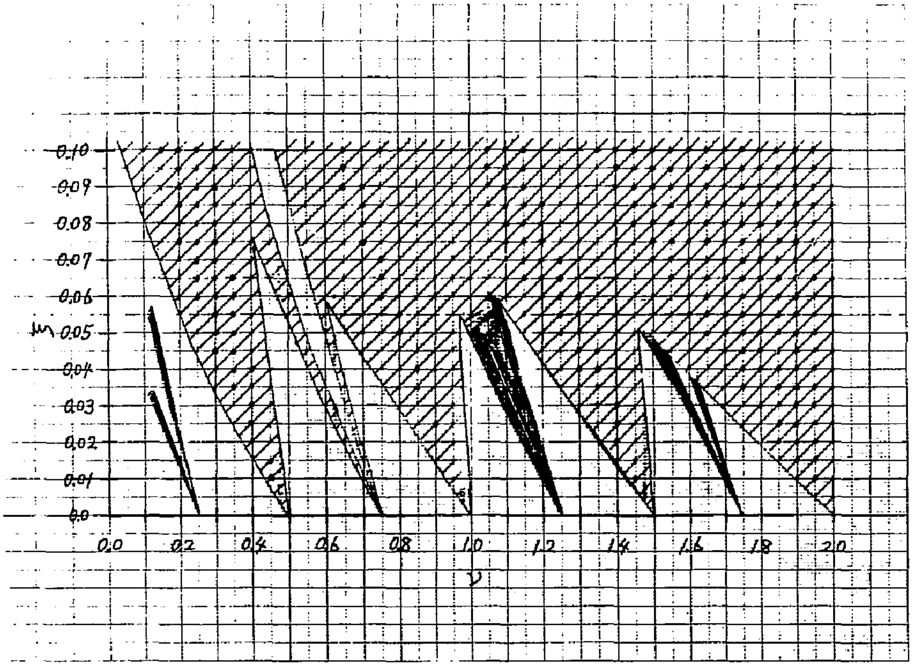
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Fig. 8

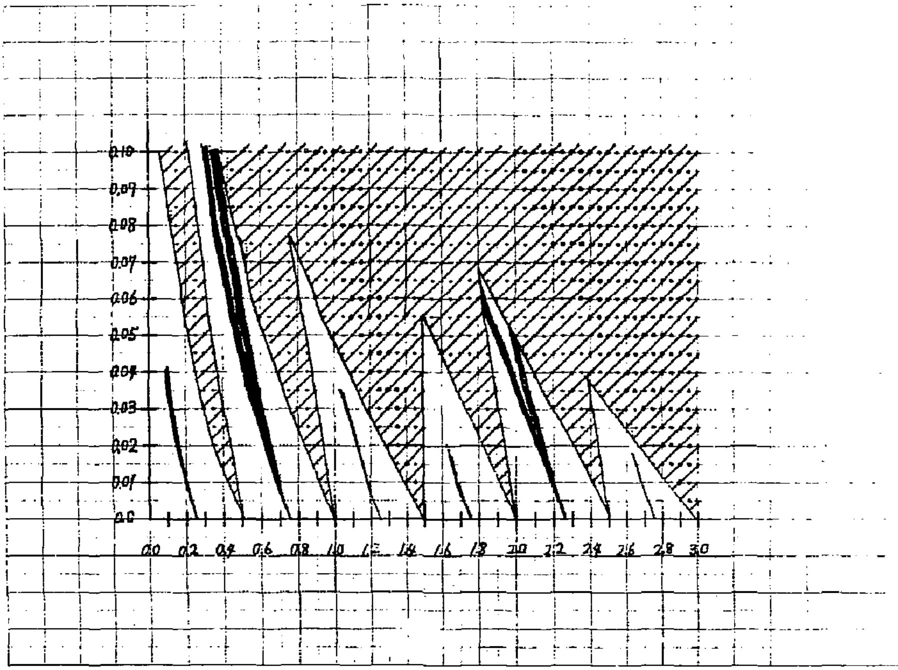
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Fig. 9

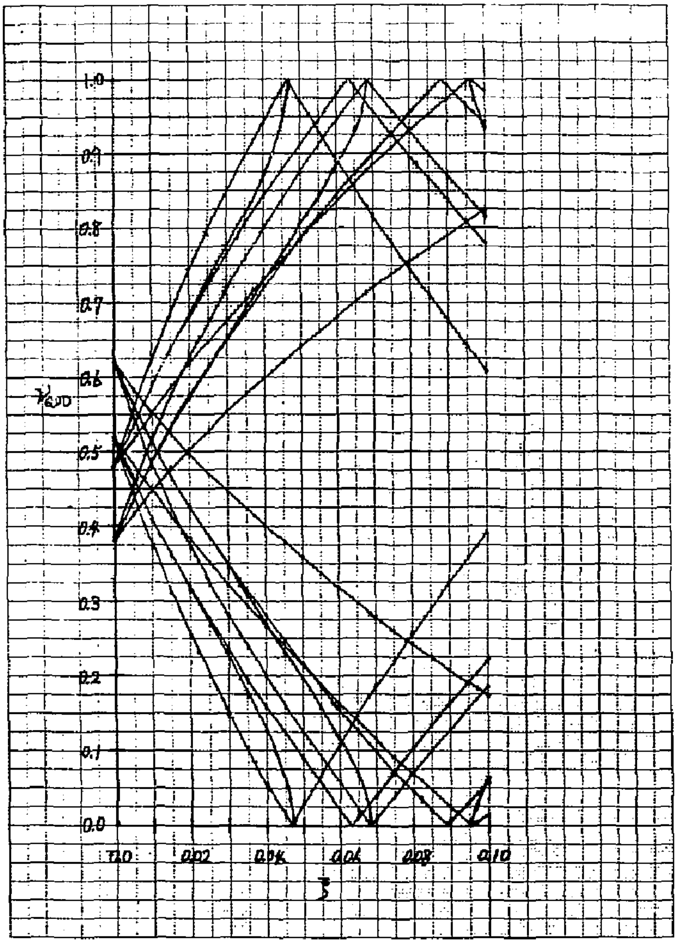

Fig. 10