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SOME DUAL RELATIONS IN TWISTOR THEORY *

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ABSTRACT

The option of employing twistors or dual twistors in integral representations, etc., is considered. In particular, dual-space analyses are presented which relate to the problem of background electromagnetic fields, and to the inverse transformation.

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1. Introduction.

In twistorial description of fields, one sometimes has the option of constructing integral representations, etc., in terms of twistors (Z^α), or in terms of dual twistors (W_α). The difference is not always a trivial one, once the basic conventions have been made. (One could say that this difference was central in some recent speculations [1], which, however, dealt with a different aspect of the subject.)

We recall for reference two basic contour-integral representations for a positive-helicity field (cf. e.g. [2]):

$$\varphi_{A'\ldots C'}(x) = (2\pi i)^{-1}\oint(\Delta\pi)\,\pi_{A'}\ldots\pi_{C'}\,\rho_x f_{-n-2}(Z) \tag{1.1a}$$

$$= (2\pi i)^{-1}\oint(\Delta\bar\pi)\,\rho_x(\partial/\partial\bar\omega^{A'})\ldots(\partial/\partial\bar\omega^{C'})f_{n-2}(W), \tag{1.1b}$$

where f_k is a suitably chosen analytic function of homogeneity k, $Z=(\omega^A,\pi_{A'})$, $W=(\bar\pi_A,\bar\omega^{A'})$, and

$$\Delta\pi := \pi^{B'}d\pi_{B'}, \qquad \rho_x f(Z) = f(ix^{AB'}\pi_{B'},\pi_{C'}), \tag{1.2a}$$

$$\Delta\bar\pi = \bar\pi^B d\bar\pi_B, \qquad \rho_x f(W) = f(\bar\pi_A, -ix^{B'C}\bar\pi_C). \tag{1.2b}$$

For a negative-helicity field, the roles of twistors and of dual twistors are interchanged.

The representation (1.1a) was discovered earlier and has been studied more extensively than (1.1b). E.g., techniques have been developed for incorporating into (1.1a) the effect of a background anti-self-dual electromagnetic field. In this note we summarize these techniques, and we indicate how a similar modification for (1.1b) could be derived. We illustrate some of the formulas with the help of an example of a constant field. Furthermore, we adapt Lerner's construction for the inverse transform, $\varphi \to f_{-n-2}$, to (1.1b).

One could say that the formulas which we obtain for the case (1.1b) are fairly direct extensions from those for (1.1a). However, we felt that they complement their (1.1a) - counterparts to a sufficient extent, so as to justify the present note.

We remark that we express our conclusions primarily in terms of integral representations, and we largely omit cohomological interpretations.

2. Background electromagnetic fields.

We first summarize a twistorial approach to the background-field problem. (This approach has been called the "twisted photon".) The following summary is based on [3], and it includes some supplementary remarks.

Let F_{AB} be an anti-self-dual electromagnetic field, which is determined by the twistor function $f_o(Z)$, the two being related by the twistor-space analogue of (1.1b). Let $\Phi^{C'}_{B}$ be an associated potential: $F_{AB} = \nabla_{C'(A}\Phi^{C'}_{B)}$. This potential is trivial, i.e. a gradient, when restricted to an α-plane (cf. [3d], proposition 3.2). We can therefore write $\Phi^{C'}_{B} = \nabla^{C'}_{B}\chi$ for the components of the potential Φ which are tangential to the plane. The field χ is called a Hertz potential.

We recall that an α-plane consists of points $y_{AA'} + \pi_{A'}\nu_{A}$, where the components $y_{AA'}$ and $\pi_{A'}$ are fixed while the ν_A are variable. In general, χ depends on the $\pi_{A'}$, and this dependence could designate an α-plane or planes under consideration. We will look for χ's which are homogeneous of degree zero in the $\pi_{A'}$, and we will write $\chi(x,\xi)$ where $\xi = \pi_{0'}/\pi_{1'}$. (The fixed point $(y_{AA'})$ of an α-plane apparently plays no role in the present analysis.) Now, the derivatives with respect to ν_A, i.e. tangential to an α-plane, are $\pi_{B'}\nabla^{B'A}$, or $\pi_{B'}\nabla^{B'}_{A}$. It would therefore be desirable to determine χ (for a given Φ) which would satisfy the following equation,

$$\pi_{B'}\nabla^{B'}_{A}\chi(x,\pi_{0'},\pi_{1'}) = \pi_{B'}\nabla^{B'}_{A}\chi(x,\xi) = \pi_{B'}\Phi^{B'}_{A}(x). \qquad (2.1)$$

It turns out that there is a convenient way of solving this equation, namely (cf. [3e] and also the next section),

$$\chi(x,\xi) = (4\pi i)^{-1}\oint d\xi'(\xi'-2\xi)^{-1}f_o(x,\xi'), \qquad (2.2)$$

where $f_o(x,\xi)$ is the function $\rho_x f_o(Z)$. (It can be shown that the l.h.s. in (2.1) is then a linear function of ξ.) This solution in turn enables us to generalize (1.1a) as follows. Set

$$\varphi_{A'\ldots C'}(\Phi;x) = (2\pi i)^{-1}\oint(\Delta\pi)\exp[-\chi(x,\pi)]\pi_{A'}\ldots\pi_{C'}\rho_x f_{-n-2}, \qquad (2.3a)$$

and (2.1) yields directly

$$(\nabla^{A'}_{A} - \Phi^{A'}_{A})\varphi_{A'\ldots C'}(\Phi;x) = 0. \qquad (2.3b)$$

The representation (2.3a) seems to be part of the folklore of twistor theory [4], but we have not seen it previously in print.

Equation (2.1), its solution (2.2), and the resulting representation (2.3a) are strikingly simple. It appears that the corresponding analysis in terms of dual twistors has to be rather more involved. In order to prepare for such an analysis, we should like to describe an alternative method of solving (2.1).

Let us return to F_{AB} and to f_o. We determine $\Phi^{C'}_{B}$ through a relation of the kind $\frac{1}{2}(\nabla_{C'A})^{-1}F_{AB}$, without summing over A. Observe that $\nabla_{C'A}$ corresponds to the following action in the integrand of (1.1b) (transformed to twistor space):

$$\nabla_{C'A} \leftrightarrow i\pi_{C'}\partial/\partial\omega^{A}. \qquad (2.4)$$

We set therefore,

$$\Phi^{C'}_{B}(x) = (2\pi i)^{-1}\oint(\Delta\pi)(2i\pi_{C'})^{-1}(\partial/\partial\omega^{B})f_o(\omega,\pi). \qquad (2.5)$$

This expression is not manifestly covariant, in view of $\pi^{-1}_{C'}$, but nonetheless it leads directly to the desired F_{AB}. We will see in the next section that it is possible for different f_o to yield the same F_{AB} but different $\Phi^{C'}_{B}$.

In order to determine χ by this kind of procedure, we first set $\pi_{0'} = 0 = \xi$ in (2.1). This equation then gives χ as $(\nabla^{1'}_{A})^{-1}\Phi^{1'}_{A}$ (without summing), and

$$\chi(x,0) = -(2\pi i)^{-1}\oint(\Delta\pi)(2\pi_{1'}\pi^{1'})^{-1}f_o(ix\pi,\pi). \qquad (2.6)$$

Since $\pi^{1'} = -\pi_{0'}$ and

$$\Delta\pi/\pi_{0'}\pi_{1'} = d\xi/\xi\,, \tag{2.7}$$

we see that (2.6) is in fact a special case of (2.2).

Next, to obtain $\chi(x,\xi)$ for a given $\xi \neq 0$, one can make a transformation of the spin basis so that $(\pi_{0'},\pi_{1'}) \to (0,\sigma_{1'})$, and proceed as before. The resulting χ will clearly depend analytically on ξ. So one obtains (in principle) a solution of (2.1) which is suitable for use in (2.3a). (It is not clear if this solution equals that in (2.2).)

We now turn to the dual situation. First we note that in place of (2.4) we have here $\nabla_{C'A} \leftrightarrow -i\bar{\pi}_A\,\partial/\partial\bar{\omega}^{C'}$, so that the analogue of (2.5) is

$$\bar{\Phi}_B^{C'}(x) = (2\pi i)^{-1}\oint(\Delta\bar{\pi})(-2i)^{-1}\bar{\pi}_B\,\rho_x\int d\bar{\omega}^{C'} f_{-4}(\bar{\pi}_A,\bar{\omega}^{A'}). \tag{2.8}$$

(We might not have $\bar{\Phi} = \Phi$.) Furthermore, we may set $\chi_1 = (\nabla_A^{1'})^{-1}\bar{\Phi}_A^{1'}$, as a Hertz potential for the $\bar{\Phi}_A^{1'}$. Then, using $\bar{\omega}_{1'} = \bar{\omega}^{0'}$,

$$\chi_1(x) = -(4\pi i)^{-1}\oint(\Delta\bar{\pi})\rho_x\int d\bar{\omega}^{0'}\int d\bar{\omega}^{1'} f_{-4}\,. \tag{2.9}$$

By introducing $\bar{\xi} = \bar{\pi}_0/\bar{\pi}_1$ and recalling (2.7), we can write this in a form resembling (2.2) with $\xi = 0$:

$$\chi_1(x) = -(4\pi i)^{-1}\oint(d\bar{\xi}/\bar{\xi})\,\bar{f}(x,\bar{\xi}), \tag{2.10a}$$

$$\bar{f}(x,\bar{\xi}) = \bar{\pi}_0\bar{\pi}_1\,\rho_x\int d\bar{\omega}^{0'}\int d\bar{\omega}^{1'} f_{-4}(\bar{\pi},\bar{\omega}). \tag{2.10b}$$

Next, let us allow χ to depend on $\bar{\xi}$, or on the $\bar{\pi}_A$, and let us write the dual to (2.3a) as an Ansatz:

$$\varphi_{A'\ldots C'}(\bar{\Phi};x) = (2\pi i)^{-1}\oint(\Delta\bar{\pi})\exp[-\chi(x,\bar{\pi})]\,\rho_x(\partial/\partial\bar{\omega}^{A'})\ldots \times(\partial/\partial\bar{\omega}^{C'})f_{n-2}\,. \tag{2.11}$$

By applying $\nabla_A^{A'}$, we see that we can fulfill (2.3b) (for the potential $\bar{F}$) if in place of (2.1), χ satisfies

$$\Pi_{A'}(x,\bar{\pi})\,\nabla_A^{A'}\chi(x,\bar{\pi}) = \Pi_{A'}(x,\bar{\pi})\,\bar{\Phi}_A^{A'}(x), \tag{2.12a}$$

$$\Pi_{A'}(x,\bar{\pi}) = \rho_x(\partial/\partial\bar{\omega}^{A'})\ldots(\partial/\partial\bar{\omega}^{C'})f_{n-2}\,, \tag{2.12}$$

where we suppressed the dependence of the $\Pi_{A'}$ on the other indices. The $\Pi_{A'}$ are known functions if f_{n-2} is given. Therefore it is possible to make a change of spin basis, so that $(\Pi_{0'},\Pi_{1'}) \to (0,\Sigma_{1'})$, and then to find the corresponding χ_1, as before. This is our proposed solution to eq. (2.12a), and to the problem of determining χ in the Ansatz (2.11) in such a way that φ fulfills (2.3b).

3. Example: the constant anti-self-dual field.

The literature on twistors contains still very few explicit examples. For this reason we feel justified in presenting the following trivial one. We consider $\underset{\sim}{E} - i\underset{\sim}{H}$, *given explicitly by*

$$F = -E(dt\wedge dx - i\,dy\wedge dz), \qquad E = \text{const}. \tag{3.1}$$

We employ the standard spinor notation,

$$\begin{pmatrix} x^{00'} & x^{01'} \\ x^{10'} & x^{11'} \end{pmatrix} = 2^{-\frac{1}{2}}\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{3.2}$$

(and the antisymmetric (ε_{AB}) with $\varepsilon_{01} = 1$, etc., yielding $\pi^{1'} = \varepsilon^{1'0'}\pi_{0'} = -\pi_{0'}$, etc.). Then $dx = 2^{-\frac{1}{2}}(dx^{00'} - dx^{11'})$, etc., and

$$F = E\,(dx^{00'}\wedge dx^{11'} - dx^{01'}\wedge dx^{10'}). \tag{3.3}$$

This form is symmetric under the interchange $0 \leftrightarrow 1$, and skew under $0' \leftrightarrow 1'$. We set therefore as usual $(F^{\mu\nu}) \leftrightarrow (F^{AB}\varepsilon^{A'B'})$ and have the correspondence

$$F \leftrightarrow (F_{11},\ F_{10} = F_{01},\ F_{00}), \quad F_{11} = F_{00} = 0, \quad F_{10} = E. \tag{3.4}$$

The normalization $F_{10} = E$ is consistent with subsequent formulas.

The contour-integral representations (1.1b,a) (but with Z and W interchanged) can now be obtained by choosing the following functions of twistor variables, respectively:

$$f_0 = E\,\omega^0\omega^1/\pi_{0'}\pi_{1'}\,, \qquad f_{-4} = E\,\bar{\pi}_0^{-2}\bar{\pi}_1^{-2}\,. \tag{3.5}$$

The contours have to separate the poles at $\pi_{0'}=0$ and at $\pi_{1'}=0$ for f_o, and those at $\bar{\pi}_0=0$ and at $\bar{\pi}_1=0$ for f_{-4}. Now, f_o determines the following potential components through (2.5),

$$\bar{\Phi}_0^{A'} = -\bar{\Phi}^{1A'} = \tfrac{1}{2}E\,x^{1A'}, \qquad \bar{\Phi}_1^{A'} = \bar{\Phi}^{0A'} = \tfrac{1}{2}E\,x^{0A'}, \tag{3.6}$$

which fulfil $\nabla_{A'B}\bar{\Phi}^{BC'}=0$ and $\nabla_{A'(B}\bar{\Phi}^{A'}_{C)}=F_{AB}$. One may also verify that if we add e.g. $\omega^1/\pi_{1'}$ to f_o, then Φ will be altered by a pure gauge term, and that here the potential components $\bar{\Phi}^{A'}_B$ specified by (2.8) equal the preceding $\bar{\Phi}^{A'}_B$. The total potential (fulfilling $d\bar{\Phi}=F$) now is:

$$\Phi = \sum \Phi^{AB'}dx_{AB'} = \tfrac{1}{2}E\left(x\,dt - t\,dx + iy\,dz - iz\,dy\right). \tag{3.7}$$

We turn to the Hertz potential. Equation (2.2) (with ξ inside the contour) gives directly

$$\chi(x,\xi) = -\tfrac{1}{2}E\left(x^{00'}x^{11'} + x^{01'}x^{10'} + 2\xi\, x^{00'}x^{10'}\right). \tag{3.8}$$

The verification of (2.1) is now direct, but the following points are worth noting. Let us write $\chi=\chi_o+\xi\chi_\xi$. Then $\bar{\Phi}^{1'}_B=\nabla^{1'}_B\chi_o$, in accordance with (2.6). However, for the $\bar{\Phi}^{0'}_B$, (2.1) yields two separate contributions:

$$\nabla^{0'}_B\chi_o = -\bar{\Phi}^{0'}_B, \qquad \nabla^{1'}_B\chi_\xi = 2\bar{\Phi}^{0'}_B. \tag{3.9a,b}$$

In fact, one can show, by extending slightly the analysis of sec. 2, that the relation (3.9a) is a general one.

(We may point out, with regard to (2.2), that in other articles other conventions are used. One sees there $(2\pi i)^{-1}$ and $(\xi'-\xi)^{-1}$ instead of our $(4\pi i)^{-1}$ and $(\xi'-2\xi)^{-1}$.)

We make a further comment. In sec. 2 we explained the origin of the relation $\bar{\Phi}^{1'}_B=\nabla^{1'}_B\chi_o$ for an anti-self-dual field. It is therefore somewhat surprising that in this example one can also find χ', χ'' such that $\bar{\Phi}^{A'}_0=\nabla^{A'}_0\chi'$ and $\bar{\Phi}^{A'}_1=\nabla^{A'}_1\chi''$. We see no clear interpretation for this fact.

4. <u>The inverse transform</u>.

We consider the maps $\varphi\to f_{-n-2}$, $\varphi\to f_{n-2}$, inverse to (1.1a,b). An explicit and elegant construction of the first was given by Lerner [5], and we should like to adapt it so as to obtain the second.

We recall a few formulas from <u>loc</u>. <u>cit</u>. Let $\varphi_{A'\ldots C'}$ be a positive-frequency field with n indices. It can be expressed in terms of its Fourier transform in the following way,

$$\varphi_{A'\ldots C'}(x) = (2\pi i)^{-1}\int_{V^+}\left[d(\Delta p\wedge\Delta\bar{p})\right]p_{A'}\cdots p_{C'}\,\tilde{\varphi}(p_{D'},\bar{p}_D) \times \exp\left(-i\,p_{E'}\bar{p}_E\,x^{EE'}\right), \tag{4.1}$$

where V^+ is the future light cone. We assume that $\tilde{\varphi}$ satisfies a condition of integrability (and of smoothness, cf. below), as well as: $\tilde{\varphi}(e^{i\theta}p_{D'}, e^{-i\theta}\bar{p}_D) = e^{-in\theta}\tilde{\varphi}(p_{D'},\bar{p}_D)$. We choose a cut in V^+ in such a way that $p_{A'},\bar{p}_A = r\pi_{A'},\bar{\pi}_A$, $r>0$. Then (4.1) becomes

$$\varphi_{A'\ldots C'}(x) = (2\pi i)^{-1}\int(\Delta\pi\wedge\Delta\bar{\pi})\,\pi_{A'}\cdots\pi_{C'}\,F_{-n-2}(x,\pi_{D'},\bar{\pi}_D), \tag{4.2a}$$

$$F_{-n-2}(x,\pi_{D'},\bar{\pi}_D) = 2\int_0^\infty dr\, r^{\frac{1}{2}(n+2)}\,\tilde{\varphi}(r^{\frac{1}{2}}\pi_{D'}, r^{\frac{1}{2}}\bar{\pi}_D)\exp\left(-ir\,\pi_{D'}\bar{\pi}_D x^{DD'}\right). \tag{4.2b}$$

One now obtains f_{-n-2} by transforming F_{-n-2}.

We see that the factors $\pi_{B'}$ in (4.2a) can be replaced by the $\hat{\pi}_{B'} := \partial/\partial(\bar{\pi}_E x^{EB'})$, provided that the additional factor $(-ir)^{-1}$ is supplied with each $\hat{\pi}_{B'}$. This replacement therefore entails replacing also $r^{\frac{1}{2}(n+2)}$ in (4.2b) by $r^{\frac{1}{2}(-n+2)}$. The integral then diverges (if $n\geqslant 4$), and has to be regularized. We write:

$$\varphi_{A'\ldots C'}(x) = (2\pi i)^{-1}\int(\Delta\pi\wedge\Delta\bar{\pi})\,\pi_{A'}\cdots\pi_{C'}\,F_{n-2}(x,\pi_{D'},\bar{\pi}_D), \tag{4.3a}$$

$$F_{n-2}(x,\pi_{D'},\bar{\pi}_D) = 2i\left[\int_0^\infty dr\, r^{\frac{1}{2}(-n+2)}\,\tilde{\varphi}(r^{\frac{1}{2}}\pi_{D'}, r^{\frac{1}{2}}\bar{\pi}_D)\exp\left(-ir\pi_{D'}\bar{\pi}_D x^{DD'}\right)\right]_{reg}. \tag{4.3b}$$

Let us set $r^{\frac{1}{2}}=s$. We then find the combination $ds\cdot s^{-n+3}$. For regularization we employ techniques described in [6], and interpret the singular factor in terms of the distribution s_+^λ. This distribution has a pole when λ is a negative integer, but the regularization s_+^{-n+3} (loc. cit.) is adequate for us. (At this point further conditions on $\tilde{\varphi}$ must be imposed. E.g., $\tilde{\varphi} \in \mathcal{S}$ is sufficient.) Now, let

$$F_{n-2}(x, \pi_{D'}, \bar{\pi}_D) = i^n \int ds \left(s_+^{-n+3}\right) \tilde{\varphi}(s\pi_{D'}, s\bar{\pi}_D) \exp\left(-is^2 \pi_{D'} \bar{\pi}_D x^{DD'}\right). \quad (4.3c)$$

The use of this expression can be justified by noting that upon applying the $\hat{\pi}_{B'}$, additional factors of s^2, or of r, will appear in the integrand. Then s_+^{-n+3} and $r_+^{\frac{1}{2}(-n+2)}$ can be identified with the original functions, and one finds agreement with eqs. (4.2).

We now proceed as in Lerner's construction [5]. A variant of his argument is as follows. The variables $\pi_{D'}$ and $\bar{\pi}_D$ define a complex bundle over the future tube of the Minkowski space. Alternately, we may consider x as fixed, and restrict our attention to the spinor variables. Now, the function F_{n-2} is homogenous, of degree $n-2$ in the $\bar{\pi}_D$, and of degree -2 in the $\pi_{D'}$. The form $F_{n-2}\Delta\pi$ is homogeneous of degree 0 in the $\pi_{D'}$, and its particular functional form shows that it is weakly $\bar{\partial}$-closed. Then, as in loc. cit., we write $F_{n-2}\Delta\pi = \bar{\partial}\gamma_j$ in a given coordinate patch U_j. By taking $\gamma_j - \gamma_k$ in the intersections, we determine f_{n-2} (cohomologically).

Let us still try to see how this construction of f_{n-2}, the original one of f_{-n-2}, and the twistor transformation (cf. [7],[8])

$$f_{n-2}(W) = \int (\Delta_3 Z) f_{-n-2}(Z) (Z^\alpha W_\alpha)^{n-2}, \qquad \Delta_3 Z = \tfrac{1}{6} \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \quad (4.4a,b)$$

could be interrelated. Let us go back to (4.3b), replacing $r^{\frac{1}{2}(-n+2)}$ by $r^{\frac{1}{2}(n+2)}(r_+^{\lambda})$, with the values $\lambda \approx -n$ being of interest to us. We may consider F_{n-2} as depending on the variable $\zeta = \pi_{E'} \bar{\pi}_E x^{EE'}$ (in addition to the dependence on the $\pi_{D'}$ and $\bar{\pi}_D$), and then, heuristically, $F_{n-2}(\zeta, \ldots)$ is a convolution. I.e., if $\Lambda_\lambda(.)$ denotes the Fourier transform of r_+^{λ}, cf. [6], then

$$F_{n-2}(\zeta, \pi_{D'}, \bar{\pi}_D) \sim (\text{const.}) \int_{-\infty}^{\infty} d\zeta' F_{-n-2}(\zeta - \zeta', \pi_{D'}, \bar{\pi}_D) \Lambda_\lambda(\zeta') \Big|_{\lambda \approx -n} \quad (4.5)$$

We will not investigate the passage from (4.5) to (4.4). However, a few comments can be made. First, ζ can be identified with one-half of $Z^\alpha W_\alpha$, after ρ_x is

applied. Next, $\Lambda_\lambda(\mu)$ contains the combination $\mu^{-\lambda-1}$, and so the possibility of such a passage becomes plausible. Furthermore, $\Lambda_\lambda(\mu)$ contains the factor $\Gamma(\lambda+1)$, which becomes singular at $\lambda = -n$. The need for such singular factors in (4.4) was pointed out in [7].

It is interesting to note that the preceding construction of f_{n-2} depended on a subtractive regularization, while in (4.4-5) a multiplicative regularization is natural.

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