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Factorial representations of path groups

by

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ABSTRACT

We give the reduction of the energy representation of the group of mappings from $I = [0,1]$, S^1, $\mathbb{R}_+$ or $\mathbb{R}$ into a compact semisimple Lie group G. For $G = SU(2)$ we prove the factoriality of the representation, which is of type III in the case $I = \mathbb{R}$.

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1. Introduction

The study of irreducible non local representations of the groups of mappings of a Riemannian manifold into a Lie group has been initiated by I.M. Gelfand, M.I. Graev and A.M. Vershik [1].
In the case where G is a compact Lie group a natural representation of such type is the so called energy representation, which was introduced in work by R.S. Ismagilov [2], K.R. Parthasarathy and K. Schmidt [3], A.M. Vershik I.M. Gelfand, M.I. Graev [4], and S. Albeverio und R. Høegh-Krohn [5].
In the case G = SU(2) Ismagilov [2] has shown the irreducibility for dim $X \geq 5$. This result has been extended to all G compact semisimple and dim $X \geq 4$ in a paper by Gelfand, Graev and Vershik [4] (containing also a correction to [6]). The irreducibility in the case dim $X \geq 3$ (and dim $X = 2$ in case all roots have sufficiently big length), G compact semisimple, was shown by Albeverio, Høegh-Krohn and Testard [7]. The method of proof is related to the one used in quantum field theory for proving the triviality of the exponential interaction in dimension ≥ 3 (resp. $d = 2$ with parameter sufficiently big) [8]. In [7] the reducibility of the energy representation in the case dim $X = 1$ was also pointed out.

This depends on the identification ([5], [7]) of the cyclic component of the vacuum of the energy representation with the representation given by left (or right-) translations along the paths of Brownian motion in G, using the quasi invariance of Brownian measure on G by left (resp. right-) translations. In the present paper we shall pursue the study of the representation in this case. Before coming to a description of our present results let us mention a few other papers connected with the study of the energy representation. Generalizations (e.g. to representations of central real extensions of the group of smooth sections of the 1-jet fibre bundle) have been discussed in [6], [9]. An interesting discussion of the representation in the general context of affine Lie algebras has been given by I.B. Frenkel in [10] (especially ch.6.4). This reference also contains a discussion of the interesting relations of affine Lie algebras with Kac-Moody graded Lie algebras and its applications to problems in number theory, topology and other fields (e.g. [11], [19-23]). Recently B. Gaveau and Ph. Trauber [12] has studied certain regular representations of the group of smooth mappings from a manifold into a compact semisimple Lie group, given by quasi-invariant Gaussian measures on the paths with values in a Sobolev space $H_k(X,G)$, $k > \dim X/2$ and conjectured an asymptotic relation with the energy representation.

Let us now describe shortly the content of this paper. For readers convenience we recall first briefly the basic construction of the energy representation (for details see [2] - [7]). Let X be a Riemann manifold with volume measure dx and a C^∞ strictly positive density ρ. Let G be a compact semisimple Lie group. Let $C_0^\infty(X;G)$ be the space of C^∞ mappings from X into G which are equal to the unit in G outside some individual compact of X. $C_0^\infty(X;G)$ is a group under pointwise multiplication We can look at $C_0^\infty(X;G)$ as a metric group in the following way ([5]). Let $\psi,\psi' \in C_0^\infty(X;G)$ and define

$$(\psi,\psi') \equiv \int_X (d\psi(x),d\psi'(x)) \quad \rho(x)dx.$$

Here the tangent planes $T_xX \equiv X_x$ and $T_{\psi(x)}G \equiv G_{\psi(x)}$ are equipped with their natural Eudidean structures, given by the Riemannian metric on X and the Eudidean structure induced on $T_{\psi(x)}G$ by the one on the Lie algebra $g \equiv T_eG$ and the natural mapping $T_eG \to T_{\psi(x)}G$. (,) is the natural scalar product in the space of bounded linear mappings $\mathcal{L}(T_xX,T_{\psi(x)}G)$ given by the Hilbert-Schmidt norm.
It was proven in [5] that

$$\psi,\psi' \to (\psi^{-1}\psi',\psi^{-1}\psi')^{1/2} \equiv |\psi^{-1}\psi'|$$

is a metric on $C_0^\infty(X;G)$, and completing $C_0^\infty(X;G)$ with respect to this metric we get a complete metric group $G^X \equiv H_1(X;G)$, the Sobolev-Lie group of mappings from X into G. $H_1(X;G)$ is an infinite dimensional Lie group (see [6][9]). We shall now describe the energy representation of this group.

Let $\Omega \equiv \Omega(TX;g)$ be the space of smooth 1-forms i.e. smooth maps ω from the tangent bundle TX into g, linear on the fibers i.e. $\omega(x)$ is a linear map from T_xX into g, for all $x \in X$. Ω is a pre-Hilbert space with respect to the scalar product $(\omega_1,\omega_2) \equiv \int_X Tr(\omega_1(x)\omega_2(x)^*)\rho(x)dx$.
We shall call $\mathcal{H}$ the completion of Ω in the corresponding norm.
Let V be the unitary representation of G^X in $\mathcal{H}$ given by the adjoint representation in each fiber i.e. given by the following formula, where $\psi \in G^X$, $\omega \in \mathcal{H}$:

$$(V(\psi)\omega)\ (x) \equiv Ad\ \psi(x)\omega(x).$$

$\psi \in G^X \to \gamma(\psi)(x) \equiv d\psi(x).\ \psi(x)^{-1}$ is a cocycle for the representation V (the so called Maurer-Cartan cocycle). By a general procedure, due to R.F. Streater[13], H. Araki[14] and K.R. Parthasarathy-K. Schmidt [15], one can

associate to V an exponential unitary representation U of G^X, and this is the energy representation. Descriptions of this representations are given in [2] - [7]. We recall here two of them. Let μ be the standard Gaussian measure associated with the real part of $\mathcal{H}$.
Then, for $\psi \in G^X$, $f \in L^2(d\mu)$, $\omega' \in \mathcal{H}$:

$$(U(\psi)f)(\omega') = e^{i(\gamma(\psi),\omega')} f\,(V^{-1}(\psi)\omega').$$

An equivalent description of U is obtained by using the Fock-Wiener-decomposition of $L^2(d\mu) \cong \bigoplus_{n=0} \mathcal{H}^{\otimes s_n}$ and writing, for $\omega' \in \mathcal{H}$:

$$U(\psi)\exp\omega' = \exp(-\tfrac{1}{2}|\gamma(\psi)|^2)\exp-(V(\psi)\omega',\gamma(\psi))\exp(V(\psi)\omega'+\gamma(\psi))$$

with $|\ |$ the norm in $\mathcal{H}$ and $\exp\omega' = 1+\omega'+\frac{\omega'\otimes\omega'}{\sqrt{2}}+\ldots$

It is shown in [7] that for $X = \mathbb{R}$ the cyclic component[1] of $1 \in L^2(d\mu)$ ("the vacuum") under U is unitary equivalent to the representations U^R, U^L given by right resp. left translations along Brownians paths in G i.e. $(U^R(\psi)f)(\eta) = f(\eta\psi)\left(\frac{d\mu_o(\eta\psi)}{d\mu_o(\eta)}\right)^{1/2}$

resp.

$$(U^L(\psi)f)(\eta) = f(\psi^{-1}\eta)\left(\frac{d\mu_o(\psi^{-1}\eta)}{d\mu_o(\eta)}\right)^{1/2},$$

$f \in L^2(d\mu_o)$, $\eta \in C(I,G)$, μ_o being the Brownian motion measure on the continuous functions from X to G.

Similarly in the cases $X = \mathbb{R}_+$, $[0,1]$, S one defines U^L, U^R by conditioning the measure μ_o on $C(R;G)$ with respect to the conditions $\eta(\tau)=e$, $\tau\leq o$ resp. $\eta(0)=\eta(1)=e$ resp. $\eta(0)=\eta(1)=e$, e being the unit in G. For more details see [5], [7].

Since U^L and U^R commute and each is equivalent to U, we have in particular that U is reducible [7]. In the present paper we shall investigate further this representation, and we shall now briefly summarize the results. In Sect. 2 we show that for any torus $T\subset G$ we have a decomposition of μ_o in $\mu_h \otimes \mu_T$, where μ_T is the Brownian measure on paths in $C(X,T)$, starting at the identity for $t = o$, and μ_h is such that $\varphi^{-1}d\varphi$ is white noise distributed,

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1) Using the method of [4] one can show [25] that 1 is cyclic.

and has horizontal increments, with φ such that $\eta = \varphi\,\alpha$, $\eta \in C(I,G)$ $\alpha \in C(I,T)$. $\dot{\mu}_h$ is mapped by the canonical mapping associating to φ its equivalence class in $C(X,G/T)$ into the Brownian measure μ_1 on $C(X,G/T)$. We also show that U^L is unitarily equivalent to V^L acting on $K = L^2(C(X,G/T)\ ,\mu_1) \otimes L^2\ (C(X,T),\mu_T)$ and given by

$$(V^L\ (\psi)F)\ (\xi,\alpha) = \left(\frac{d\mu_1\ (\psi^{-1}\ \xi)}{d\mu_1(\xi)}\right)^{1/2}\left(\frac{d\mu_T\ (\beta^{-1}\ \alpha)}{d\mu_T\ (\alpha)}\right)^{1/2}\ F(\psi^{-1}\xi,\ \beta^{-1}\alpha),$$

$\psi \in G^X$, $\xi \in C(X,G/T)$, $\alpha \in C(X,T)$, $F \in K$, $\beta \in C(X,T)$ s.t. $\beta\ (0) = 0$ and $\beta^{-1}\ d\beta = P(\varphi^{-1}\ \psi^{-1}(d\ \psi)\varphi)$, with φ a unique path equivalent ξ.

In the direct decomposition in which $U^R(H(I,T))$ generates the diagonalized operators has

$$U^L = \int^{\oplus} U^{\alpha^{-1}d\alpha} d\mu_T(\alpha),$$

with

$$(U^{\alpha^{-1}d\alpha}(\psi)F)(\xi) = \left(\frac{d\mu_T(\psi^{-1}\xi)}{d\mu_T(\xi)}\right)^{1/2}\ F(\psi^{-1}\xi)\ \exp i\ (\alpha^{-1}d\alpha,\ \phi^{-1}\psi^{-1}(d\psi)\phi).$$

We also show that the action of $C^\infty(X,G)$ by left multiplication on the measure spaces $(C_0(X,G),\mu_0)$ resp. $(C(X,G/T),\mu_0)$ of Brownian motion on G resp. G/T is ergodic. In sect. 3 we decompose the representation $U^{\alpha^{-1}d\alpha}$ described above. This gives in particular that U^L, U^R restricted to T_L^X, T_L some torus, are direct integral representations. In the case $G = SU(2)$ a suitable parametrization of the paths ϕ as paths in $T_L\backslash G/T_R$, T_L, T_R tori, yields that the representation of $(T_L \times T_R)^X$ given by $(\phi,\psi) \to U^L(\phi)\ \ U^R(\psi)$ has simple spectrum. We then prove in particular that the components $U^{\alpha^{-1}d\alpha}$ are a.s. irreducible, that $(U^L)' = U^R$, $(U^R)' = U^L$ and that U^L and U^R are factor representations. In the case $X = \mathbb{R}$ we also show that the corresponding factors are of type III.

*The possibility of the occurrence of factors not of type I had been pointed out in the fall 1974 by Prof. I. M. Gelfand in conversations with the fourth author .

2. A reduction of the energy representation

In this section we recall briefly the main facts concerning the energy representation of interest for the problem we want to discuss here. For more details we refer to [2] - [7].

Let G be a compact semi-simple Lie group. Let I be one of the following sets: $[0,1]$, S^1, $\mathbb{R}^+$, $\mathbb{R}$ and let $C(I,G)$ be the set of continuous mappings from I into G. As explained and proven in [7] (sect.5), $C(I,G)$ bears a measure μ_0, the standard Brownian measure, which is useful to describe the cyclic component of the vacuum in the energy representation of the Sobolev-Lie group $H(I,G)$. μ_0 is quasi-invariant with respect to the left and right actions of $H(I,G)$ on $C(I,G)$ and the right resp. left regular representations U^R resp. U^L, given for $\phi \in H(I,G)$, $\eta \in C(I,G)$, $f \in L^2(C(I,G),\mu_0)$ by

$$(U^R(\phi)f)(\eta) = f(\eta\phi)\left(\frac{d\mu_0(\eta\phi)}{d\mu_0(\eta)}\right)^{1/2}$$

resp.

$$(U^L(\phi)f)(\eta) = f(\phi^{-1}\eta)\left(\frac{d\mu_0(\phi^{-1}\eta)}{d\mu_0(\eta)}\right)^{1/2}$$

are equivalent to the cyclic component of the vacuum in the energy representation. As remarked in [7], U^L and U^R commute and are thus reducible.

Let us fix a torus T in G. In this section we shall give a direct integral decomposition of the Hilbert space $\mathcal{H} = L^2(C(I,G),\mu_0)$, relative to T,

$$\mathcal{H} = \int^{\oplus} \mathcal{H}^{\chi} d\mu(\chi),$$

such that the operators $U^R(\alpha)$, $\alpha \in H(I,T)$ are weakly dense in the set of diagonal operators for this decomposition. Since U^L and U^R commute we have that $U^L(\phi)$ will appear in this integral decomposition as a decomposable operator

$$U^L(\phi) = \int^{\oplus} U^{\chi}(\phi) d\mu(\chi).$$

Our aim is the study of the representation U^{χ}. The main tool for this is the following proposition

Proposition 2.1

Consider G as a principal fiber bundle with the right action of T and the connection on G such that the vertical subspace in the tangent space G_x at $x \in G$ is xt, where t is the Lie algebra of T, and the horizontal subspace is $xt^{\perp}$ (with $t^{\perp}$ the orthogonal complement of t with respect to the Euclidean structure on the Lie algebra g of G given by the opposite of the Killing form).

For almost all $\eta \in C(I,G)$ (with respect to the standard Brownian measure μ_0) there exists a canonical decomposition

$$\eta = \phi \quad \alpha \tag{2.1}$$

where $\alpha \in C(I,T)$ and ϕ has horizontal increments i.e. or any smooth function δ from I into t with compact support one has

$$\int_I (\delta(\tau), \phi^{-1}(\tau) d\phi(\tau)) = 0, \tag{2.2}$$

where $(\, , \,)$ is the scalar product in t. The decomposition (2.1) is unique and the mapping $\eta \to (\phi,\alpha)$ is measurable and transforms μ_0 in $\mu_h \times \mu_T$, where μ_T is the Brownian measure on path in $C(I,T)$, starting at the identity for $t = 0$, and μ_h is such that $\phi^{-1} d\phi$ is white noise distributed in $t^{\perp}$.

Moreover the mapping $\phi \to \phi^0$, where ϕ^0 is the class of ϕ in $C(I,G/T)$, is injective on the paths with horizontal increments and sends μ_h into the Brownian measure μ_1 on $C(I,G/T)$.

Proof: Let P be the orthogonal projection in g with range t. For μ_0-almost all $\eta \in C(I,G)$ we have that $P(\eta^{-1} d\eta)$ is a measure which is the derivative of a continuous function with values in t, and there exists a function α in $C(I,T)$ with $\alpha(0) = 0$, such that

$$\alpha^{-1} d\alpha = P(\eta^{-1} d\eta). \tag{2.3}$$

The mapping $\eta \to \alpha$ transforms μ_0 into the Brownian motion measure on $C(I,T)$. Let us take $\phi = \eta\alpha^{-1}$, then, for δ as in the statement of the proposition, we have easily $\langle \delta, \phi^{-1} d\phi \rangle \equiv \int_I (\delta, \phi^{-1} d\phi) = \int (\delta, \alpha\eta^{-1} d\eta\alpha^{-1} - \alpha^{-1} d\alpha) = 0,$

since $\delta(\tau) \in t$ and $\alpha \cdot \alpha^{-1}$ is trivial on t and we use (2.3). The unitarity of $\alpha \cdot \alpha^{-1}$ and its triviality on t yield

$$P(\alpha^{-1} \phi^{-1} d\phi \, \alpha) = P(\phi^{-1} d\phi)$$

and hence the orthogonality of $\alpha^{-1} \varphi^{-1} d\varphi \, \alpha$ and $\alpha^{-1} d\alpha$, which then yields the tensorial splitting $\mu_o = \mu_h \times \mu_T$ of μ_o. The injectivity of the canonical mapping from C(I,G) onto C(I,G/T) restricted to the paths with horizontal increments is a consequence of the unicity of the decomposition (2.1). As a consequence we have that it is possible to look at a function on C(I,G) as a function in two variables i.e. as a function on C(I,G/T)XC(I,T).

In this picture the right translation invariant functions are just the functions independent of the second variable, which implies the last assertion in the proposition. □

In the sequel we shall refer to the decomposition (2.1) as "<u>the orthogonal decomposition of η</u>".
In order to perform the decomposition of U^L we need to relate the orthogonal decompositions of η and $\psi^{-1}\eta$ with $\psi \in H(I,G)$, which is the object of the following Lemma.

<u>Lemma 2.2.</u> If $\eta = \phi\alpha$ is the orthogonal decomposition of η, then the orthogonal decomposition of $\psi^{-1} \eta$ with $\psi \in H(I,G)$ is

$$\psi^{-1}\eta = \phi'\alpha'$$

with $\phi' = \psi^{-1} \phi\beta$, $\alpha' = \beta^{-1}\alpha$, $\beta \in C(I,T)$ such that $\beta(0) = 0$, $\beta^{-1}d\beta = P(\phi^{-1}\psi^{-1}d\psi\phi)$.

<u>Proof:</u> The existence of β can be obtained as in Prop. 2.1 (in this case β is obtained as a function with derivative in $L^2(d\mu_o)$). The Lemma follows from the formula

$$\phi'^{-1} d\phi' = \beta^{-1}d\beta + \beta^{-1}\phi^{-1} d\phi \, \beta - \beta^{-1}\phi^{-1}\psi^{-1}d\psi \, \psi\beta,$$

using the fact that $\phi^{-1}d\phi$ is horizontal, together with the properties of P.□

We can now describe the representation U^L by the following

Theorem 2.3 U^L is unitarily equivalent to the representation V^L acting on $L^2(C(I,{}^G/T),\mu_1) \otimes L^2(C(I,T),\mu_T)$ and defined for $\psi \in H(I,G)$, $\xi \in C(I,{}^G/T)$, $\alpha \in C(I,T)$, $F \in L^2(C(I,{}^G/T),\mu_1) \otimes L^2(C(I,T),\mu_T)$ by:

$$(V^L(\psi)F)(\xi,\alpha) = \left(\frac{d\mu_1(\psi^{-1}\xi)}{d\mu_1(\xi)}\right)^{1/2} \left(\frac{d\mu_T(\beta^{-1}\alpha)}{d\mu_T(\alpha)}\right)^{1/2} F(\psi^{-1}\xi,\beta^{-1}\alpha),$$

with $\beta \in C(I,T)$ s.t. $\beta(0) = 0$ and $\beta^{-1}d\beta = P(\phi^{-1}\psi^{-1}d\psi\phi)$ for the unique path ϕ with horizontal increments s.t. $\phi^0 = \xi$.
In the same unitary equivalence $U^R \restriction C(I,T)$ is equivalent to V^R, where V^R is given by

$$(V^R(\gamma)F)(\xi,\alpha) = \left(\frac{d\mu_T(\alpha\gamma)}{d\mu_T(\alpha)}\right)^{1/2} F(\xi,\alpha\gamma), \gamma \in C(I,T).$$

Proof: We have
$$WU^L(\psi)W^{-1} = V^L(\psi)$$
$$WU^R(\gamma)W^{-1} = V^R(\gamma)$$
with
$$(WF)(\xi,\alpha) \equiv F(\phi\alpha).$$

This, together with the statements about the measures in Prop. 2.1 and Lemma 2.2 yields then the theorem. □

We remark that V^L and V^R commute, which is related to the fact that the β appearing in Theor. 2.3 depends on Ψ,ξ but not on α. This permits to prove the following decomposition theorem, by conditioning with respect to the second projection $(\xi,\alpha) \to \alpha$.

Theorem 2.4 In the decomposition in which $U^R(C(I,T))$ generates the diagonalized operators, one has

$$U^L = \int^{\oplus} U^{\alpha^{-1}d\alpha} d\mu_T(\alpha)$$

where

$$(U^{\alpha^{-1}d\alpha}(\psi)F)(\xi) = \left(\frac{d\mu_T(\psi^{-1}\xi)}{d\mu_T(\xi)}\right)^{1/2} F(\psi^{-1}\xi) \exp\left(i < \alpha^{-1}d\alpha, \phi^{-1}\psi^{-1}d\psi\phi >\right),$$

where Φ is an arbitrary element of the class ξ.

Proof: This follows from the decomposition of U^R acting on $L^2(C(I,T),\mu_T)$, given by Fourier transform, and the definition of β in the statement of Lemma 2.2 □

Later on we shall also use the following result.

Prop. 2.5 Then the action of $H(I,G)$ by left multiplication on the measure spaces $(C(I,G),\mu_0)$ resp. $(C(I,G/T),\mu_1)$ of Brownian motion on G resp. G/T is ergodic.

Proof: The case of G/T follows from the one of G because the image under the left multiplication action of a right coset is a right coset.
Let now f be a measurable function on C (I,G) which is invariant under left multiplication by smooth functions on I with values in some one-dimensional torus T. From the proceding analysis one has, for μ_0-almost all η:

$$f(\eta) = f(\eta_1),$$

where η_1 is such that $\eta_1^{-1}d\eta_1 = (1-P_t)(\eta^{-1}d\eta)$,

where P_t now denotes the orthogonal projection of g onto t. Repeating this argument for sufficiently many tori $T_1,\ldots,T_p$, such that the corresponding one-dimensional sub-Lie-algebras $t_1,\ldots,t_p$ are mutually orthogonal and generate g, one obtains that

$$f(\eta) = f(\eta_p)$$

for μ_0 - a.e. η, with η_p such that

$$\eta_p^{-1}d\eta_p = (1-P_{t_p})\ldots(1-P_{t_1})(\eta^{-1}d\eta).$$

From this it follows that f is a constant μ_0 - a.e., which ends the proof. □

3. Further decomposition of the energy representation.

In this section we shall obtain a further decomposition of the representation U^χ of Theor. 2.4, $\chi = \alpha^{-1}d\alpha$. A first step is the following

Prop. 3.1 Consider G as a principal fiber bundle with fiber $T_L \times T_R$, where T_L and T_R are arbitrary tori in G, with Lie t_L and t_R, respectively. $T_L \times T_R$ acts on G by $h \to \alpha^{-1}h\beta$, for $(\alpha,\beta) \in T_L \times T_R$. Consider the connection on G with vertical subspace $t_L h + h t_R \equiv V_h$ and horizontal subspace the orthogonal complement to V_h in G_h, where G_h is the tangent space of G at h. Then we have:

i) for μ_0-almost all $\eta \in C(I,G)$ there exists a decomposition

$$\eta = \alpha^{-1}\phi\beta,$$

where $\alpha \in C(I,T_L)$, $\beta \in C(I,T_R)$ and ϕ having horizontal increments, i.e. such that for any smooth functions $\delta_L : I \to t_L$ and $\delta_R : I \to t_R$ with compact support, one has

$$\langle\delta_R, \phi^{-1}d\phi\rangle = \langle\delta_L, d\phi\phi^{-1}\rangle = 0,$$

ii) the decomposition in i) is unique and the mapping $\eta \to (\alpha,\phi,\beta)$ is measurable and transforms μ_0 in $\int \nu^{\phi}\, d\mu(\phi)$, where μ is the unique measure on horizontal paths with image by the injection $\phi \to \phi^{oo}$ (canonical projection from $C(I,G)$ to $C(I,T_L\backslash G/T_R)$) the Brownian measure on $C(I,T_L\backslash G/T_R)$. ν^{ϕ} is a measure on the pairs of paths $(\alpha,\beta) \in C(I,T_L) \times C(I,T_R) \equiv C(I,T_L \times T_R)$ such that $(\alpha^{-1}d\alpha, \beta^{-1}d\beta)$ has Gaussian distribution with covariance given by the matrix

$$\begin{pmatrix} \mathbb{1} & -\mathrm{Ad}\phi \\ -\mathrm{Ad}\phi^{-1} & \mathbb{1} \end{pmatrix}$$

acting pointwise in L^2 $(t_L \times t_R)$ valued functions on I.

<u>Proof:</u> Let P_L, P_R be orthogonal projections from g onto t_L, t_R.
For almost every $\eta \in C(I,G)$, $\mathrm{ad}\eta$ is unitary in $L^2(I,g)$ and $P_L(\eta^{-1}d\eta)$ resp. $P_R(d\eta\, \eta^{-1})$ are measures which are derivatives of continuous functions with values in t_L resp. t_R. Let us first remark that the set of $\eta \in C(I,G)$ such that $\mathrm{ad}\eta(C(I,t_L)) \subset C(I,t_R)$ is of μ_0-measure zero, because $\eta^{-1}d\eta$ is Gaussian and t_R is a proper subspace of g.
It follows that the matrix

$$\begin{pmatrix} \mathbb{1} & -P_R\mathrm{ad}\eta^{-1} \\ -P_L\mathrm{ad}\eta & \mathbb{1} \end{pmatrix}, \tag{3.1}$$

acting on L^2-functions with values in $t_R \times t_L$, is, for μ_0-a.a. η, invertible and the following set of equations

$$\alpha^{-1}d\alpha - P_L(\eta\beta^{-1}d\beta\eta^{-1}) = -P_L(d\eta\eta^{-1})$$
$$\beta^{-1}d\beta - P_R(\eta^{-1}\alpha^{-1}d\alpha\eta) = P_R(\eta^{-1}d\eta)$$

has solution $\alpha,\beta \in C(I,T_L) \times C(I,T_R)$ such that $\alpha(0) = 0$, $\beta(0) = 0$. Putting now $\phi = \alpha\eta\beta^{-1}$ one easily sees that i) is satisfied. The unicity comes from the μ_o- a.e. invertibility of the matrix (3.1) and the fact that for $\eta=\alpha^{-1}\phi\beta$, $\alpha,\gamma \in T_L$, $\beta,\delta \in T_R$

$$P_L(\eta\gamma^{-1}d\gamma\ \eta^{-1}) = P_L(\phi\gamma^{-1}\ d\gamma\ \phi^{-1})$$
$$P_R(\eta^{-1}\delta^{-1}d\delta\ \eta) = P_R(\phi^{-1}\delta^{-1}d\delta\ \phi).$$

The mapping $\eta \to (\alpha,\beta,\phi)$ is thus one - to - one and transforms the T_L - left and T_R - right invariant functions into functions which only depend on ϕ. Therefore the conditioning with respect to the mapping $\eta \to \eta^{oo}$ is just obtained by specifying a value of ϕ. Finally the statements about measures are consequences of the following relation, which holds for $\eta=\alpha^{-1}\phi\beta$ as in i):

$$|\eta^{-1}d\eta|^2 = |\alpha^{-1}d\alpha|^2 + |\beta^{-1}d\beta|^2 +$$
$$+|\phi^{-1}d\phi|^2 - 2\ (\alpha^{-1}d\alpha,\phi^{-1}\beta^{-1}d\beta\ \phi). \qquad \square$$

The following Corollay describes the algebraic properties of U^L and U^R that follow from the preceding results:

<u>Corollay II.2</u> The restriction of U^L to $H(I,T_L)$ and the restriction of U^R to $H(I,T_R)$ are direct integral representations

$$U^L(\alpha) = \int^{\oplus} U^{L,\phi}\ (\alpha)d\mu\ (\phi)$$
$$U^R(\beta) = \int^{\oplus} U^{R,\phi}\ (\beta)d\mu\ (\phi).$$

The representation $(\alpha,\beta) \in H(I,T_L x T_R) \to U^{L,\phi}(\alpha)\ U^{R,\phi}(\beta)$ is Gaussian, namely $U^{L,\phi}$ and $U^{R,\phi}$ act in $L^2\ (C(I,T_L x T_R), \nu^\phi)$ and , with (,) the scalar product in $L^2(C(I,T_L x T_R), \nu^\phi)$:

$$(1,\ U^{L,\phi}(\alpha)\ U^{R,\phi}\ (\beta)1) =$$
$$\exp[-\tfrac{1}{2}\ (|\alpha^{-1}d\alpha|^2 + |\beta^{-1}d\beta|^2 - 2(\alpha^{-1}d\alpha,\phi^{-1}\beta^{-1}d\beta\phi)]\ \text{with}\ |\cdot|^2 = \langle\cdot,\cdot\rangle.$$

We remark that one can think of the horizontal path ϕ as representing a path in the double coset space of G(denoted by $T_L \backslash G/T_R$). In this picture μ is just the Brownian measure on $C(I,T_L\backslash G/T_R)$ i.e. the image of μ_o by the canonical projection of $C(I,G)$ onto $C(I,T_L\backslash G/T_R)$.

In the case $G = SU(2)$ we are able to give a useful parametrization of paths in $C(I, T_L\backslash G/T_R)$ and we shall from now on always assume $G = SU(2)$. The parametrization is given in the following Lemma.

Lemma 3.3 For $G = SU(2)$ we have the following:

i) If $\psi, \phi \in C(I,G)$ are such that for any $\alpha \in C(I,T_L), \beta \in C(I,T_R)$:

$\langle \psi^{-1}\alpha^{-1}d\alpha\, \psi, \beta^{-1}d\beta\rangle = \langle \phi^{-1}\alpha^{-1}d\alpha\, \phi, \beta^{-1}d\beta\rangle$,

then ψ, ϕ define the same path in $C(I, T_L\backslash G/T_R)$, where T_L and T_R are arbitrary tori in G;

ii) If $\psi, \phi \in C(I,G)$ are such that for any $\xi \in C(I,G)$, $\beta \in C(I,T_L)$

$\langle \psi^{-1}\beta^{-1}d\beta\psi, \xi^{-1}d\xi\rangle = \langle \phi^{-1}\beta^{-1}d\beta\, \phi, \xi^{-1}d\xi\rangle$,

then ψ, ϕ define the same path in $C(I, T_L\backslash G)$.

Proof: Using approximations of Dirac measures at points $x \in I$ by functions of the form $\alpha^{-1}d\alpha$ and $\beta^{-1}d\beta$, we get

$$Tr(\psi(x)^{-1}a\psi(x)b) = Tr(\phi^{-1}(x)a\phi(x)b)$$

for any $a \in t_L$, $b \in t_R$.

Let D be the set of diagonal matrices with zero trace and let u resp. v be such that t_L resp. t_R are conjugate with D by the conjugation $u.\, u^{-1}$ resp. $v \cdot v^{-1}$. Then with $f \equiv u\psi(x)u^{-1}$, $g \equiv v\phi(x)v^{-1}$ we have

$$Tr(f^{-1}\gamma f\delta) = Tr(g^{-1}\gamma g\delta) \qquad (3.2)$$

for any $\gamma, \delta \in D$. This relation is also true for γ or δ the unit matrix, hence it is true for all diagonal matrices γ, δ. Using matrix units we have then $|f_{ij}| = |g_{ij}|$. From this and the unitarity of f and g we have for $c_{ij} \equiv f_{ij}/g_{ij}$:

$$c_{11}/c_{12} = c_{12}/c_{22}.$$

It follows from this that there exist $s_i, t_j, i,j = 1,2$ with $c_{ij} = s_i t_j$, $/s_i/=/t_j/=1$. Thus we have the relation for matrices on C^2:

$$f = sgt,$$

with s,t diagonal in $SU(2)$ with diagonal elements s_i resp. t_i, $i = 1,2$.

It follows then that

$$\phi(x) = u^{-1}su\phi(x)v^{-1}tv.$$

Since s_i, t_i depend continuously on x, we have completed the proof of i). The rest of Lemma 3.3 is easy), using the same method. □

We shall also need the following

<u>Lemma 3.4</u> For $G = SU(2)$ there exists a basis B for the Borel sets in $C(I, T_L\backslash G/T_R)$ such that for $A \in B$ the measures

$\int_A \nu^\phi d\mu_h(\phi)$ and $\int_{CA} \nu^\phi d\mu_h(\phi)$, with CA the complement of A, are mutually disjoint.

<u>Proof:</u> Using a basis in t_R and t_L one sees that the measures ν^ϕ are measures on $\mathcal{S}'(\mathbb{R}^2)$ with covariance given by the quadratic form $\phi, \psi \in \mathcal{S}(\mathbb{R}^2) \to$

$$\int \Phi A \Phi d\tau, \text{ with } \Phi(\tau) \equiv \binom{\phi(\tau)}{\psi(\tau)}, \quad A_{ii} = 1, \; A_{ij} = a(\tau), \; i \neq j, \; i,j = 1,2$$

where $a(\tau)$ is a real function with $|a(\tau)|<1$, only depending on $\phi(\tau)$ and satisfying $a_1(\tau) \neq a_2(\tau)$ iff $\phi_1(\tau) \neq \phi_2(\tau)$ (this follows from Lemma 3.3). $a(\tau)$ is continuous for almost all ϕ. Using another basis in $t_L \times t_R$ one may assume that the matrix $\begin{pmatrix} 1 & a(\tau) \\ a(\tau) & 1 \end{pmatrix}$ can be replaced by a diagonal matrix with $1 \pm a(\tau)$ on the diagonal. A set in B is a set of ϕ's such that the corresponding a – denoted by a_ϕ – takes its value $a_\phi(t_0)$ in a given Borel set for some $t_0 \in I$. By Lemma 3.3 such sets are separating in $C(I, T_L\backslash G/T_R)$ and consequently they generate the Borel structure. It is sufficient to prove the Lemma for A given by the condition $a_\phi(t_0) < d$, for any $d > 0$. By the almost sure continuity of trajectories we have that there exists an open set $\mathcal{O}$ and a number $\eta > 0$ such that for $\tau \in \mathcal{O}$, $\phi \in A$, $\phi' \in CA$ we have

$$a_\phi(\tau) \leq d - \eta < d \leq a_{\phi'}(\tau).$$

The Lemma is then proven using the result of Prop. A. 1 of the Appendix, which actually shows that the two measures in the statement of the present Lemma have disjoint projections on the set of $T_L \times T_R$ – valued functions with Logarithmic derivatives at $(\alpha,\beta) \in T_L \times T_R$, $(\alpha^{-1}d\alpha, \beta^{-1}d\beta)$ which are distributions with support in $\mathcal{O}$. □

We are now able to prove the following

Theorem 3.5 Let $G = SU(2)$, then the representation of $H(I, T_L \times T_R)$ in $L^2(C(I,SU(2)))$ given by $(\phi,\psi) \in H(I,T_L \times T_R) \to U^L(\phi)U^R(\psi)$ has simple spectrum. The representation space can be identified with the set of L^2-functions on $t_L \times t_R$ - valued measures on I with respect to the measure ν_0 whose Fourier transform is given by

$$\int e^{i(f,\chi_L)} e^{i(g,\chi_R)} d\nu_0 (\chi_L,\chi_R) =$$

$$= e^{-\frac{|f|^2}{2}} e^{-\frac{|g|^2}{2}} \int e^{-(f,\eta g^{-1}\eta)} d\mu_0(\eta),$$

where μ_0 is the standard Brownian measure on $C(I,G)$.

Proof: The second assertion follows from the first one and the fact that $(f,\eta g \eta^{-1})$ only depends on the $T_L \backslash G/T_R$ double cosets. The cyclicity of the vacuum vector $(1 \in L^2(C(I,SU(2)),\mu_0))$ follows from Lemma 3.4 and the cyclicity of the vacuum for the representation $(\phi,\psi) \to U^{L,\rho}(\phi)U^{L,\rho}(\psi)$ for a given path ρ in $T_L \backslash G/T_R$, this being a consequence of the Gaussian character of this representation. □

Corollay 3.6 Let $G = SU(2)$. Then we have the following

i) For almost all $\chi_L \cong \alpha^{-1} d\alpha$, with $\alpha \in C(I,T_L)$, $\alpha(0)=e$, with respect to μ_{T_R}, μ_{T_R} being as in Theor. 2.4, the operator

$$(W(\gamma)f) \quad (\xi) = \left(\frac{d\mu(\gamma^{-1}\xi)}{d\mu(\xi)}\right)^{1/2} \quad f(\gamma^{-1}\xi)$$

is in the von Neumann algebra generated by $U^{\chi_L}(H(I,T_L))$.

ii) The same statement as in ii) holds also for $\gamma \in H(I,G)$ and $U^{\chi_L}(H(I,T_L))$ replaced by $U^{\chi_L}(H(I,G))$.

iii) The double commutant $U^{\chi_L}(H(I,G))''$ contains all operators of multiplication by functions of $\xi, \xi \in C(I,G/T_R)$.

PROOF:
The decomposable operator $(\gamma \in H(I,T_L)) \to \underline{W}(\gamma) = \int^{\oplus} W(\gamma)\, d\mu(\alpha)$, associated to the (constant) field $\alpha \to W(\gamma)$ in the Hilbert decomposition of Th.2.4, commutes with $U^R(H(I,T_R))$. It also cummutes with $U^L(H(I,T_L))$ because $W(\gamma)$ does commute with $U^{\chi_L}(H(I,T_L))$, as seen using the fact that the function :

$$\xi \to \langle \chi_L , \xi^{-1} \psi^{-1} d\psi \ \xi \rangle$$

is invariant by left multiplication by $\gamma \in H(I,T_L)$. It follows from Th. 3.5 that $\underline{W}(\gamma)$ is in $(U^L(H(I,T_L)), U^R(H(I,T_R)))''$ and i) follows.

ii) is an immediate consequence of i) since the tori of G generate G.

iii) From Theor.2.4 and ii) we have that $U^{\chi_L}(H(I,G))''$ contains the operators multiplication by functions of the form

$$\xi \to \exp i \langle \chi_L, \xi^{-1} \phi^{-1} d\phi \xi \rangle.$$

As a consequence of Lemma 3.3 this set of functions is an algebra and separates the points, which then ends the proof of the Corollary. □

We conclude with a Theorem stating the factoriality of the representations U^L, U^R.

<u>Theorem 3.7</u> For $G = SU(2)$ we have

i) The restriction of U^R to $H(I,T_R)$ has double commutant $U^R(H(I,T_R))''$ which is maximal abelian in $U^R(H(I,G))''$;

ii) $U^L(H(I,T_L))''$ is maximal abelian in $U^L(H(I,G))''$

iii) In the decomposition of Th.2.4 in which $U^R(C(I,T_R))$ generates the diagonal operators, i.e.

$$U^L = \int^{\oplus} U^{\alpha^{-1}d\alpha} d\mu_{T_R}(\alpha),$$

the components U^{χ}, $\chi = \alpha^{-1} d\alpha$, $\alpha(0)=e$ are irreducible for μ_{T_R}- almost all χ.

iv) The commutant of U^L is U^R and the one of U^R is U^L, i.e.

$$(U^L)' = U^R, \ (U^R)' = U^L .$$

v) U^L and U^R are factor representations.

<u>Proof:</u> Let $Q \in U^R(H(I,T_R))' \cap U^R(H(I,G))''$.
Then Q is decomposable with respect to the Hilbert decomposition of Theor.2.4 i.e.

$$Q = \int^{\oplus} Q^{\chi} d\mu_{T_R}(\chi).$$

Since $U^R(H(I,G))'' \subset U^L(H(I,G))'$ for almost all χ, Q^χ commutes with $U^{L,\chi}(H(I,G))$. Q^χ is the multiplication by some function of $\xi \in C\ (I,G/T_R)$ which is invariant under translations, by Coroll.3.6, iii)iv). Hence by Prop.2.5 we have that Q^χ is a multiple of the identity and thus $Q \in U^R(H(I,T_R))''$. This proves i), ii) is obtained in a similar way as i), interchanging U^R and U^L. iii) is an immediate consequence of i) (using e.g. Th.8.32 in [24]). To prove iv), let us consider a $Q \in U^R(H(I,G))'$. Then Q is decomposable in the decomposition of Theor.2.4 i.e.

$$Q = \int^{\oplus} Q^\chi d\mu_{T_R}(\chi).$$

By the irreducibility of $U^{L,\chi}$ one has that $Q^\chi \in U^{L,\chi}(H(I,G))''$ for μ_{T_R}-a.a.χ and thus $Q \in U^L(H(I,G))''$, which proves iv). To prove v) let us consider $Q \in U^R(H(I,G))'' \cap U^R(H(I,G))'$. By iv) and i) we see that for any torus T_R we have $Q \in U^R(H(I,T_R))''$ and symmetricaly for any T_L, $Q \in U^L(H(I,T_L))''$. On the other hand by Theor. 3.5 the representation space can be described as a set of functions of two variables χ_L, χ_R in such a way that $U_R(H(I,T_R))$ and $U_L(H(I,T_L))$ are multiplications by functions of χ_R (but not of χ_L) resp. by functions of χ_L (but not of χ_R). This together with $Q \in U^R(H(I,T_R))'' \cap U^L(H(I,T_L))''$, implies that Q is a constant, hence $U^R(H(I,G))'' \cap U^R(H(I,G))'$ are the multiples of the identity, which proves that U^R, and similarly U^L, are factor representations. □

Let again $G = SU(2)$ and X let now X be $\mathbb{R}$. The von Neumann algebras $U^L(H(I,G))''$ and $U^R(H(I,G))''$ are asymptotically abelian as seen by restricting first to functions of compact support and noticing that by translations one has eventually commutation (since $U^L(\psi)$ and $U^L(\psi')$ commute when ψ and ψ' have disjoint support).

One easily verifies that $\mathbb{1}$ is, up to scalar multiplication, the unique invariant vector, using asymptotic abelianness and clustering for the vacuum expectation $(\mathbb{1}, .\mathbb{1})$. This, by a result of Størmer [18], implies that U^L and U^R generate Von Neumann algebras of typeIII or of type II_1 and, in the latter case, the vacuum expectation is a trace. But a routine computation shows that, for $G = SU_2$, with the Pauli matrices notation, taking

$$\psi(t) = \exp\left(\frac{i}{2}\sigma(\vec{u}(t))\alpha(t)\right)$$

$$\varphi(t) = \exp\left(\frac{i}{2}\sigma(\vec{w})\,\alpha(t)\right).$$

One has :

$$(1, U^L(\psi) \;\; U^L(\varphi)\, 1) \neq (1, U^L(\varphi) \;\; U^L(\psi)\, 1),$$

if one chooses $\vec{w}.\vec{w} = 1$, $\vec{w}.\vec{u} = 0$, $\vec{u}.\vec{u} = 1$, $\frac{d\vec{u}}{dt} = \vec{w} \wedge \vec{u}$ and $\alpha \neq 0$ with compact support.

This rules out the case II_1, hence we have type III. We formulate this in the follpwing.

<u>Theorem 3.8</u> Let $G = SU(2)$ and $I = \mathbb{R}$. Then the von Neumann algebras $U^L(H(I,G))''$ and $U^R(H(I,G))''$ generated by U^L, U^R are asymptotic abelian factors of type III.

Appendix

In this appendix we give an elementary proof of a "uniform disjointness" result, namely the following

Proposition A 1 Let A and B be bounded invertible operators on $L^2(\mathbb{R})$ satisfying

$$A \leq \lambda \mathbb{1} < \lambda' \mathbb{1} \leq B$$

for some constants λ, λ'. Let μ_A, μ_B be the corresponding gaussian measures with covariances given by A resp. B. Then there exists a Borel set N in $S'(\mathbb{R})$, only depending on λ and λ', such that $\mu_A(N)=0$ and $\mu_B(N)=1$.

Proof: Let us first of all remark that if Δ is the ball of center 0 and of radius R in $\mathbb{R}^2$ and if μ_a and $\mu_{\lambda 1}$ are the gaussian measures with covariance matrices a and $\lambda \mathbb{1}$ resp., with $a \leq \lambda \mathbb{1}$ then one has

$$\mu_a(\Delta) \geq \mu_{\lambda \mathbb{1}}(\Delta) .$$

Moreover if $b \geq \lambda' \mathbb{1}$ we have

$$\mu_b(\Delta) \leq \mu_{\lambda' \mathbb{1}}(\Delta) .$$

Using an arbitrary basis in $L^2(\mathbb{R}^3)$ one can assume that the measures are on $\Omega = (\mathbb{R} \cup \{\infty\})^{\mathbb{N}}$ and one can describe sets on Ω by prescriptions on the canonical projections x_i, $i = 1, \ldots, N$.

One has:

$$\mu_A\{|x_i|^2 + |x_j|^2 \leq R^2\} \geq \mu_{\lambda \mathbb{1}}\{|x_i|^2 + |x_j|^2 \leq R^2\} = 1 - e^{-R^2/2\lambda}$$

$$\mu_B\{|x_i|^2 + |x_j|^2 \leq R^2\} \leq \mu_{\lambda' \mathbb{1}}\{|x_i|^2 + |x_j|^2 \leq R^2\} = 1 - e^{-R^2/2\lambda'} .$$

Choosing now sequences of real numbers R_n such that $e^{-R^2/2\lambda'} = n^{-1}$

and sets

$$N^k = \{|x_{2n}|^2 + |x_{2n+1}|^2\} \leq R_{nk} \; \forall n\}$$

one easily verifies that $N = \bigcup_{k \geq 1} N^k$ satisfies the statements in the

Propositions. □

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