FR8400797

MINIMAL GRAVITATIONAL COUPLING IN THE NEWTONIAN THEORY AND THE COVARIANT SCHRODINGER EQUATION (*)

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CNRS - CPT-83/P.1488
February 1983

Abstract : The role of the Bargmann group (11-dimensional extended Galilei group) in non relativistic gravitation theory is investigated. The generalized Newtonian gravitation theory (Newton-Cartan theory) achieves the status of a gauge theory about as much as General Relativity and couples minimally to a complex scalar field leading to a fourdimensionally covariant Schrödinger equation. Matter current and stress-energy tensor follow correctly from the Lagrangian. This theory on curved Newtonian space-time is also shown to be a limit of the Einstein-Klein-Gordon theory.

(*) Partially supported by the Natural Sciences and Engineering Research Council of Canada, Grant no. A8059

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I - INTRODUCTION

Much work has been done during the last decade on the general structure of gauge theories and also the sense in which General Relativity and similar gravitation theories can be interpreted as such. For reviews see, for example, [28] and [12]. This latter issue has probably not yet been fully settled so that it may seem worthwhile to investigate for comparison purposes what happens in the only other physically well understood gravitation theory, namely in Newtonian gravity.

On the other hand, as Kuchar [15] has pointed out, it may also be instructive to study nonrelativistic quantum mechanics in a fully covariant geometrical form in order to gain insight that could be useful for relativistic quantum theories in curved space-time.

In this paper, we simply wish to show how easily the minimal interaction principle can be applied to the free one-particle Schrödinger equation to get a fully fourdimensionally covariant theory. Kuchar [15] has already obtained essentially the same equation by considering various observer and coordinate transformations. Our starting point is the geometrical (and slightly generalized) Newtonian theory in the form of [16] and [6].

This theory describes the Newtonian gravitational field with a rather elaborate geometrical structure in order to make it fully co-variant and as similar as possible to the relativistic theory. Instead of one scalar potential function V , one has a contravariant space metric $\gamma = \gamma^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, a time metric $\psi = \psi_\alpha dx^\alpha$ and a pair (A, u), $A = A_\alpha dx^\alpha$, $u = u^\alpha \partial_\alpha$ determining the connection $\Gamma_\alpha{}^\gamma{}_\beta$ with constraints $(\gamma^{\alpha\beta}\psi_\beta = 0 , u^\alpha \psi_\alpha = 1)$ and a correspondingly large gauge group. There are a few benefits, however. As was shown in [6] the equations of motion for continua and test particles now follow from a general covariance postulate and, as we will show here, gravitational minimal coupling can be achieved in a very simple and natural way. On the other hand, the description appears to be too degenerate for the existence of a variational formulation of the gravitiational field equations in terms of (γ, ψ, A, u).

While we do not expect to find any new physical phenomena (1) in this very classical domain, it seems to us that still many insights can be gained from a systematic study of this classical field theory. As a test case for certain ideasin field theory formalisms, it has the advantage that physically the situation is so well understood that any wrong approach is very quickly revealed as such.

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(1) See ref [27] for a discussion of past and proposed experiments concerning the interaction of matter waves and gravitational field.

In section II, we review parts of the geometric Newtonian gravitation theory (sometimes called Newton-Cartan theory) in the form developed in [16] and [6] where references are given to its history, back to Cartan [3]. But we assume that the reader has access to [16] and [6], at least for motivation purposes and develop the subject from a different angle. We attempt to present Newtonian gravity as much as possible as a gauge theory of the Bargmann group (i.e. the 11-dimensional non trivial extension of the inhomogeneous Galilei group [1]). This cannot fully succeed, at least not in the narrow sense of a Yang-Mills type gauge theory, just as General Relativity is not simply the gauge theory of the Poincaré (or the Lorentz) group.

On the other hand, there are some clear indications that the Bargmann group $\mathcal{B}$ has a role to play in Newtonian gravitation (independently of quantum mechanical considerations). In [6] an evolution space for a test particle was constructed with the help of a Bargmann bundle in a manner similar to the evolution spaces for particles in relativistic gravitational, electromagnetic [17] and Yang-Mills fields [7],[8]. In [9] and, in a more intrinsic form, in [23] the Newtonian connections [16] were characterized in a simple geometric way as connections on a Bargmann bundle.

Our aim being minimal gravitational coupling of a complex scalar field we would like to associate a bundle with standard fiber $\mathbb{C}$

to a principal Bargmann bundle in such a way that $\mathcal{B}$ acts via its phase rotation on $\mathbb{C}$. But this is impossible so that we are forced to break the $\mathcal{B}$-symmetry and have only the abelian subgroup consisting of boosts and phase rotations act on $\mathbb{C}$. It turns out that the corresponding infinite dimensional gauge group of section changes is just the one previously used in the principle of general covariance [6]. Moreover, it also leads to the Bargmann group itself as a transformation group of space-time and the phase of a wave function. See, for example, Levy-Leblond [20]. The reader not interested in these group theoretical considerations should be able to skip section II-a without great loss of continuity.

In section III, we associate $\mathbb{C}$ thus at least to this subgroup of $\mathcal{B}$ having it act only via the $\mathcal{B}$-centre on $\mathbb{C}$. This now looks like a $U(1)$-gauge theory on space-time coupling to a complex scalar field, i.e. just like the quantum mechanical wave function in an electromagnetic field. Taking a Lagrangian for the free Schrödinger equation and replacing $\partial_\alpha \Phi$ by $\mathcal{D}_\alpha \Phi := \partial_\alpha \Phi - im/\hbar\, A_\alpha \Phi$ (and writing $u^\alpha \mathcal{D}_\alpha \Phi$ for $\partial_t \Phi$) we find, quite remarkably, an expression that is invariant under the full automorphism group (space-time diffeomorphisms and the above mentioned section changes). The associated Euler-Lagrange equations are a 4-covariant form of Schrödinger's equation on a general Newtonian space-time and agree in form with those of Kuchar [15]. Applying the method of [6] to this Lagrangian, we derive a matter flow vector J^α and a "Hilbert" stress-energy tensor $T^{\alpha}_{\cdot\beta}$ that are gauge invariant, satisfy the general balance equations and agree on flat space with the classical expressions. (We call this a "Hilbert" stress-energy tensor since it corresponds to the one obtained in relativistic theories by variations with respect to the metric. The canonical stress-energy tensor is different and not gauge invariant.)

Finally, in section IV, we show how this field theory is naturally the Newtonian limit $(c \longrightarrow \infty)$ of the Klein-Gordon field on curved Lorentzian space-time. The standard (WKB-like) method of decoupling a phase factor $\exp(-imc^2 t/\hbar)$, where t is a world time, still works and the Lagrangian, the Schrödinger equation as well as the Newtonian matter current are obtained as limits of the corresponding relativistic quantities. We also discuss the limit of the Klein-Gordon stress-energy tensor and find that some restrictions on the c -dependent family of relativistic metrics and Klein-Gordon fields are needed in order to produce the exact Schrödinger stress-energy tensor in the limit.

II - The NEWTONIAN GRAVITATIONAL FIELD and ITS GAUGE GROUP

a - The Bargmann bundle and Newtonian structures

The Newtonian gravitational field can be described by a connection on a certain principal bundle over a fourdimensional manifold M. In the relativistic gravitation theory the Lorentz (or the Poincaré) group plays a central role. It turns out that the essential features of Newtonian gravity follow more naturally if the (inhomogeneous) Galilei group G is replaced by its *non* trivial central extension, the Bargmann group B [1], [21], [25] which appears in the exact sequence

$$1 \longrightarrow U(1) \longrightarrow B \longrightarrow G \longrightarrow 1 .$$

The group B can be described by its 6x6-matrix representation [22]

$$\begin{pmatrix} \underline{R} & \underline{b} & \underline{c} & 0 \\ 0 & 1 & c^0 & 0 \\ 0 & 0 & 1 & 0 \\ \underline{b}^T\underline{R} & \underline{b}^2/2 & \lambda & 1 \end{pmatrix} \qquad \begin{array}{l} \underline{R} \in SO(3) \\ \underline{b}, \underline{c} \in \mathbb{R}^3 \\ c^0 \in \mathbb{R} \\ \lambda \in \mathbb{R}/2\pi\mathbb{Z} . \end{array} \qquad \text{(II.1)}$$

If $G\ell(M)$ is the principal bundle of linear frames over M, locally described by (x^α, e^α_a) [2], a <u>Galilei structure</u> is a reduction of $G\ell(M)$ to the homogeneous Galilei group $H = SO(3) \underline{\times} \mathbb{R}^3$ which can be given by tensor fields $(\gamma^{\alpha\beta}, \psi_\alpha)$ on M such that

$$H(M) = \left\{ (x^\alpha, e^\alpha_a) / \gamma^{\alpha\beta} = \delta^{AB} e^\alpha_A e^\beta_B \ , \ \psi_\alpha = \theta^0_\alpha \right\}$$

where $\theta^a_\lambda e^\lambda_b = \delta^a_b$. Similarly, $G(M) = \left\{ (x^\alpha, e^\alpha_a, \xi^\alpha) \right\}$ is the pullback of $H(M)$ by the canonical projection $TM \to M$. It is a subbundle of the bundle of affine frames of M. Denote by $\iota : H(M) \hookrightarrow G(M)$ its imbedding through the zero section of TM. Let $B(M)$ be a $U(1)$-extension of $G(M)$ with structure group B so that we have a surjective principal bundle homomorphism $B(M) \longrightarrow G(M)$ and let $\tilde{H}(M)$ be the pull back of $B(M)$ by ι. We then have the commutative diagramme

$$\begin{tikzcd}
B(M) & \tilde{H}(M) \arrow[l, hook', "\tilde{\iota}"'] & \\
(x,e,\xi,\kappa) \arrow[d] & (x,e,\kappa) \arrow[d] & \tilde{\iota}(x,e,\kappa) = (x,e,0,\kappa) \\
G(M) \arrow[r, shift right] & H(M) \arrow[l, shift right, "\iota"'] & \\
(x,e,\xi) \arrow[d] & (x,e) \arrow[d] & \iota(x,e) = (x,e,0) \\
TM \arrow[r] & M & \\
(x,\xi) & (x) &
\end{tikzcd}$$

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(2) $\alpha, \beta, \cdots, a, b, \ldots = 0,1,2,3 \ ; A, B, \ldots = 1,2,3$. Greek indices refer to coordinates in M, Latin indices are frame labels.

In the general theory of principal bundles [14], one trivializes the bundle locally by means of reference sections and uses x^α and coordinates of the structure group manifold. Here we prefer to keep the standard notations $(x^\alpha, e^\alpha_a, \xi^\alpha)$ and treat only the phase factor κ as an intrinsic group coordinate. Of a local reference section, we need only explicitely give the timelike unit vector which we call u^α. We can then express the right action of B in a matrix form by describing the general element $p_x \in B(M)$ as

$$p_x = \begin{pmatrix} e^\alpha_a & \xi^\alpha & 0 \\ 0 & 1 & 0 \\ \overset{u}{\gamma}_a & \kappa & 1 \end{pmatrix} \tag{II.2}$$

with $$\overset{u}{\gamma}_a := \overset{u}{\gamma}_{\alpha\beta}\, e^\alpha_0 e^\beta_a - 1/2\, \overset{u}{\gamma}_{\alpha\beta}\, e^\alpha_0 e^\beta_0\, \delta^0_a \tag{II.3}$$

and $\overset{u}{\gamma}_{\alpha\beta}$ defined by $\overset{u}{\gamma}_{\alpha\beta}\, u^\beta = 0\,,\ \overset{u}{\gamma}_{\alpha\lambda}\, \gamma^{\lambda\beta} = \delta^\beta_\alpha - u^\beta \psi_\alpha$.
Explicitely, if $\hat{p}_x = p_x\, g$ where $g \in B$ is given by (II.1), one finds

$$\hat{e}^\alpha_A = e^\alpha_B\, R^B_{\cdot A}\,,\quad \hat{e}^\alpha_0 = e^\alpha_0 + e^\alpha_K\, b^K$$
$$\hat{\xi}^\alpha = \xi^\alpha + e^\alpha_r\, c^r\,,\quad \hat{\kappa} = \kappa + \lambda + \overset{u}{\gamma}_r\, c^r .$$

It is found that the image point $\hat{p}_x$ is then given by (II.2) with the same u^α.

It was shown in [9] that the most general connection form on $\mathcal{B}(M)$ is given by

$$\omega = \begin{pmatrix} \omega^{A}_{\cdot B} & \omega^{A}_{\cdot 0} & \omega^{A} & 0 \\ 0 & 0 & \omega^{0} & 0 \\ 0 & 0 & 0 & 0 \\ \omega_{B0} & 0 & \omega_{*} & 0 \end{pmatrix} \tag{II.4}$$

where

$$\omega^{a}_{\cdot b} = \theta^{a}_{\alpha}\left(de^{\alpha}_{b} + \Gamma^{\alpha}_{\beta\gamma}\, e^{\gamma}_{b}\, dx^{\beta}\right) , \tag{II.5}$$

$$\omega^{a} = \theta^{a}_{\alpha}\left[d\xi^{\alpha} + \left(\Gamma^{\alpha}_{\beta\gamma}\,\xi^{\gamma} + \varphi^{\alpha}_{\cdot\beta}\right)dx^{\beta}\right] , \tag{II.6}$$

$$\omega_{*} = d\chi + \xi^{r}\omega_{r0} - \left(\overset{u}{\gamma}_{\beta}\,\varphi^{\beta}_{\cdot\alpha} + A_{\alpha}\right)dx^{\alpha} \tag{II.7}$$

with $\overset{u}{\gamma}_{\alpha} := \overset{u}{\gamma}_{r}\,\theta^{r}_{\alpha}$, $\xi^{a} := \theta^{a}_{\alpha}\,\xi^{\alpha}$, $\chi := \kappa - \overset{u}{\gamma}_{r}\,\xi^{r}$.
Here the $\Gamma^{\alpha}_{\beta\gamma}$, A_{α} and $\varphi^{\alpha}_{\cdot\beta}$ depend on x^{α} only and A_{α} and $\varphi^{\alpha}_{\cdot\beta}$ transform as tensor fields on M .

We are only interested in <u>*Newtonian connections*</u> *which according* to [9] are characterized as those for which $\varphi^{\alpha}_{\cdot\beta} = \delta^{\alpha}_{\beta}$ (3) and the

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(3) This condition expresses the fact that the pull back of the translation component of the connection ω is the soldering form , i.e. $\iota^{*}\omega^{a} = \theta^{a}$.

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pull back of the curvature form Ω by $\tilde{\iota}$ takes its value in the Lie algebra of the homogeneous Galilei group. This means that, locally,

$$\left.\begin{array}{c} \nabla_\alpha \gamma^{\beta\gamma} = 0 \ , \ \nabla_\alpha \psi_\beta = 0 \ , \ \Gamma^{\alpha}_{[\beta\gamma]} = 0 \\ \partial_{[\alpha} A_{\beta]} + \overset{u}{\gamma}_{\lambda[\alpha} \nabla_{\beta]} u^\lambda = 0 . \end{array}\right\} \quad \text{(II.8)}$$

The last condition (II.8) is equivalent to the intrinsic form [16],[5]

$$R^{\alpha\ \cdot\ \gamma\ \cdot}_{\ \cdot\ \beta\ \cdot\ \delta} = R^{\gamma\ \cdot\ \alpha\ \cdot}_{\ \cdot\ \delta\ \cdot\ \beta} . \qquad \text{(II.9)}$$

See also [23] for a more intrinsic characterization of the Newtonian connection on $B(M)$ where no reference vector field u^α is introduced. We will need this vector field in the following, however.

Next we discuss the notion of <u>covariance of a Galilei-relativistic field theory</u>. In a relativistic theory covariance under space-time diffeomorphisms (or minimal gravitational coupling) is normally achieved by replacing partial by covariant derivatives. The range of physical fields need not be considered as a (nontrivially) associated bundle of the Lorentz or Poincaré frame bundle. Neither will the Bargmann group act nontrivially on the range of physical fields.

Continuing with the preceding diagramme, we have

$$\begin{tikzcd}
\tilde{H}(M) \arrow[r] \arrow[d, "U(1)"'] & \widetilde{T_1M} \arrow[r] \arrow[d] & \tilde{M} \arrow[d] \\
H(M) \arrow[r, "SO(3)"] & T_1M \arrow[r, "\mathbb{R}^3"] & M
\end{tikzcd}$$

(x, e, χ) (x, e_0, χ) (x, x^0)

(x, e) (x, e_0) (x)

where $T_1M = H(M)/SO(3)$ is the unit tangent bundle of (M, γ, ψ) and $\tilde{M} := \tilde{H}(M)/H$ is a $U(1)$-principal bundle over M. If ω defines a Newtonian connection, $\tilde{\omega}_* := \tilde{\imath}^* \omega_*$ passes to the quotient $\widetilde{T_1M} := \tilde{H}(M)/SO(3)$ (use the structural equation $d\tilde{\omega}_* = \tilde{\Omega}_* + \theta^A \wedge \omega^B{}_0\, \delta_{AB}$ and remember that $\tilde{\Omega}_* = 0$).
Given a mapping $\hat{u} : H(M) \to \mathbb{R}^3 \subset \mathbb{R}^4$ such that $\hat{u}(e_x a) = a^{-1} \hat{u}(e_x)$ for all $a \in H$, i.e. a section u of $(T_1M \to M)$ such that

$$\hat{u}(e_x) = u_x \lrcorner\, \theta_x$$

we can define on $H(M)$ the following 1-form associated with u

$$\overset{u}{\gamma} := -\hat{u}_A\, \theta^A + 1/2\, \hat{u}_A\, \hat{u}^A\, \theta^0 .$$

Clearly $\overset{u}{\gamma}$ passes to the quotient T_1M and can be lifted to $\widetilde{T_1M}$.
It can be checked (using (II.3) and (II.7) that

$$\alpha := \tilde{\omega}_* + \overset{u}{\gamma} \qquad \text{(II.10)}$$

is a connection form on $\widetilde{M}$ [4]. Thus, any Newtonian connection ω defines on $\widetilde{M}$ a whole family of connections which are in 1-1 correspondence with the sections u of $T_1 M$ (the "observer" fields).
A field theory should therefore be covariant not only under automorphims of $\widetilde{M}$ but also under changes of u.

If $$u^\alpha \mapsto \hat{u}^\alpha = u^\alpha + \gamma^{\alpha\lambda} w_\lambda \quad , \quad \chi \mapsto \hat{\chi} = \chi + f \tag{II.11}$$

then the Newtonian connection does not change, provided

$$A_\alpha \mapsto \hat{A}_\alpha = A_\alpha + \partial_\alpha f + W_\alpha - (u^\lambda w_\lambda + 1/2\, \gamma^{\lambda\mu} w_\lambda w_\mu)\, \psi_\alpha \, . \tag{II.12}$$

Note that the 1-form W is only defined modulo ψ.
We can consider (II.11) as the vertical automorphism of $\widetilde{T_1 M}$. Together with the diffeomorphisms of M, they constitute the full automorphisms group $\mathcal{A}$ of the theory. This group $\mathcal{A}$ plays about the same fundamental role for Newtonian covariance as $\mathrm{Diff}(M)$ does for general relativistic covariance (we restrict, for simplicity, to time and space orientation preserving mappings).

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(4) $\alpha = d\chi - A_\beta \, dx^\beta$, locally.

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b - The gauge group of the Newtonian gravitation theory

The group $\hat{\mathcal{A}} := \{(a,W,f)/a \in \mathrm{Diff}(M),\, W \in \mathfrak{X}^*(M),\, f \in C^\infty(M,\mathbb{R})\}$ acts as the set $\{(\gamma,\psi,u,A)\}$ by

$$(a,W,f) : \begin{pmatrix} \gamma \\ \psi \\ u \\ A \end{pmatrix} \mapsto \begin{pmatrix} a_*\gamma \\ (a^{-1})^*\psi \\ a_*[u + \gamma(W)] \\ (a^{-1})^*\left[A + df + W - \left(W(u) + \tfrac{1}{2}\gamma(W,W)\right)\psi\right] \end{pmatrix} \tag{II.13}$$

from which one easily derives the group law

$$(a,W,f)\cdot(\bar{a},\bar{W},\bar{f}) = \left(a\bar{a},\, \bar{a}^*W + \bar{W},\, \bar{a}^*f + \bar{f}\right). \tag{II.14}$$

This shows that $\hat{\mathcal{A}}$ has the structure of a semi-direct product $\mathrm{Diff}(M) \underline{\times} \left(\mathfrak{X}^*(M) \times C^\infty(M,\mathbb{R})\right)$. To obtain the automorphism group $\mathcal{A}$ itself, $\mathfrak{X}^*(M) \times C^\infty(M,\mathbb{R})$ must first be factored with respect to the relation : $W \sim \bar{W}$ iff $\bar{W} = W + \sigma\psi$ for some function σ on M .

The Bargmann group is still hidden in this new gauge group, for we have the

Proposition 1 :

The isotropy subgroup (of $\mathcal{A}$) of the standard flat structure

$$\left(\gamma=\delta^{AB}\partial_A\otimes\partial_B\,,\ \psi=dt\,,\ u=\partial_t\,,\ A=0\right)$$

is the Bargmann group B.

Proof :

Diffeomorphisms leaving (γ,ψ) and the flat connection invariant must be Galilei transformations, i.e.

$$a:\begin{pmatrix}\underline{x}\\ t\end{pmatrix}\longmapsto\begin{pmatrix}\underline{R}\cdot\underline{x}+\underline{b}\,t+\underline{c}\\ t+c^{0}\end{pmatrix} \tag{II.15}$$

with $\underline{R}\in SO(3)$; $\underline{b},\underline{c}\in\mathbb{R}^3$; $c^{0}\in\mathbb{R}$. Now $u=\partial_t$ maps into $\partial_t+b^A\partial_A+R^{A}_{\cdot B}W^B\partial_A$ and therefore is invariant provided $W_A=-b_BR^{B}_{\cdot A}$. On the other hand, $A=0$ maps into

$$(\partial_t f-1/2\,W_AW^A)dt+(\partial_A f+W_A)\,dx^A$$

This vanishes if

$$f=\underline{b}^T\cdot\underline{R}\cdot\underline{x}+1/2\,\underline{b}^2t+\lambda\quad(\lambda=\text{Const.}) \tag{II.16}$$

Writing (II.15) and (II.16) together in matrix form, we recover (II.1) while W_0 remains undetermined as it should. $\square$

To achieve minimal coupling to Newtonian gravity the Lagrangian $\mathcal{L}_m$ for matter field must be written in a form so that the variation of the action $S = \int_M \mathcal{L}_m d^4x$ is invariant under the group $\mathcal{G}$. Then, not only the (material) field equations will be invariant, but there is also a systematic way to derive the associated matter current vector J^α and "Hilbert" stress-tensor $T^\alpha_{\cdot\beta}$. Let $\nu\, d^4x$ denote the space-time volume element of the Galilei structure (γ, ψ).

Proposition 2 :

If the functional

$$\delta S = \int_M \left(1/2\, P_{\alpha\beta}\, \delta\gamma^{\alpha\beta} + Q^\alpha \delta\psi_\alpha + J^\alpha \delta A_\alpha + K_\alpha \delta u^\alpha\right) \nu\, d^4x \tag{II.17}$$

of the variations $\delta\gamma^{\alpha\beta}$, $\delta\psi_\alpha$, δA_α, δu^α is invariant under the group $\mathcal{G}$, then the matter current J^α and the stress-energy tensor

$$T^\alpha_{\cdot\beta} := \gamma^{\alpha\lambda} P_{\lambda\beta} - Q^\alpha \psi_\beta + u^\alpha K_\beta + J^\alpha \left(V_\beta - V^2/2\, \psi_\beta\right) \tag{II.18}$$

where $\rho := J^\lambda \psi_\lambda$, $J^\alpha =: \rho V^\alpha$, $V_\alpha := \overset{u}{\gamma}_{\alpha\lambda} V^\lambda$, $V^2 := V^\lambda V_\lambda$ are invariant under $\mathcal{G}$ and satisfy the balance equations

$$\nabla_\alpha J^\alpha = 0 \quad \text{(mass conservation)} \tag{II.19}$$

and $$\nabla_\lambda T^\lambda_{\cdot\alpha} = \rho\, \overset{\vee}{\gamma}_{\alpha\lambda} V^\mu \nabla_\mu V^\lambda \quad \text{(energy-momentum conservation)}. \tag{II.20}$$

Proof :

All quantities used being tensorial it is clear that $\mathcal{P}_{(\alpha\beta)}, Q^{\alpha}, J^{\alpha}, K_{\alpha}$ and hence $T^{\alpha}_{\cdot\beta}$ are invariant under diffeomorphisms. To check invariance under vertical automorphisms in $\mathcal{A}$, note that (II.11) and (II.12) imply $\delta\hat{u}^{\alpha} = \delta u^{\alpha} + W_{\lambda}\,\delta\gamma^{\alpha\lambda}$ and

$$\delta\hat{A}_{\alpha} = \delta A_{\alpha} - (u^{\lambda}W_{\lambda} + 1/2\,\gamma^{\lambda\mu}W_{\lambda}W_{\mu})\,\delta\psi_{\alpha} - \psi_{\alpha}W_{\lambda}\,\delta u^{\lambda}$$
$$- 1/2\,\psi_{\alpha}W_{\lambda}W_{\mu}\,\delta\gamma^{\lambda\mu}\,.$$

Substituting this into the transformed equation (II.17), we find that

$$\begin{aligned}
\mathcal{P}_{\alpha\beta} &= \hat{\mathcal{P}}_{\alpha\beta} - \rho W_{\alpha}W_{\beta} + 2\,\hat{K}_{(\alpha}W_{\beta)}\,,\\
Q^{\alpha} &= \hat{Q}^{\alpha} - \hat{J}^{\alpha}(u^{\lambda}W_{\lambda} + 1/2\,\gamma^{\lambda\mu}W_{\lambda}W_{\mu})\,,\\
K_{\alpha} &= \hat{K}_{\alpha} - \hat{J}^{\lambda}\psi_{\lambda}W_{\alpha}\,,\quad J^{\alpha} = \hat{J}^{\alpha}.
\end{aligned} \tag{II.21}$$

This shows the invariance of J^{α} and hence of ρ and V^{α}. It was shown in [6] that

$$K_{\alpha} = -\overset{u}{\gamma}_{\alpha\lambda}J^{\lambda} + \kappa\,\psi_{\alpha}\,. \tag{II.22}$$

Substituting (II.21) and (II.22) into (II.28), then gives $T^{\alpha}_{\cdot\beta} = \hat{T}^{\alpha}_{\cdot\beta}$. The balance equations (II.19) and (II.20) were already derived in [6]. □

Note that this is the closest we can come to the Hilbert stress-energy tensor of relativistic theories since, for example, the quantities $\mathcal{P}_{\alpha\beta}$ and Q^{α} cannot have an invariant meaning in view of the degeneracy of $\gamma^{\alpha\beta}$ and the constraints between $\gamma^{\alpha\beta}$ and ψ_{α}. A specific matter Lagrangian satisfying the conditions of proposition 2 and describing a perfect fluid was given in [18], another will be introduced in the next section. In neither of these cases, at least, the canonical stress-energy tensor, or one formed by variation with respect to tetrad fields is acceptable, i.e. invariant under $\mathcal{A}$.

The principle of general covariance used in proposition 2 has also been applied to Yang-Mills type gauge theories [11],[10] and can be used to obtain equations of motion of test particles [26], [6], [11], [7], [8].

III - SCHRÖDINGER EQUATION on a CURVED NEWTONIAN SPACE-TIME

Consider now a vector bundle E with standard fiber $\mathbb{C}$ associated with the $U(1)$-principal bundle $\widetilde{M}$. This means that under the gauge transformations (II.13) a section Φ of E transforms like

$$(a, W, f): \quad \Phi \longmapsto (a^{-1})^{*}\left(e^{imf/\hbar}\, \Phi\right) \tag{III.1}$$

(we have assumed here that every tensor has a unique physical dimension, namely $[\gamma^{\alpha\beta}] = L^{-2}$, $[\psi_\alpha] = T$, $[u^\alpha] = T^{-1}$, $[A_\alpha] = L^2 T^{-1}$ (5) and that coordinates have no dimension. Then f must have the same dimension as A which justifies the factor $m/\hbar$ in (III.1)).

Similarly, as in electromagnetic theory, the covariant derivative induced by the connection form α of eq(II.10) on sections of E is

$$D_\alpha \Phi = \partial_\alpha \Phi - im/\hbar\, A_\alpha \Phi\,. \tag{III.2}$$

To make now the free one-particle Schrödinger equation,

$$\hbar^2/(2m)\, \Delta \Phi + i\hbar\, \partial_t \Phi = 0 \tag{III.3}$$

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(5) $[\Phi] = L^{-3/2}$

covariant on a curved Newtonain space-time it is, as always, best to apply the minimal interaction principle to a Lagrangian for (III.3), such as

$$\mathcal{L} = \hbar^2/(2m)\left|\nabla\Phi\right|^2 + i\hbar/2\left(\Phi\partial_t\bar{\Phi} - \bar{\Phi}\partial_t\Phi\right). \qquad \text{(III.4)}$$

Clearly, the time derivative can only be expressed covariantly by means of a timelike vector field like u^α, for the Laplacian the contravariant 3-metric $\gamma^{\alpha\beta}$ must be used and the partial derivatives ∂_α replaced by D_α. These considerations already lead to

$$\mathcal{L}_{Sch.} = \left\{\hbar^2/(2m)\gamma^{\alpha\beta} D_\alpha\Phi\overline{D_\beta\Phi} + i\hbar/2\, u^\alpha\left(\Phi\overline{D_\alpha\Phi} - \bar{\Phi}D_\alpha\Phi\right)\right\}\nu. \qquad \text{(III.5)}$$

Since $\gamma^{\alpha\beta}$, ψ_α, A_α, u^α and Φ are all tensor fields on space-time, the Lagrangian is manifestly covariant with respect to diffeomorphisms of M. It is less obvious, but can be verified by straightforward calculations that $\mathcal{L}_{Sch.}$ is also invariant under simultaneous transformations of A, u and Φ by vertical gauge transformations. It follows that the whole theory will be independent of the particular choice of the Galilei frame represented by u and that the gravitational interaction depends only on the Newtonian structure (γ, ψ, Γ). This is so in spite of the fact that the

Euler-Lagrange equations to the Lagrangian (III.5) obtained by varying independently with respect to Φ and $\overline{\Phi}$ contain u^α explicitely. They are

$$E_{Sch}[\Phi] := \hbar^2/(2m)\, D^\alpha D_\alpha \Phi + i\hbar\, u^\alpha D_\alpha \Phi + i\hbar/2\, \nabla_\alpha u^\alpha \Phi = 0 \qquad \text{(III.6)}$$

and its complex conjugate.

One recovers again the classical result that, in the standard flat case, this equation is invariant under the action of the Bargmann group on space-time and the wave function as given in (II.15), (II.16) and (III.1). If the covariant derivatives are worked out (III.6) becomes

$$\begin{aligned} &\hbar^2/(2m)\, \nabla^\alpha \partial_\alpha \Phi + i\hbar\,(u^\alpha - \gamma^{\alpha\beta} A_\beta)\, \partial_\alpha \Phi \\ &+ m\,(u^\alpha A_\alpha - 1/2\, \gamma^{\alpha\beta} A_\alpha A_\beta)\, \Phi \\ &+ i\hbar/2\, \nabla_\alpha (u^\alpha - \gamma^{\alpha\beta} A_\beta)\, \Phi = 0 . \end{aligned} \qquad \text{(III.7)}$$

It can be seen to be the same as the equation Kuchar [15] obtained by some frame changes on the standard Newtonian space-time.

Equation (III.7) can be written in a simpler form once a preferred "observer" u has been singled out. In fact, any Newtonian connection determines a (locally) unique unit vector-field u such that u is geodesic and curlfree [16]. This particular choice of Newtonian gauge amounts to putting $A = 0$ in (II.7). Then (III.7) reduces to

$$\hbar^2/(2m)\,\gamma^{\alpha\beta}\overset{u}{\nabla}_\alpha\partial_\beta\Phi + i\hbar\left(u^\alpha\partial_\alpha\Phi + 1/2\,\overset{u}{\nabla}_\alpha u^\alpha\Phi\right) = 0\,.$$

The wave function is now replaced by the <u>half-density</u>

$$\Phi^\# := \Phi\,|\nu|^{1/2}$$

where $|\nu|$ denotes the Galileian density of (M, γ, ψ).
Since

$$\mathcal{L}_u \nu = \nabla_\alpha u^\alpha\,\nu$$

the Schrödinger equation becomes

$$\hbar^2/(2m)\,\gamma^{\alpha\beta}\overset{u}{\nabla}_\alpha\overset{u}{\nabla}_\beta\Phi^\# + i\hbar\,\mathcal{L}_u\Phi^\# = 0\,.$$

Proposition 2 can now be applied to the Lagrangian (III.5). Let first

$$\tau_{\alpha\beta} := \hbar^2/m\,D_{(\alpha}\Phi\,\overline{D_{\beta)}\Phi}\;,\;\varphi_\alpha := i\hbar/2\left(\Phi\overline{D_\alpha\Phi} - \overline{\Phi}D_\alpha\Phi\right) \qquad \text{(III.8)}$$

then it follows by straightforward calculation that

$$J^\alpha = m\,\Phi\overline{\Phi}\,u^\alpha + \gamma^{\alpha\lambda}\varphi_\lambda\,. \qquad \text{(III.9)}$$

In particular, the mass density is $\rho = m\,\Phi\,\bar{\Phi}$ and also the spacelike part of J reduces to $m \times$ ("probability" current) in the flat case. The stress-energy tensor becomes

$$T^{\alpha}_{\cdot\beta} = -\tau^{\alpha}_{\cdot\beta} + \left(1/2\,\tau^{\lambda}_{\cdot\lambda} + u^{\lambda}\varphi_{\lambda}\right)\delta^{\alpha}_{\beta} - u^{\alpha}\varphi_{\beta} + J^{\alpha}\left(v_{\beta} - v^{2}/2\;\psi_{\beta}\right) \tag{III.10}$$

which may be compared to expressions found in textbooks like [4]. For example, the timelike eigenvalue of T, i.e. the energy density, becomes (in the generic case)

$$\rho\,\epsilon = \hbar^{2}/(8m\rho)\,\gamma^{\alpha\beta}\partial_{\alpha}\rho\,\partial_{\beta}\rho = \hbar^{2}/(2m)\left(\nabla|\Phi|\right)^{2}. \tag{III.11}$$

This theory can easily be extended to include the electromagnetic in addition to the gravitational field. There would be an additional $U(1)$-gauge potential $\overset{em}{A}_{\alpha}\,dx^{\alpha}$ which will, however, transform not like A, but in the usual simple way. It is even possible that this (Galileian!) theory can be generalized to multiparticle systems, perhaps in the framework of the Newtonian configuration space-time formalism of Trümper [29].

IV - NEWTONIAN LIMIT of the KLEIN-GORDON THEORY on CURVED SPACE-TIME

While at the beginnings of quantum theory the status of the Klein-Gordon equation was somewhat controversial it has now long been recognized that the latter is as valid a classical model for particles with spin zero as the Dirac equation is for particles with spin 1/2. In particular, it provides a way to minimally couple this field to the gravitational field if the equation is made generally covariant under diffeomorphisms. This procedure is in general less arbitrary when performed on the Lagrangian (although for scalar fields there is no difference).

If we consider the Schrödinger and the Klein-Gordon field both on curved space-time we should expect the latter to result in the former when $c \longrightarrow \infty$. However, since the Klein-Gordon equation describes a massive particle it is not unexpected that a term mc^2 must be split off somewhere. We therefore try the method of a high frequency phase factor, described in [24].

We start with the Lagrangian

$$\mathcal{L}_{K.G.} = 1/2 \left(\hbar^2/m \, \nabla^\lambda \psi \, \overline{\nabla_\lambda \psi} + mc^2 \psi \overline{\psi} \right) \nu \qquad \text{(IV.1)}$$

where $\nu = c^{-1} \left| \det (g_{\alpha\beta}) \right|^{1/2}$, which, when varied with respect to $\overline{\Psi}$, leads to the Klein-Gordon equation,

$$E_{K.G}[\Psi] := \nabla^\lambda \nabla_\lambda \Psi - m^2 c^2/\hbar^2 \Psi = 0 \tag{IV.2}$$

and, when varied with respect to the space-time metric g, to the Hilbert stress-energy tensor

$$T^{\alpha\beta}_{K.G.} = \hbar^2/(2m)\left[2\, \nabla^{(\alpha} \Psi \overline{\nabla^{\beta)} \Psi} - \nabla^\lambda \Psi \overline{\nabla_\lambda \Psi}\, g^{\alpha\beta}\right] - mc^2/2\, \Psi \overline{\Psi}\, g^{\alpha\beta}. \tag{IV.3}$$

The quantity

$$j^\alpha := i\hbar/2\, g^{\alpha\beta}\left(\Psi \overline{\nabla_\beta \Psi} - \overline{\Psi} \nabla_\beta \Psi\right) \tag{IV.4}$$

can be considered as the associated mass current [2] although it cannot be geometrically or otherwise derived from the theory. (The electromagnetic current $j^\alpha_{e.m.}$ on the other hand is the dual quantity to the electromagnetic vector potential and well justified. The above j^α is simply chosen parallel to $j^\alpha_{e.m.}$.)

We now assume that there is a family of Lorentz metrics and complex scalar fields Ψ parametrized by c such that for $c^{-2} \longrightarrow 0$, the following hold [19]

$$g^{\alpha\beta} = \gamma^{\alpha\beta} + c^{-2}\overset{1}{k}{}^{\alpha\beta} + c^{-4}\overset{2}{k}{}^{\alpha\beta} + O(c^{-6}) \qquad \text{(IV.5)}$$

$$g_{\alpha\beta} = -c^2\psi_\alpha\psi_\beta + \overset{1}{\ell}_{\alpha\beta} + c^{-2}\overset{2}{\ell}_{\alpha\beta} + O(c^{-4}) \qquad \text{(IV.6)}$$

$$\left\{{\alpha \atop \beta\gamma}\right\} = \Gamma^\alpha_{\beta\gamma} + c^{-2}\overset{1}{C}{}^\alpha_{\beta\gamma} + O(c^{-4}) \qquad \text{(IV.7)}$$

$$\Psi = e^{-imc^2/\hbar N}\left(1 + c^{-2}\overset{1}{\omega} + O(c^{-4})\right)\Phi \qquad \text{(IV.8)}$$

where $(\gamma^{\alpha\beta}, \psi_\alpha, \Gamma^\alpha_{\beta\gamma})$ is a Newtonian structure, N is a world-time (lapse function) such that

$$N = t + c^{-2}\overset{1}{N} + c^{-4}\overset{2}{N} + O(c^{-6}) \qquad \text{(IV.9)}$$

with $dt = \psi$ and where all quantities in these expansions are considered tensors on the Newtonian space-time (the terms $\overset{1}{\omega}$ and $\overset{k}{N}$ can be chosen real since the phase is contained in N).

We assume that $\overset{1}{k}{}^{\alpha\beta}$ is timelike and future oriented with respect to $g^{\alpha\beta}$. It then follows that

$$u^\alpha := -\overset{1}{k}{}^{\alpha\lambda}\psi_\lambda \qquad \text{(IV.10)}$$

satisfies $u^\alpha\psi_\alpha = 1$. From $g^{\alpha\lambda}g_{\lambda\beta} = \delta^\alpha_\beta$ one then derives that

$$\overset{1}{k}{}^{\alpha\beta} = -u^\alpha u^\beta + K^{\alpha\beta} \quad \text{with} \quad K^{\alpha\lambda}\psi_\lambda = 0 . \qquad \text{(IV.11)}$$

If we write

$$\overset{2}{k}{}^{\alpha\beta} = 2Vu^{\alpha}u^{\beta} + 2u^{(\alpha}L^{\beta)} + M^{\alpha\beta} \qquad \text{(IV.12)}$$

for space-like L^{α} and $M^{\alpha\beta}$ then

$$\overset{1}{\ell}_{\alpha\beta} = -2V\psi_{\alpha}\psi_{\beta} + \overset{u}{\gamma}_{\alpha\beta} \qquad \text{(IV.13)}$$

and

$$\overset{2}{\ell}_{\alpha\beta} = -2W\psi_{\alpha}\psi_{\beta} + 2\psi_{(\alpha}L_{\beta)} - K_{\alpha\beta} \qquad \text{(IV.14)}$$

for some scalar W where indices of spacelike tensors are lowered with $\overset{u}{\gamma}_{\alpha\beta}$.

To derive the expansion of the Christoffel symbols, it now takes a straighforward but long calculation which gives to lowest order [19]

$$\Gamma^{\alpha}_{\beta\gamma} = \overset{u}{\Gamma}{}^{\alpha}_{\beta\gamma} + \gamma^{\alpha\lambda}\partial_{\lambda}V\psi_{\beta}\psi_{\gamma} \qquad \text{(IV.15)}$$

where $\overset{u}{\Gamma}{}^{\alpha}_{\beta\gamma}$ is the Newtonian connection such that u^{α} is goedesic and satisfies $\gamma^{\lambda[\alpha}\nabla_{\lambda}u^{\beta]}=0$. The limiting Newtonian connection is therefore described, for example, by u^{α} and $A_{\alpha} = -V\psi_{\alpha}$. This particular unit vector field u, which describes "into which direction the light cones open up", is somewhat special. It follows from (IV.15) that its derivative has the (u,ψ)-orthogonal decomposition

$$\nabla_{\beta}u^{\alpha} = D^{\alpha}_{\cdot\beta} + \dot{u}^{\alpha}\psi_{\beta} \qquad \text{(IV.16)}$$

where $D_{\alpha\beta}$ is u-orthogonal and symmetric and $\dot{u}^{\alpha} = \gamma^{\alpha\lambda}\partial_{\lambda} V$.

For the first order terms of the connection coefficients one then finds, also in (u,ψ)-orthogonally decomposed form,

$$\overset{1}{C}{}^{\alpha}_{\beta\gamma} = u^{\alpha}(u^{\lambda}\partial_{\lambda} V \psi_{\beta}\psi_{\gamma} + 2\psi_{(\beta}\dot{u}_{\gamma)} + D_{\beta\gamma}) + (D^{\alpha}_{\cdot\lambda} L^{\lambda} + K^{\alpha\lambda}\dot{u}_{\lambda} + u^{\lambda}\nabla_{\lambda} L^{\alpha})\psi_{\beta}\psi_{\gamma} + \psi_{(\beta}(F^{\alpha}_{\cdot\gamma)} - \overset{u}{\gamma}_{\gamma)\lambda} u^{\mu}\nabla_{\mu} K^{\alpha\lambda}) + 1/2\, K_{\beta\gamma},^{\alpha} - K^{\alpha}_{\cdot(\beta,\gamma)} \qquad \text{(IV.17)}$$

where $K^{\alpha\beta},_{\gamma} := \gamma^{\lambda\mu}\overset{u}{\gamma}_{\mu\gamma}\nabla_{\lambda} K^{\alpha\beta}$ and $F^{\alpha\beta} := 2K^{\lambda[\alpha} D^{\beta]}_{\lambda}$ are spacelike and u-orthogonal. In particular we have

$$\gamma^{\lambda\mu}\,\overset{1}{C}{}^{\nu}_{\lambda\mu}\psi_{\nu} = D^{\lambda}_{\lambda} = \nabla_{\lambda} u^{\lambda} \qquad \text{(IV.18)}$$

Using these results about the expansion of metric and connection we can now calculate the limit of the Klein-Gordon Lagrangian and equation. Since the Schrödinger theory is invariant under gauge transformations, we may without loss of generality describe the Newtonian connection by the u^{α} defined in (IV.10) and define the covariant derivative D_{α} with respect to the correponding $A_{\alpha} = -V\psi_{\alpha}$. The result is

$$E_{K.G.}[\Psi] = e^{-imc^2/\hbar N}\{2m/\hbar^2\, E_{Sch.}[\Phi] - 2im/\hbar[\gamma^{\lambda\mu}\partial_{\lambda}\overset{1}{N} D_{\mu}\Phi + 1/2(\nabla^{\lambda}\nabla_{\lambda}\overset{1}{N} - im/\hbar\,\gamma^{\lambda\mu}\partial_{\lambda}\overset{1}{N}\partial_{\mu}\overset{1}{N})\Phi + O(c^{-2})]\}. \qquad \text{(IV.19)}$$

It follows easily that

<u>the Klein-Gordon equation for ψ goes over into the Schrödinger equation for Φ iff $\overset{1}{N}$ is constant.</u>

For a given space-time metric $g_{\alpha\beta}$ and Φ satisfying the Schrödinger equation in the Newtonian limit one checks that the terms of order c^{-2} of the Klein-Gordon equation result in two parabolic type second order equations for the two real scalars $\overset{1}{\omega}$ and $\overset{2}{N}$.

If $\overset{1}{N}$ is constant, it is found that also

$$\mathcal{L}_{K.G.}[\psi] = \mathcal{L}_{Sch.}[\Phi] + O(c^{-2}) \tag{IV.20}$$

The Newtonian limit of the relativistic stress-energy tensor is much more difficult to obtain. We first discuss the situation for completely <u>general matter</u>. If one accepts that mass and energy are not the same thing in Newtonian physics and that only the mass density (and not the pressure or internal energy density etc.) act as sources of the gravitational field one can show (see, for example, [19]) that Einstein's field equations only go over into the Newtonian gravitational equations provided

$$\overset{c}{T}{}^{\alpha}{}_{\cdot\beta} = -c^{2}\,\tilde{J}^{\alpha}\psi_{\beta} + \tilde{T}^{\alpha}{}_{\cdot\beta} + O(c^{-2}) \tag{IV.21}$$

where $\rho' = \tilde{J}^{\lambda}\psi_{\lambda}$ is the Newtonian matter density and $\tilde{J}^{\alpha}$ and $\tilde{T}^{\alpha}_{\cdot\beta}$ are regular in the limit. It is tempting to think that $\tilde{J}^{\alpha}$ and $\tilde{T}^{\alpha}_{\cdot\beta}$ are the Newtonian matter current J^{α} and stress-energy tensor $T^{\alpha}_{\cdot\beta}$, respectively. Then, however, they would have to satisfy the conservation equations (II.19) and (II.20), as well as

$$T^{[\alpha}_{\cdot\lambda}\,\gamma^{\beta]\lambda} = 0 \tag{IV.22}$$

(symmetry of the stress tensor). This is not the case. Instead one finds from (IV.21) that

$$\tilde{T}^{[\alpha}_{\cdot\lambda}\,\gamma^{\beta]\lambda} = -\tilde{J}^{[\alpha}u^{\beta]} \tag{IV.23}$$

and, by substituting (IV.7) and (IV.21) into the relativistic equation $\overset{c}{\nabla}_{\lambda}\overset{c}{T}{}^{\lambda}_{\cdot\alpha} = 0$, that

$$\nabla_{\lambda}\tilde{J}^{\lambda} = 0 \tag{IV.24}$$

and $$\nabla_{\lambda}\tilde{T}^{\lambda}_{\cdot\alpha} = \overset{1}{C}{}^{\lambda}_{\lambda\mu}\tilde{J}^{\mu}\psi_{\alpha} - \psi_{\lambda}\overset{1}{C}{}^{\lambda}_{\alpha\mu}\tilde{J}^{\mu}. \tag{IV.25}$$

In view of (IV.24), we can indeed interpret $\tilde{J}^{\alpha}$ as the Newtonian matter current J^{α} while (IV.23) suggests that we try

$$T^{\alpha}_{\cdot\beta} = -\tilde{T}^{\alpha}_{\cdot\beta} + J^{\alpha}\overset{u}{\gamma}_{\beta\lambda}V^{\lambda} + E\,J^{\alpha}\psi_{\beta}\,. \tag{IV.26}$$

This $T^{\alpha}_{\cdot\beta}$ will satisfy (IV.22) for any E. Substituting into (IV.25) and using (IV.17) gives

$$\nabla_\lambda T^{\lambda}_{\cdot\alpha} = \rho \overset{v}{\gamma}_{\alpha\lambda} V^\mu \nabla_\mu V^\lambda + \rho \mathcal{E} \psi_\alpha \qquad \text{(IV.27)}$$

where

$$\mathcal{E} = V^\mu \partial_\mu \left(E + V^2/2 + K^\lambda_\lambda \right). \qquad \text{(IV.28)}$$

By choosing $E = -V^2/2 - K^\lambda_\lambda + f$ with $J^\lambda \partial_\lambda f = 0$ we therefore obtain a Newtonian stress-energy tensor that satisfies all conditions. In summary, we have

Proposition 3 :

If a one-parameter family of relativistic space-times $(\overset{c}{g}_{\alpha\beta}, \overset{c}{T}^{\alpha}_{\cdot\beta})$ tends to a Newtonian limit $(\gamma^{\alpha\beta}, \psi_\alpha, \Gamma^\alpha_{\beta\gamma}, J^\alpha = \rho V^\alpha, T^\alpha_{\cdot\beta})$ for $c \longrightarrow \infty$ such that

$$\overset{c}{g}^{\alpha\beta} = \gamma^{\alpha\beta} + c^{-2} \overset{1}{k}^{\alpha\beta} + O(c^{-4})$$

with $u^\alpha = -\overset{1}{k}^{\alpha\lambda} \psi_\lambda$ and

$$\overset{c}{T}^{\alpha}_{\cdot\beta} = -c^2 J^\alpha \psi_\beta + \tilde{T}^{\alpha}_{\cdot\beta} + O(c^{-2})$$

then, the Newtonian matter current is given by J^α and the stress-energy tensor by

$$T^{\alpha}_{\cdot\beta} = -\tilde{T}^{\alpha}_{\cdot\beta} + J^\alpha \overset{u}{\gamma}_{\beta\lambda} V^\lambda - \overset{u}{\gamma}_{\lambda\mu} (\overset{1}{k}^{\lambda\mu} + 1/2\, V^\lambda V^\mu) J^\alpha \psi_\beta .$$

The Newtonian proper energy density is only defined up to a constant on each streamline of J^α.

When developing $T^{\alpha}_{KG.\beta}$, using (IV.5) to (IV.9) (with $\partial_{\alpha}\overset{1}{N}=0$) we find $J^{\alpha}= m\,\Phi\bar{\Phi}\,u^{\alpha}+\gamma^{\alpha\lambda}\varphi_{\lambda}$ which agrees with the Schrödinger current (III.9) (and also with the limit of j^{α} defined in (IV.4)).

On the other hand, we find

$$\tilde{T}^{\alpha}_{\cdot\beta} = \tau^{\alpha}_{\cdot\beta} - (1/2\,\tau^{\lambda}_{\cdot\lambda} + u^{\lambda}\varphi_{\lambda})\delta^{\alpha}_{\beta} + u^{\alpha}\varphi_{\beta}$$
$$+\left[\rho(L^{\alpha}+\gamma^{\alpha\lambda}\partial_{\lambda}\overset{2}{N}) - (V\gamma^{\alpha\lambda}+\overset{1}{k}{}^{\alpha\lambda})\varphi_{\lambda}\right.$$
$$\left.-\,2\overset{1}{\omega}J^{\alpha}\right]\psi_{\beta} \qquad \text{(IV.29)}$$

which leads, according to proposition 3, to

$$T^{\alpha}_{\cdot\beta} \doteq T^{\alpha}_{Sch.\beta} - Q^{\alpha}\psi_{\beta} \qquad \text{(IV.30)}$$

where

$$Q^{\alpha} = J^{\alpha}(K^{\lambda}_{\lambda} - 2\overset{1}{\omega}) + \rho(L^{\alpha}+\gamma^{\alpha\lambda}\partial_{\lambda}\overset{2}{N}) - (\overset{1}{k}{}^{\alpha\lambda}+V\gamma^{\alpha\lambda})\varphi_{\lambda}. \qquad \text{(IV.31)}$$

The conditions that Q^{α} vanishes are

$$\overset{1}{\omega} = 1/2\,K^{\lambda}_{\lambda} + \rho^{-1}u^{\lambda}\varphi_{\lambda} \qquad \text{(IV.32)}$$

and

$$\gamma^{\alpha\lambda}\partial_{\lambda}\overset{2}{N} = -L^{\alpha} + \rho^{-1}\left[K^{\alpha\lambda} + (V+\rho^{-1}u^{\mu}\varphi_{\mu})\gamma^{\alpha\lambda}\right]\varphi_{\lambda}. \qquad \text{(IV.33)}$$

However, if ψ is to satisfy the Klein-Gordon equation on the relativistic space-time $\overset{1}{\omega}$ and $\overset{2}{N}$ are already subject to parabolic type second order equations.

It is thus not evident that the specific energy and heat flux part of $T^{\alpha}_{K.G.\,\beta}$ tend to the corresponding Schrödinger terms.

This problem has not yet been solved. Possibly the gravitational field equations should also be used. But it seems likely that the fully fourdimensionally covariant formalism used in this paper is not the most appropriate to investigate this type of question since there is too much gauge freedom in the expansion in terms of powers of c^{-2}. A post-Newtonian expansion based on the initial value formulation of the relativistic equations, as used in [18], might be more suitable since it is easier to impose (asymptotic) boundary and gauge conditions.

ACKNOWLEDGEMENTS

We thank Guy Burdet for several discussions and Jürgen Ehlers for useful comments on the manuscript. The second author (H.P.K.) acknowledges the hospitality and the support of the Faculté des Sciences de Luminy, support of the Centre de Physique Théorique, C.N.R.S., Marseille and the Max-Planck-Institut für Astrophysik, Garching bei München.

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