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UNITARITY OR ASYMPTOTIC COMPLETENESS EQUATIONS AND

ANALYTIC STRUCTURE OF THE S MATRIX AND GREEN FUNCTIONS

by

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UNITARITY OR ASYMPTOTIC COMPLETENESS EQUATIONS AND ANALYTIC STRUCTURE OF THE S MATRIX AND GREEN FUNCTIONS

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Abstract : Recent axiomatic results on the (non holonomic) analytic structure of the multiparticle S matrix and Green functions are reviewed and related general conjectures are described : (i) formal expansions of Green functions in terms of (holonomic) Feynman-type integrals in which each vertex represents an irreducible kernel, and (ii) "graph by graph unitarity" and other discontinuity formulae of the latter.

These conjectures are closely linked with unitarity or asymptotic completeness equations, which they yield in a formal sense. In constructive field theory, a direct proof of the first conjecture (together with an independent proof of the second) would thus imply, as a first step, asymptotic completeness in that sense.

1. Introduction

The purpose of this text is to present recent results and conjectures on the (physical-region) momentum space analytic structure of the S matrix and Green functions in a theory of massive particles with short-range interactions. The subject is not new and goes back in fact to the first part of the sixties. While important ideas were already proposed at that time, results were, however, limited in the multiparticle case by both conceptual and mathematical difficulties. It was then developed in various ways, in connection with related mathematical studies such as (analytic) essential support theory [6,13], the theory of (regular) holonomic functions [24,20] (in the sense of Sato), and Fredholm theory in complex space and with complex parameters [1,2,7]. It has on the other hand very close links with the question of asymptotic completeness in constructive field theory, as we shall discuss in more detail.

We shall consider for simplicity, unless otherwise stated, a theory with only one type of stable particle, a boson of mass $\mu > 0$ and denote by s the total initial (=final) squared center of mass energy. Sections 2 and 3 present results of S-matrix theory and axiomatic field theory respectively, based on unitarity or asymptotic completeness equations, with emphasis on aspects that will be of interest for the later introduction of the conjectures in Sect.4, such as the notions of "separation of singularities" in unitarity equations and of irreducible kernels[1].

We first introduce in Sect.2 unitarity and extended unitarity equations, and recall general results of S-matrix theory on macrocausality and physical-region $+\alpha$-Landau singularities and on the derivation of local or global discontinuity formulae around the Landau surfaces [2]. A first property of separation of singularities, by classes of topological diagrams, recently proved in some cases and more generally assumed, is used in this derivation. We then present the more detailed analysis of the m-particle threshold in a simplified $m\to m$ scattering theory with no subchannel interaction, carried out [4] in connection in particular with a conjecture made by M. Sato in 1975 on the holonomicity [3] of the S matrix. If $2\beta = (m-1)\nu-m-1$, where ν is the dimension of space-time, is even, e.g. m=3, ν=4, the S-matrix is not holonomic. It admits, on the other hand, local expansions in terms of (holonomic) contributions in $(\ln\sigma)^n$, $\sigma = (m\mu)^2-s$, associated with the multiple scattering graphs  and involving, in the S-matrix framework, an irreducible (=locally analytic) kernel U which replaces the K-matrix of the case m=2, ν=4. A new property of strong separation of singularities, graph by graph, will be exhibited in that framework.

In the non simplified theory, the S matrix is again expected [4] to be non holonomic in general, e.g. at 3-particle thresholds, for the same reasons as above and in view of further difficulties. It has not been possible so far to define kernels of type U in the non simplified theory and to introduce corresponding expansions in terms of well specified (holonomic) contributions. Such expansions, involving Bethe-Salpeter type kernels, will be introduced in the field theoretical framework (see below and Sect.4).

We present in Sect.3 results of axiomatic field theory. The results of the linear program, based on microcausality and the spectral condition, are limited (and unsatisfactory) on the (complex) mass-shell in the multiparticle case and in particular do not allow one to "extricate" the Landau singularities. The subject has then been mainly developed in the non linear program, from the supplementary axiom of asymptotic completeness which provides off-shell unitarity-type equations. Two different approaches have been proposed : that of [10] which is a priori more direct and that of Bros [2,3], which is an extension of his previous work [1] on the 2-body case, and introduces Bethe-Salpeter type irreducible kernels. Both approaches have allowed one to extricate the Landau singularities for $3\to3$ processes in the low energy region, and to obtain there some results analogous to those of Sect.2, including a first type of decomposition of the $3\to3$ Green function in terms of Feynman-type convolution integrals in which each vertex is a $2\to2$ Green function. The analysis of Bros (which uses slightly stronger regularity assumptions) leads to more detailed results. It is based, as in the 2-body case, on the proof of a "quasi"-equivalence (up to problems arising from zeroes of denominators in solutions of Fredholm-type equations) between the asymptotic completeness equations in the region considered and the irreducibility (=analyticity) properties of the Bethe-Salpeter type kernels involved. His method can thus be applied conversely in constructive field theory, where irreducibility properties can be directly established, in order to derive asymptotic completeness properties, as already done for $2\to2$ [26,11] and $3\to3$ [8] processes in the low energy region. The analysis will be outlined here in the $2\to2$ case and in the simplified theory of the m-particle threshold, where it leads [7] to results analogous to those of Sect.2, and where strong separation of singularities [17] will again be checked

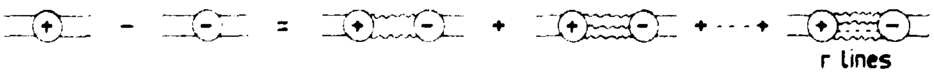
Sect.4 presents two general conjectures [17], which are directly linked and are complementary to the previous results. The first one, is, for each given process, a set of (formal) expansions $F = \Sigma_G F_G$ of the Green function F (whose mass-shell restriction is the scattering function T) in terms of Feynman-type convolution integrals F_G associated with multiple scattering graphs G and in which each vertex v represents a "totally r_v-particle irreducible" kernel ; the class of graphs G over which the sum is made and the degree r_v of irreducibility at each vertex depend on the energy region $s < [(r+1)\mu]^2$ considered (the number of graphs G increases in particular with r). Each term F_G has, in that region, a well defined analytic and monodromic structure, analogous to that of of the corresponding Feynman integral (and is as the latter holonomic in the sense of Sato). The second conjecture, which includes "graph by graph unitarity", expresses discontinuities of each term F_G around relevant sets of Landau singularities as a sum of on mass-shell convolution integrals which are associated with various ways of dividing G into successive multiple scattering subgraphs. Its proof should depend only on the irreducibility (=analyticity) properties of the kernels involved at each vertex.

These conjectures, which will appear to be natural from the viewpoint of perturbation theory, are believed to hold independently, e.g. in axiomatic field theory where conjecture 1 follows in fact from the results of Bros for the 3→3 case in the low energy region and should follow more generally from the development of this program. On the other hand, if we assume Conjecture 2, then one checks conversely that Green functions satisfying Conjecture 1, do satisfy unitarity and asymptotic completeness equations (and further discontinuity formulae) in the sense of formal expansions, strong separation of singularities being satisfied, and appear in fact as natural solutions of these equations. In constructive field theory, a direct proof of Conj.1 in a given region would thus provide asymptotic completeness, as a first step, in the sense of formal expansions in terms of irreducible kernels [4].

2. S-Matrix Theory

2.1 Unitarity and extended unitarity equations

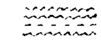
The traditional starting point in S-matrix theory is the (infinite) set of non linear "unitarity equations", derived in the physical region from $SS^{*}=1$ (or $SS^{-1}=1$), between connected momentum-space collision amplitudes of various physical processes. Two examples are :



(1)

in the part of the physical region defined by $s<[(r+1)\mu]^2$ and :


(2)

in the part of the 3→3 physical region defined by $s<(4\mu)^2$. In these equations, + and − refer to connected kernels of S and $-S^{-1}=-S^{*}$ respectively and  denotes on-mass-shell convolution. For instance :

$$(p_1,p_2,p_3;p_4,p_5,p_6) = \int S^c_{3,3}(p_1,p_2,p_3;k_1,k_2,k_3) \times$$

$$\times\; (-S^{-1})^c_{3,3}(k_1,k_2,k_3;p_4,p_5,p_6)\,\frac{1}{3!}\prod_{\ell=1,2,3}\delta_+(k_\ell^2-\mu^2)\; d^\nu k_\ell$$

The scattering function $T_{m,n}$ of a given $m\to n$ process, initially defined in the physical region of that process, is obtained from the connected S-matrix kernel by factorizing out its (global) energy-momentum conservation δ-function. We note on the other hand that $s \geq (m\mu)^2, (n\mu)^2$ in the physical region.

The above system of equations clearly imposes constraints on the structure of its possible solutions. In the beginning of the sixties, the analytic S-matrix theory has proposed the principle of "maximal analyticity" according to which scattering functions should have (in and outside the physical region) the best analyticity properties compatible with these equations. While this idea remains basically correct, this approach is somewhat too loose by itself to derive precise results. It can be completed (or replaced) in the physical region by a causality requirement as described in Sect.2.2 (For the situation in field theory, see Sect.3). We first conclude this subsection by mentioning the *extended unitarity* equations, which are part of the results one ultimately wishes to establish. These equations, which are analogous to the previous ones, are no longer restricted to the real mass-shell, but refer to various analytic continuations of scattering functions on the complex mass-shell. Their heuristic derivation in S-matrix theory is based on the knowledge of the physical region structure together with assumptions on analytic continuations. The most simple example is

 -  = 0 (3)

in the region s real, $s < (2\mu)^2$ (=no possible intermediate physical states in that region), where $T_{2,2}$ and $T^-_{2,2}$ are analytic continuation of the physical region functions around $s=4\mu^2$ in the upper and lower Im s-half-plane respectively. This relation entails that $T^-_{2,2}$ is at s real, $s > 4\mu^2$ the analytic continuation of $T_{2,2}$ around $s = 4\mu^2$, Eq.(1) thus becoming a discontinuity equation for $T_{2,2}$. Similarly, only the first term in the r.h.s. of Eq.(2) remains for s real, $s < (3\mu)^2$ and moreover $T_{3,3}$ and $T^-_{3,3}$ coincide at $s < (2\mu)^2$, Eq.(2) thus becoming also a global discontinuity equation for $T_{3,3}$ around relevant singularities.

Further discontinuity formulae are introduced in Sect.2.3, 2.4.

2.2 Macrocausality and physical-region $+\alpha$-Laudau singularities

Macrocausality [18,12] states exponential fall-off properties of (connected) amplitudes between adequate sets of displaced initial and final wave functions, if the corresponding configuration of initial and final particles is non causal, i.e. if they cannot be linked by a (connected) classical multiple scattering diagram. (More precisely, the wave functions considered correspond to particles that are well localized asymptotically, modulo exponential fall-off, along classical straight-line trajectories in space-time with well defined on-mass-shell energy-momenta). This condition is expressed mathematically in terms of the notion of (analytic) essential support 5), and entails in particular that each (physical-region) scattering function $T_{m,n}$ is analytic outside $+\alpha$-Landau surfaces of connected graphs and is at almost all $+\alpha$-Landau points the boundary value of an analytic function from well specified "plus $i\varepsilon$" directions dual, at each $+\alpha$-Landau point, to the causal direction(s) at this point [This is no longer true e.g. at points that belong to several $+\alpha$-Landau surfaces with conflicting plus $i\varepsilon$ rules 6)].

Examples of multiple scattering graphs and corresponding $+\alpha$-Landau surfaces are the following . For a $2\to2$ process, the latter are defined by the conditions $s = (r\mu)^2$, $r = 2,3,4,\ldots$ For r=2, relevant graphs are :

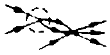
(4)

For r=3 (3-particle threshold), relevant graphs are successions of $2\to2$, $2\to3$, $3\to3$, $3\to2$ elementary parts (in arbitrary number), e.g.

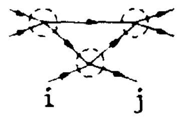
(5)

For a $3\to3$ process, one finds in the part $s<(4\mu)^2$ of the physical region :

i) the 3-particle threshold $s=(3\mu)^2$ associated with graphs similar to above, ending with 3 initial and 3 final lines.

ii) the 2-particle thresholds of two initial, or two final, particles. For instance the surface $(p_1+p_2) = (2\mu)^2$, associated with the graphs  where  denotes either  or anyone of the graphs (4).

iii) the surfaces associated with the graphs of the form :

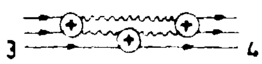
a)  b)  (6)

In the part R of the 3→3 physical region defined by the conditions $(3\mu)^2 < s < (4\mu)^2$ and excluding 2-particle thresholds, one encounters exactly 18 +α-Landau surfaces, each of which being associated with one of the classes of graphs (6a) or (6b), for a given choice of (i,j).

2.3 Separation of singularities, local discontinuity formulae and nature of Landau singularities

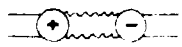
An important program in S-matrix theory (see [27,12] and references therein) is the further derivation from unitarity equations of general, local or "microlocal" [in the mathematical sense of hyperfunction or essential support theory] discontinuity formulae of scattering functions around their physical-region +α-Landau singularities. These formulae are in turn equivalent to properties of macrocausal factorization for causal configurations of displaced particles : see [12]. If the elementary graph that gives rise to the +α-Landau surface considered is "simple", i.e. includes no set of more than one line between two vertices, the discontinuity is equal to the on-mass-shell convolution integral obtained by replacing each vertex of G by a + bubble. For instance, if

G =  , then the local discontinuity of $T_{3,3}$ around $L^+(G)$ is equal to



(after factorization of an overall energy-momentum conservation δ-function).

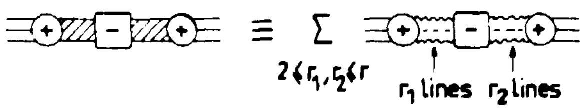
Some modification has to be made in the case of non simple graphs. Without giving details, we note that a crucial role is played in this derivation by a (micro)-local property of "separation of singularities in unitarity equations". More precisely, macrocausality and the general theorems of essential support or hyperfunction theory on products and integrals of distributions entail that the singularities of each term occurring in a unitarity equation, such as Eq.(2), are associated in a well specified way with space-time diagrams. "Separation of singularities" asserts that the singularities of the various terms arising from diagrams belonging to a common topological class (e.g. in the region R of a $3\to3$ process, the class of graphs of Fig.3a) or those of Fig.3b for a given choice of i,j) should cancel (micro)-locally among themselves. This property does not directly follow from the general edge-of-the-wedge theorems of essential support or hyperfunction theory. It has been recently established in the cases needed in the region R from refined forms of macrocausality (see [15] or [12 b,c] . This analysis is probably linked to more recent mathematical studies on "second microlocalization". See also the related works [21,22] in a "holonomic" framework). It is, so far, more generally assumed and can be conversely established, at least in many cases, from the discontinuity formulae.

The discontinuity formulae can give direct information on the nature of Landau singularities if the structure of individual bubbles in the integral that gives the discontinuity is already known : for instance, the singularity of $T_{3,3}$ in the region R is a pole (with a factorized residue) for the graphs of Fig.3a) and is a ℓn (in space-time dimension $\nu=4$) for those of Fig.3b) (see e.g.[12c]). More general results of this type, based on "holonomic" methods, for singularities of graphs with at most one or two lines between two vertices can be found in [23]. However, the structure of individual bubbles is not known in advance in general, i.e. one still remains with integral equations. The most simple case is Eq.(1) considered here at $s<(3\mu)^2$, i.e. with only one term in the r.h.s.. As explained in Sect.2.1 (and admitted here), it states that the (local) discontinuity of $T_{2,2}$ around $s = (2\mu)^2$ is equal to  (after factorization of $\delta^{\nu}(p_1+p_2-p_3-p_4)$). This integral equation is easily solved, e.g. via a partial wave decomposition as already done in 1960, or by direct analysis [10] : see Sect.2.5. As well known, one obtains (in dimension 4) a two-sheeted, square-root singularity (a result used in [23]). However, the situation is already much more complicated for 3-particle thresholds, such as those mentioned in Sect.2.2 for $2\to2$ and $3\to3$ processes, and as a consequence for derived singularities of graphs with sets of 3 lines between some vertices. The m-particle threshold is treated in Sect.2.5 in a simplified theory.

2.4 Global discontinuity formulae

The formulae discussed in Sect.2.3 give information on the local structure of the S matrix in the neighborhood of individual, physical-region singularities. Other formulae of interest [2)] are global discontinuity formulae around sets of singularities which do not all lie in the physical-region. They relate in particular (see examples below) boundary values of analytic continuations of scattering functions "above" certain subsets and "below" other subsets of normal threshold singularities. The latter are of the form $s_g = \mu^2$, $(2\mu)^2$,... where g is a given channel (= partition of the set of initial and final particles into two subsets, each containing at least two initial or two final particles), and above or below refer respectively to plus $i\varepsilon$ and minus $i\varepsilon$ distortions in Im s_g (The rule is the same for all thresholds corresponding to the same g). The rule of analytic continuation around other singularities (e.g. the triangle singularity in Sect.2.2) is also specified, but is not described here.

As already mentioned in Sect.2.1, Eq.(2) is an example of such a discontinuity, $T_{3,3}$ and $T^-_{3,3}$ being (boundary values of) analytic continuations, respectively above and below all normal thresholds, of the common function with which they coincide at $s < (2\mu)^2$. As another example, the discontinuity between $T_{3,3}$ and the analytic continuation below the normal thresholds $s = (2\mu)^2$, $(3\mu)^2$,... and above all other normal thresholds can be expressed in the region $s < [(r+1)\mu]^2$ in the form :



where the box minus stands for the non connected kernel of $S^{-1} = S^+$. (The overall δ-function $\delta^{\nu}(p_1+p_2+p_3-p_4-p_5-p_6)$ is again to be factored out).

For details on discontinuity, and multiple discontinuity, formulae of this type, see [27] and references therein. We note that all these formulae are expressed in terms of S-matrix elements (or their complex conjugates). All these results are derived from unitarity equations, together again with assumptions of separation of singularities and further assumptions on analytic continuations away from the physical region.

2.5 2-particle threshold and simplified theory of the m-particle threshold

We describe below the results on the 2-particle threshold of a $2\to2$ process in the more general framework[4] of a simplified theory of the m-particle threshold in a $m\to m$ process with no subchannel interaction, i.e. in which the unitarity equation reads locally at $s>(m\mu)^2$:

(7)

T_- is again assumed to be an analytic continuation of $T=T_+$: more precisely, T_+ and T_- are at $s>(m\mu)^2$ boundary values from the directions Im $s>0$ and Im $s<0$ respectively of a function $\underline{T}$ assumed to be analytic in a cut neighborhood of $s=(m\mu)^2$, with the cut along $s\geq(m\mu)^2$. T_+ and T_-, also denoted below T_0 and T_1, will only be assumed to be continuous in the arguments below, in view of the further application in field theory (Sect.3) : analyticity (e.g. analyticity for $(2\mu)^2<s<(3\mu)^2$ at m=2 already known in the S-matrix approach from macrocausality) is then part of the results.

Eq.(7), in which T_0 and T_1 can be expressed in terms of the variable s and of angular variables Ω, Ω' for the initial and final particles respectively, appears as a Fredholm-type resolvent equation in Ω-space, depending on the further parameter s. It is then shown, by Fredholm theory with complex parameters (and a simple application of the edge-of-the-wedge theorem) that $\underline{T}$ admits analytic or meromorphic continuations (with possible poles in s) into new sheets around $s=(m\mu)^2$. A two-sheeted, square root structure is obtained if $2\beta=(m-1)\nu-m-1$ is odd e.g. $m=2,\nu=4$, but the situation is different if 2β is even, e.g. $m=3$, $\nu=4$ or $m=2$, $\nu=5$; the number of sheets around $s=(m\mu)^2$ is then infinite and moreover T is not holonomic[7)] at $s=(m\mu)^2$. The difference between the two cases is due to the factor $\sigma^\beta,\sigma=(m\mu)^2-s$, in the convolution operation $*$ obtained in the r.h.s. of (7) after elimination of δ-functions : $\sigma^\beta\to-\sigma^\beta$, thus $*\to*_1=-*$, after one turn around $s=(m\mu)^2$ in the first case, whereas $\sigma^\beta\to\sigma^\beta$ and $*\to*_1=*$ in the second.

One way to characterize the local structure of T at $s=(m\mu)^2$ is then to introduce the kernel U via the integral equation :

$$T = U + T \circledast U \tag{8}$$

where $\Theta = \frac{1}{2} *$ if 2β is odd, $\Theta = \frac{1}{2i\pi} \ell n\, \sigma\, *$ if 2β is even. This definition ensures in both parity cases the basic relation used below :

$$\Theta_o - \Theta_1 = * \ (\equiv *_o) \tag{9}$$

where Θ_o is the initial value of the operator Θ at $s>(m\mu)^2$ and Θ_1 is the value obtained after one turn around $s=(m\mu)^2$. First, by Fredholm theory with complex parameters, U is shown to exist (e.g. if T is locally bounded) and to be a well defined analytic or meromorphic function around $s=(m\mu)^2$. Then, a simple argument (analogous to that used first in [1] for Bethe-Salpeter kernels) showsthe *equivalence* between the unitarity equation (7) ($T_o - T_1 = T_o * T_1$) and the *uniformity* of U ($U_o = U_1$). It is based on the relation :

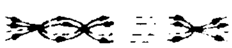
$$(T_o - T_1 - T_o * T_1)\ \Theta_1 (1_1 - U_1) = (1_o + T_o)\ \Theta_o (U_o - U_1) \tag{10}$$

easily checked from (8) [which provides $T_i = U_i + T_i\, \Theta_i\, U_i$, hence also $T_i\, \Theta_i (1_i - U_i) = U_i$, $i=0,1$], from the associativity of the operations involved and from (9).

If $\underline{T}$ is locally bounded near $\sigma=0$ in its first sheet and if $\beta>0$, one shows moreover that U is itself bounded, hence analytic at $\sigma=0$, i.e. (locally) irreducible, and that the Neumann series of T in Eq.(8) is locally convergent. Thus one obtains, for 2β even, the locally convergent expansion of T in powers of $\sigma^\beta\, \ell n\, \sigma$:

$$T = \sum_{n\geq 0} U^{\Theta(n+1)} = \sum_{n\geq 0} U^{\hat{*}(n+1)} \left[\frac{1}{2i\pi}\, \sigma^\beta\, \ell n\, \sigma\right]^n \tag{11}$$

where $* = \sigma^\beta\, \hat{*}$, $U^{\Theta(n+1)} = U\Theta U..\Theta U$ (n+1 factors U) and $U^{\hat{*}(n+1)}$ is, like U, locally analytic at $\sigma=0$. Each term in the r.h.s. of (11) can be associated with the graph :

$G_n^{(m)} =$  (12)

with n sets of m internal lines and n+1 vertices : an irreducible kernel U is associated to each vertex and a convolution operator Θ to each set of internal lines.

We finally note that $U^{\Theta(n+1)}$ satisfies, for each n, the "unitarity equation" :

$$(U^{\Theta(n+1)})_o - (U^{\Theta(n+1)})_1 = \sum_{0\leq r\leq n} (U^{\Theta(n-r)})_o * (U^{\Theta(r+1)})_1 \tag{13}$$

where $(U^{\circledast(n+1)})_i \equiv U^{\circledast_i(n+1)}$, $i=0,1$ in view of the local analyticity of U, and where the sum in the r.h.s. runs over all ways of dividing $G_n^{(m)}$ into two successive multiple scattering subgraphs. Eq.(13) follows in a straightforward way from successive applications of the basic relation (9), and provides an alternative way of showing how the irreducibility of U yields the unitarity equation $T_o - T_1 = T_o * T_1$. Namely, one checks [8] that :

$$(\sum_n U^{\circledast(n+1)})_+ - (\sum_n U^{\circledast(n+1)})_- = (\sum_n U^{\circledast(n+1)})_+ * (\sum_n U^{\circledast(n+1)})_- \qquad (14)$$

strong separation of singularities, graph by graph, being in fact satisfied in view of (13).

3. Axiomatic Field Theory

3.1 Asymptotic completeness equations [2,5]

Asymptotic completeness is expressed in axiomatic field theory in the form of discontinuity equations analogous to those of Sect.2.1, 2.4 (and involving as before *on-mass-shell* convolution integrals), with the following differences. On the one hand, the *external* energy-momenta are no longer restricted to the mass-shell. On the other hand, the formulae refer to particular analytic continuations of Green functions and these continuations are made in (complex) *off-shell* domains : it is desirable (e.g. in view of the derivation of multiparticle dispersion relations) and it should be ultimately possible (as already known in the two-body case) to extend analyticity domains in a way such that these functions, when restricted to the mass-shell be obtained from each other by analytic continuations on the complex mass-shell analogous to those considered in Sect.2 and to derive in a precise way all formulae described there. But this is not known in general at the outset, and is part of the results to be established.

3.2 2-particle threshold and simplified theory of the m-particle threshold

For a 2→2 process, the linear program entails analyticity in a cut domain and asymptotic completeness provides the corresponding equation $F_o - F_1 = F_o * F_1$ in the region $(2\mu)^2 < s < (3\mu)^2$. The analysis of Sect.2.5 on T can then be extended off-shell [10] and provides in particular the two-sheeted, square-root structure of F at $s = (2\mu)^2$ in dimension $\nu=4$. If we consider the off-shell version of the simplified theory of

Sect.2.5, in which the off-shell unitarity-type equation $F_O - F_1 = F_O * F_1$ holds for the m→m Green function in the region $(m\mu)^2 < s < [(m+1)\mu]^2$, one obtains similarly [16] a two-sheeted structure at $s = (m\mu)^2$ if 2β is odd, and an infinite-sheeted structure if 2β is even : the kernel U can be extended off shell and the locally convergent expansion (11) still holds for F in the case 2β even.

Similar results on F in the two-body case [1] at $\nu=4$, and more generally in the simplified theory [7], have been also obtained from slightly stronger assumptions in the alternative approach based on Bethe-Salpeter type kernels G, related to F via the integral equation

$$F = G + F \circ G \qquad (15)$$

Here o denotes a Feynman-type convolution, with Feynman propagators attached to each internal line, and with an integration contour $\Gamma(k)$ depending on the total initial (=final) energy-momentum k and going to infinity in euclidean directions. An analytic cut-off factor, equal to one on the mass-shell is possibly attached also to each internal line to ensure convergence of integrals (and is included in the definition of o). When s turns around $(m\mu)^2$, the following basic relation, analogous to (9), holds :

$$0_o - 0_1 = *_o \qquad (16)$$

where 0_o and 0_1 correspond to the integration contours obtained at s real, $s > (m\mu)^2$ from the respective directions Im s > 0 and Im s > 0 of the initial (physical) sheet.

Eq.(15) is formally a Fredholm resolvent equation. Its treatment requires, however, an extension of Fredholm theory in complex space (and with again a complex parameter) : see [7]. It entails that G is analytic, or meromorphic (with possible poles in s), in the same cut domain as F. On the other hand, in view of (16), the algebraic argument used in Sect.2.5 (with T and Θ replaced here by F and o) shows the equivalence between the off-shell unitarity equation $F_O - F_1 = F_O * F_1$ and the uniformity of G ($G_O = G_1$) around $s = (m\mu)^2$. Let us here *assume* that G is moreover analytic (i.e. has no poles) in the region $s < [(m+1)\mu]^2$ (i.e. G is two-particle irreducible in the case m=2). Then Fredholm theory in complex space (and with complex parameters) applied in Eq.(15) where F is now considered to be defined in terms of G, allows one in turn to show that F has a two-sheeted, resp. infinite sheeted, structure for

2β odd, resp. even. The result is based here on the analyticity of G and on Eq.(16) which, in view of the equality $*_1 = -*_o$ if 2β odd, resp. $*_1 = *_o$ for 2β even, yields $O_2 = O_o$ in the first case, $O_2 = O_o + 2*_o$ and more generally $O_n = O_o + n*_o$ in the second one.

The formal expansion of F in terms of G (possibly convergent for adequate analytic cut offs in the definition of o) reads here :

$$F = \sum_n G^{o(n+1)} \tag{17}$$

where the terms in the r.h.s. can again be associated with the graphs $G_n^{(m)}$ of Eq.(12), with now an irreducible kernel G at each vertex and a Feynman-type convolution o for each set of internal lines. The link between the (off-shell) irreducible kernels U and G and the corresponding expansions of F has been studied in [16]. In particular, we note that for 2β even, $G^{o(n+1)}$ is for each n of the form :

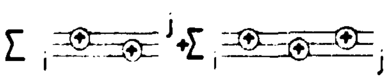
$$G^{o(n+1)} = \sum_{0 \leq r \leq n} a_{n,r}(p)\ (\sigma^{\beta}\ \ell n\ \sigma)^r \tag{18}$$

with locally analytic coefficients $a_{n,r}$, i.e. $G^{o(n+1)}$ has, as the Feynman integral $I(G_n^{(m)})$, a dominant contribution in $(\sigma^{\beta}\ \ell n\ \sigma)^n$ plus subdominant contributions. The expansion (11) (for F) then appears as a reorganization of the terms of (17) according to the powers of $\sigma^{\beta}\ \ell n\ \sigma$.

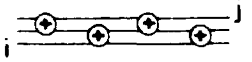
To conclude with the simplified theory, we note that the irreducibility of G and the same algebraic argument as in Sect.2.5 gives again graph by graph unitarity, namely here Eq.(13) with U and $\circledast$ replaced by G and o, which directly yields again (at least in the sense of formal expansions with respect to G) the off-shell unitarity equation $F_+ - F_- = F_+ * F_-$ with strong (graph by graph) separation of singularities.

3.3 $3 \to 3$ process in the low energy region

We finally come back to the non simplified theory and to the results obtained for a $3 \to 3$ process in the low energy region. The results of [10] are based on the analyticity of $F_{2,2}$ previously established at s real, $(2\mu)^2 < s < (3\mu)^2$ and on the discontinuity formulae for $F_{3,3}$ provided by asymptotic completeness with respect to 2-particle thresholds. Methods of essential support theory then allow one to show, up to some technical limitations (see [11]) that :

$F_{3,3} = $  $+ A$ (19)

where A is analytic [9] in the real part R of the physical region (see Sect.2.2) and where the terms in the r.h.s. are Feynman-type convolution integrals in which each vertex is a Green function $F_{2,2}$. This result is very close to those of Sect.2.3 in R (and is equivalent to off-shell versions of macrocausality and macrocausal factorization) : apart from A, each term in the r.h.s. of (19) is singular in R only along the corresponding $+\alpha$-Landau surface and admits as its discontinuity around that surface the corresponding on mass-shell convolution integral.

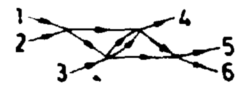
The analysis of [2,3] (presented in [2] for an even theory) gives analogous results in a first "physical-sheet", with the supplementary terms  included off-shell (they are analytic on-shell and are thus part of A in (19)) and more precise informations on the remainder term (decomposition into a finite number of terms with local analytic continuation around corresponding 2-particle thresholds). Details are omitted here for conciseness and will be found in the presentation of [2]. The analysis gives also potential information in further sheets (see [3]) and, as already mentioned in Sect.1, is a basis for Conjecture 1/ of Sect.4 .

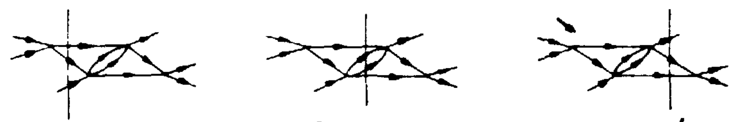
4. Expansions in Terms of Irreducible Kernels and Graph by Graph Unitarity

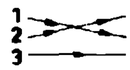
The ideas presented below are those of [17]. Details and partial proofs of Conjecture 2 proposed recently by J. Bros and A. Katz (see Sect.4.2) may be found in later works.

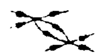
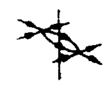
4.1 r-particle irreducible kernels and expansions $F = \Sigma_G F_G$

The irreducible kernels $F^{(r)}_{m,n}$ introduced below are, for each given $m \to n$ process and each r ($\geq$ m,n), "totally r-particle irreducible". It is known (see Sect.2,3) that the singularities of F are associated with multiple scattering connected graphs. In contrast to F, the only singularities of $F^{(r)}$ should be those associated with "totally r-particle irreducible" graphs : by definition, a graph G is totally r-particle irreducible if, for any way of dividing it into two successive (connected or not connected) multiple scattering subgraphs, the number ℓ of intermediate lines (= outgoing in G_1, incoming in G_2) is > r. [Intermediate lines may include some incoming or outgoing lines of G but have to include internal lines of G].

Examples : The graph  is 2-particle irreducible and 3-particle reducible. In fact, there are 3 ways of dividing it into two successive subgraphs G_1, G_2



and, in e.g. the first case G_1 =  , G_2 = 

The graphs  and  are 3-particle irreducible and 4-particle reducible ( and ).

Remarks : (i) $F^{(r)}$ will in particular have no singularity in the variable s in the region $s < [(r+1)\mu]^2$. It has further analyticity properties, not detailed here, with respect to other variables.
(ii) The kernels $F^{(r)}_{m,n}$ for different values of m,n,r are not independent, but should be linked by a system of integral equations.

Conjecture 1 : There exists a system of (totally) r-particle irreducible kernels such that the Green function F of each given process admits, for each r, the formal expansion :

$$F = \Sigma_{(G)} F_G \tag{20}$$

where the sum Σ runs over all connected multiple scattering graphs [10] G between the m initial and n final particles such that the number ℓ of intermediate lines in any way of dividing G into two successive subgraphs be $\leq$ r. For each graph G, F_G is the G-convolution Feynman-type integral (see remark (i) below) in which each vertex is the r_v-particle irreducible (and r_v+1 reducible) kernel $F^{(r_v)}_{m_v,n_v}$ where $r_v = r - \ell'_v$; ℓ'_v is the smallest possible number of remaining intermediate lines in the various ways of dividing G into two successive subgraphs such that the m_v incoming, or the n_v outgoing, lines of v be intermediate lines (not to be counted in ℓ'_v).

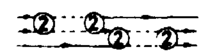
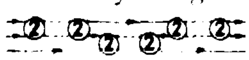
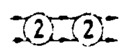
Example : m=n=2, r=3. An example of a term F_G, corresponding to the last graph G of Eq.(5) is (only the first part is shown) :

(21)

Remarks : (i) it is considered in the above statement that an adequate choice of analytic cut-off factor $\chi(k)$ (equal to one on the mass-shell), attached to each internal line ℓ of G, has been made in the definition of Feynman-type integrals, as in Sect.3, so that convergence of integrals be ensured. The irreducible kernels and the terms F_G depend on that choice.

(ii) from the viewpoint of perturbation theory, $F^{(r)}$ can be considered as the sum of Feynman integrals of (totally) r-particle irreducible graphs. Then, each term F_G in the formal expansion (20) appears as a partial resummation of Feynman integrals. As a matter of fact, the rules stated in Conjecture 1 are such that, if in each F_G the kernels $F^{(r_v)}_{m_v,n_v}$ are formally replaced by sums of Feynman integrals of r_v-particle irreducible graphs, then each Feynman integral is obtained, and is obtained once-

The expansion (20) is, for each r, of interest mainly in the region $s < [(r+1)\mu]^2$: the irreducibility (= analyticity) properties of the kernels $F^{(r_v)}$ are such that, in that region, each term F_G has the same analytic and monodromic structure as if these kernels were constants (= Feynman integrals) ; as the latter, it is "holonomic with regular singularities" [20] and its singularities are Landau surfaces associated with G and with the graphs obtained from G by contraction [only the $+\alpha$-parts of these surfaces are effective singularities in the physical region, but other parts may be singular in other sheets].

In a given region, the expansion (20) thus gives in particular a decomposition of F in terms that have a well defined analytic and monodromic structure. In the neighborhood of a singular point, or in a given subregion, these terms may be grouped in various ways : those which are not singular and whose sum is (formally) a (locally) analytic background, and the others, which can themselves be grouped in various subclasses of interest. For instance, in the region R of a 3→3 process, the only singular terms are those of the type  and . Their (formal) sums provide the different terms of (19), since  =  +  + ---.One can similarly reobtain the results of [3]. Note, however, that an infinite number of terms, each one having its own analytic and monodromic structure is needed in general, e.g. at the 3-particle threshold $s = (3\mu)^2$ of a 3→3 process : this already appears in the simplified theory, and there are many more terms in the non simplified theory.

A final remark : we have not discussed here the question of convergence of the expansions (20) or the possible existence of related (locally) convergent expansions in which the terms would have only "dominant" singularities (as was the case in the simplified theory for the expansion in terms of the kernel U). These questions are related to ideas discussed in [22] (in which the heuristic arguments, relying on comparisons between Feynman integrals and phase-space integrals, remain, however, preliminary in the absence of an actual structural analysis).

4.2 Graph by Graph unitarity and unitarity or asymptotic completeness equations

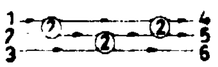
For simplicity, we first state below Conjecture 2 for the "total" basic discontinuity $(F_G)_+ - (F_G)_-$ of each term F_G around all singularities (up to the lowest) with respective $+i\varepsilon$ and $-i\varepsilon$ distortions. Related formulae will be conjectured for other simple or multiple discontinuities at the end of this Section. Note that the analytic continuation $(F_G)_-$ of $(F_G)_+ \equiv F_G$ can be defined in a way similar to $(F_G)_+$ but with minus $i\varepsilon$ propagators for each internal line and integration over corresponding contours Γ_-.

Conjecture 2

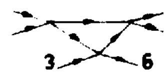
$$(F_G)_+ - (F_G)_- = \sum_{(G_1,G_2)} (F_{\hat{G}_1})_+ * (F_{\hat{G}_2})_- \qquad (22)$$

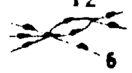
where the sum runs over all ways G can be divided into two (connected or not connected) successive multiple scattering subgraphs G_1, G_2, $\hat{G}_1$ and $\hat{G}_2$ are the parts of G_1, G_2 from which subgraphs of the form ——•—— (one incoming and one outgoing line) have been excluded, and $*$ denotes on-mass-shell convolution with respect to intermediate lines belonging both to $\hat{G}_1$ and $\hat{G}_2$ (If $\hat{G}_1$ or $\hat{G}_2$ is not connected, $(F_{\hat{G}_1})_+$, resp. $(F_{\hat{G}_2})_-$ is the product of the corresponding connected functions).

Example : Consider, in the region $s < (4\mu)^2$, the term

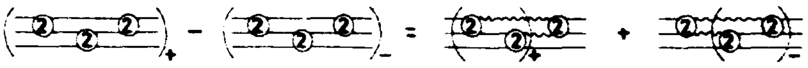
F_G = 

Its singularities in the region considered are

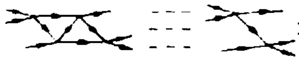
(i) the main + -Landau singularity of the graph 

(ii) the two-particle thresholds $s_{12} = (2\mu)^2$ and $s_{45} = (2\mu)^2$ of the contracted graphs  and 

The discontinuity of F_G around these *three* singularities is, according to Conjecture 2 :

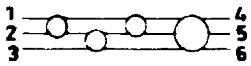
 (23)

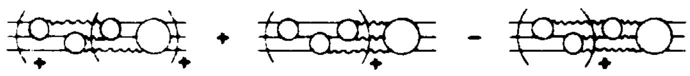
The term  is not present. We recall that it is on the hand the discontinuity of F_G around the main triangle singularity alone.

Conjecture 2 can be proved by direct inspection in some cases, such as the example above. It has been recently established by Bros, using a field theory method, for "truss-bridge" graphs G of the form  in a 3→3 process. His method can presumably be extended to the more general case. Another method of interest, which does not require a field theory background, but relies on analyticity properties and is based on the Picard-Lefschetz theory and the analysis of intersection indices, has been proposed by A. Katz and might also lead to the general result. Alternative methods, based e.g. on holonomy theory, might also be considered. Finally, we note that graph by graph unitarity can be established [9] for (possibly renormalized) Feynman integrals by an adaptation of the method used by these authors for proving unitarity at each order in perturbation theory. If we remember that, from the viewpoint of perturbation theory, each kernel $F^{(r_V)}$ and hence each term F_G can be considered (formally) as a sum of Feynman integrals, it is not difficult to recover (formally) Conjecture 2 on F_G. On the other hand, the method of [9] (T-product formalism) might also provide an actual proof of Conjecture 2, but this is not clear.

If we assume Conjecture 2, or the related other discontinuity formulae, then it can be checked by algebraic arguments that Green functions F satisfying Conjecture 1 do satisfy in each energy region unitarity, extended unitarity and asymptotic completeness equations, in the sense that corresponding formal expansions are satisfied, and are in fact satisfied graph by graph (= strong separation of singularities in these equations). As already mentioned in Sect.1, this might be applied in constructive field theory, where some results of Glimm -Jaffe, Dunlop-Combescure and Magnen-Sénéor [25] might be a first step towards a proof of Conjecture 1.

To conclude , we briefly mention other conjectures related to the formulae of Sect.2.4 (see below). The discontinuity $F_G-(F_G)_g$ of F_G relative to a channel g, (i.e. $(F_G)_g$ is the analytic continuation of F_G, from the region $s_g < \mu^2$, "below" normal threshold singularities relative to g and above others) is expressed in such conjectures in two equivalent forms : either as a sum over specified subgraphs (G_1,G_2) of on-mass-shell convolutions involving adequate analytic continuations of F_{G_1} or F_{G_2}, or alternatively as a sum (up to signs ±1) of on-mass-shell convolutions $(F_{G_1})_+ * (F_{G_2})_+ \ldots * (F_{G_\ell})_+$ where $G_1,\ldots,G_\ell$ are obtained by dividing G into successive multiple scattering subgraphs such that each cutting of internal lines is consistent with g, namely divides G into two connected parts whose external lines are those of the channel g. For instance, if g = (1,2,3;4,5,6) and

F_G =  , then :

$F_G-(F_G)_g$ = 

Conjectures on multiple discontinuities of F_G for some sets of channels $g_1,g_2,\ldots,g_r$ are similar : the latter are obtained by considering all $(G_1,\ldots,G_\ell)$ corresponding to cuttings including one or more cuttings consistent with g_i, for each i=1,...,r.

These conjectures allow one easily to obtain (formally) e.g. all simple or multiple discontinuity formulae of the type mentioned in Sect.2.4 and given in [27] for 3→3, 2→4 or 4→2 processes, by using the usual formal development of minus boxes in terms of plus bubbles that arises from S=1+T.

Endnotes

1) The analysis is made for an arbitrary dimension ν of space-time. The existence of theories in dimension 4 is not discussed. We believe the analysis should remain valid, modulo possibly some adaptations, in physical theories. For special phenomena at $\nu=2$, see note 6.

2) Global formulae are e.g. those needed for multiparticle optical theorems and dispersion relations.

3) Holonomic functions (in the sense of Sato) are solutions of maximally overdetermined systems of pseudodifferential equations and have special simplicity properties : see e.g. note 7 in Sect.2.3. The holonomicity of Feynman integrals has been generally established in [19,20].

4) Stronger results that would not involve formal expansions might be ultimately obtained in general in axiomatic or constructive theory. This is, however, still far from being achieved. The intermediate program proposed here will in any case keep its own interest in the understanding of the general structure of the theory.

5) The (analytic) essential support [6,13] of a distribution f is, at each real point, a set of "singular" direction in the space of dual, Fourier-transformed variables along which a "localized" Fourier transform of f does not decrease exponentially in a well-specified sense. As well known today, this notion issued from [18] coincides with the singular spectrum introduced independently by different methods in hyperfunction theory [24] and with Hormander's analytic wave front set (also introduced independently).

6) Such points do exist, e.g. in the region R of a $3\to3$ process introduced below, in arbitrary space-time dimension ν. In dimension $\nu=2$, special phenomena occur : different sets of $+\alpha$-Landau surfaces *coincide* but have opposite causal or plus $i\varepsilon$ directions. In this case, macrocausality and macrocausal factorization (Sect.2.3) -which are fall-off or factorization properties of transition amplitudes between sets of displaced wave functions, depending on the way initial and final particles are displaced from each other- reduce (see [14]) for some models to a factorization property of the multiparticle S-matrix itself in momentum space (including in fact factorization equations). This structure is, however, very specific of the dimension 2.

7) Holonomicity would mean essentially here that the vector space generated by the successive determination $T_0, T_1, T_2, \ldots$ at $s > (m\mu)^2$ should be finite-dimensional whereas this is not the case.

8) An alternative argument is given (for 2β even) in [22], where it is shown that functions of the form $\Sigma\, a_n(p) \left(\frac{1}{2i\pi}\, \sigma^\beta \ln \sigma\right)^n$ with locally analytic coefficients a_n satisfy $T_0 - T_1 = T_0 * T_1$ iff $a_n = a^{*(n+1)}$ for some locally analytic a. The arguments that we present here are more general and depend only on the algebraic relation (9) : they will thus apply equally in Sect.3.2 in field theory.

9) A is only shown in [10] to be a boundary value of analytic function from the directions Im $s > 0$. A new regularity assumption would be needed to avoid à la Martin pathologies and get analyticity.

10) G is allowed here to include $1\to m$ or $m\to1$ vertices (This type of vertex is forbidden on-shell, but not off-shell). We thus consider multiple scattering graphs in an "enlarged sense". On the other hand, the trivial graph, with only one vertex is included, in which case $F_G = F^{(r)}$.

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