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INSTABILITY OF HIGHER DIMENSIONAL YANG-MILLS SYSTEMS *

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ABSTRACT

We investigate the stability of Poincaré × 0(3) invariant solutions for a pure semi-simple Yang-Mills, as well as Yang-Mills coupled to gravity in 6-dimensional space-time compactified over $M^4 \times S^2$. In contrast to the Maxwell U(1) theory [6] in six dimensions coupled with gravity and investigated previously, the present theory exhibits tachyonic excitations and is unstable.

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I. INTRODUCTION

Much effort has been devoted over the past years to the proposition that space-time may have more than four dimensions. In its present form, starting with the speculation of Kaluza and Klein [1], the invariance group of the metric of the extra dimensions is invoked to explain the internal symmetries of elementary particle physics. It is, however, perfectly possible to imagine that some or all of the observed symmetries have some other origin and that the symmetries associated with the extra dimensions are not directly visible to us except through the spectrum of higher excitations.

One of the simplest types of such field-theoretical models that can be envisioned is a Yang-Mills theory in a space-time of (4 + K)-dimensions [2]. It may be assumed that this space-time has the product structure $M^4 \times B^K$ where M^4 denotes the 4-dimensional Minkowski space-time and B^K is a K-dimensional compact Riemanian manifold. Underlying such a model one could imagine that (4 + K)-dimensional general relativity is coupled to the Yang-Mills system and is the agency responsible for the "spontaneous compactification" of the extra K-dimensions [3]. One of the criteria for the consistency of such a theory is the stability of the vacuum solution.

If it is assumed that the internal space B^K is a quotient space $G|H$, then the invariance group of its metric can be used to classify the excitation modes. The harmonic analysis of the excitation spectrum can be pursued in detail [4] and it is possible to test the stability of the (assumed) ground state. As an illustrative example of this we consider in this note, the case of a pure SU(3)-Yang-Mills theory on a 6-dimensional space-time of the form $M^4 \times S^2$, assuming a Poincaré × SO(3) invariant configuration for the vacuum expectation value of the 6-vector potential. Our results are quite general and apply to SO(3)-invariant solutions of any non-Abelian classical gauge group. We find that the theory is unstable against the formation of tachyons *).

*) SU(3) is one of the cases which were investigated by Manton [5] in his construction of the electro-weak SU(2) x U(1) theory from a higher dimensional starting point. The instability of the SO(3)-invariant solution found in this note is not manifest in Ref.[5], because there a prescription is used, whereby only SO(3) invariant excitations, corresponding to $\ell = 0$, are retained (for the definition of ℓ see Eq.(1)). The harmonic expansion is reduced to a kind of Ansatz, which defines a 4-dimensional theory, but not a theory which would arise from spontaneous compactification. Physically, since there is one mass scale in the problem, the excitations which are excluded include some which are comparable in mass to those which are kept.

To stabilize the theory we allowed, as the next step, for the excitations of the 6-dimensional metric tensor, which would generally mix with the vector excitations. This also fails to stabilize the background except for the case of Maxwell U(1) theory which we had investigated earlier [6]. The conclusion we reach is that Yang-Mills and Yang-Mills-Einstein theories in 4+K dimensions generally fail to compactify on a stable background of the form $M^4 \times B^K$ where B^K is a quotient space $G|H$ except in special cases which we shall briefly attempt to motivate.

II. PURE YANG-MILLS CASE

The action for pure Yang-Mills theory in 5 + 1 space-time is

$$S = \int d^4x\, d^2y \sqrt{g} \left[-\tfrac{1}{4}\vec{F}_{AB}^{\,2}\right] ,$$

where x^0, x^1, x^2, x^3 are the usual Minkowski coordinates and $y^\mu = (\theta, \varphi)$ are coordinates on S^2 with $d^2y \sqrt{g} = \sin\theta\, d\theta\, d\varphi$. The field strength tensor is

$$\vec{F}_{AB} = \nabla_A \vec{A}_B - \nabla_B \vec{A}_A + \vec{A}_A \times \vec{A}_B$$

with ∇_A representing the Riemanian covariant derivative on $M^4 \times S^2$, $A = 0,1,2,3,4,5$. (For the rest of our conventions see [6].)

The ground state configuration must satisfy the equations of motion

$$\begin{aligned} 0 &= D_A \vec{F}_{AB} \\ &= \nabla_A \vec{F}_{AB} + \vec{A}_A \times \vec{F}_{AB} \end{aligned}$$

and, if the ground state is Poincaré invariant then

$$\langle A_a \rangle = 0 \qquad a = 0,1,2,3$$

$$\langle \vec{A}_\alpha \rangle = \vec{A}_\alpha(y) \qquad \alpha = 4,5 .$$

Further, if the ground state is SO(3)-invariant then $\vec{A}_\alpha$ must be invariant up to a gauge transformation. Typically

$$\langle A \rangle = \tfrac{1}{2} H\, d\varphi(\cos\theta \mp 1) ,$$

where H belongs to the Cartan subalgebra of the gauge group [7]. It is constrained by the requirement that A be patched smoothly on S^2, i.e.

$$\exp(4\pi i H) = k ,$$

where $k = 1$ (or more generally, belongs to the centre of the gauge group [8]). To test the stability of such a solution write $A = \langle A \rangle + V$ and treat V as a perturbation. The bilinear part of the action reads

$$S_2 = \int d^4x\, d^2y \sqrt{g}\, \tfrac{1}{2} \Big[-D_A \vec{V}_B \cdot D_A \vec{V}_B + D_A \vec{V}_A \cdot D_B \vec{V}_B + \vec{V}_A [D_A, D_B] \vec{V}_B + \vec{V}_A \cdot \langle \vec{F}_{AB} \rangle \times \vec{V}_B \Big] ,$$

where $D_A \vec{V}_B = \nabla_A \vec{V}_B + \langle \vec{A}_A \rangle \times \vec{V}_B$.

Since the background is SO(3) x Poincaré invariant, it is natural to expand in 4-dimensional fields:

$$\vec{V}_a(x,\theta,\varphi) = \sum_{\ell \geq |\lambda|} \sum_m \sqrt{2\ell+1}\, D^{\ell}_{\lambda,m}(\varphi,\theta)\, V^{\ell}_{am}(x)$$

$$\vec{V}_\pm(x,\theta,\varphi) = \frac{1}{\sqrt{2}} (\vec{V}_4 \mp i\vec{V}_5) = \sum_{\ell \geq |\lambda\pm|} \sum_m \sqrt{2\ell+1}\, D^{\ell}_{\lambda\pm,m}(\varphi,\theta)\, V^{\ell}_{\pm m}(x),$$

where $D^{\ell}_{\lambda m}$ are the SU(2)-representation matrices with Euler angles $\varphi, \theta, \pm\varphi$. To determine the appropriate helicity values, λ, it is necessary to consider the U(1) gauge transformation associated with O(3)-action on S^2. This takes the form

$$V_a \to e^{i\zeta H} V_a e^{-i\zeta H}$$

$$V_\pm \to e^{i\zeta H} V_\pm e^{-i\zeta(H\pm 1)} ,$$

where tha angle $\zeta = \zeta(\theta,\varphi)$ is determined such that $\langle A \rangle$ is invariant. On decomposing V into irreducible representations of this U(1), the component transforming as $\exp(i\lambda\zeta)$ is expanded in terms of $D^{\ell}_{\lambda,m}$. (For more details see [4] and [6].)

With H belonging to SU(3) and given by

$$H = \mathrm{diag}(n_1, n_2, -n_1 - n_2),$$

(where $3n_1, 3n_2$ and $n_1 - n_2$ are integers), the values of λ associated with the octet components V_a, $V_\pm$ can be displayed as 3×3 matrices:

$$\lambda(V_a) = \begin{pmatrix} 0 & n_1 - n_2 & 2n_1 + n_2 \\ -n_1 + n_2 & 0 & n_1 + 2n_2 \\ -2n_1 - n_2 & -n_1 - 2n_2 & 0 \end{pmatrix}$$

$$\lambda(V_\pm) = \begin{pmatrix} \pm 1 & n_1 - n_2 \pm 1 & 2n_1 + n_2 \pm 1 \\ -n_1 + n_2 \pm 1 & \pm 1 & n_1 + 2n_2 \pm 1 \\ -2n_1 - n_2 \pm 1 & -n_1 - 2n_2 \pm 1 & \pm 1 \end{pmatrix} .$$

The linearized Yang-Mills equations, in the covariant gauge $D_A V_A = 0$, satisfied by the component fields $V_{bm}{}^{\ell}(x)$, $V_{\pm m}{}^{\ell}(x)$ are

$$\left[\partial^2 - \frac{\ell(\ell+1) - \lambda^2}{a^2}\right] \vec{V}_b = 0 \qquad \ell \geq |\lambda|,\ |\lambda| + 1, \ldots \qquad (1a)$$

$$\left[\partial^2 - \frac{\ell(\ell+1) - (\lambda_\pm \mp 1)^2}{a^2}\right] \vec{V}_\pm = 0 \qquad \ell \geq |\lambda_\pm|,\ |\lambda_\pm| + 1, \ldots \qquad (1b)$$

where a is the radius of S^2.

The components $V_\pm$ clearly exhibit tachyons. For example, in $\vec{V}_+$ the component $\ell = |\lambda_+|$ for $\lambda_+ \leq 0$ carries negative $(\text{mass})^2$

$$- \frac{|\lambda_+| + 1}{a^2} . \qquad (1c)$$

We therefore conclude that O(3)-invariant Yang-Mills ground state is unstable.

III. INCLUSION OF GRAVITY

One might conjecture that stability will be restored if the Yang-Mills system is coupled to gravity, so that the vacuum geometry, $M^4 \times S^2$, arises spontaneously. We have examined this possibility for the gauge group SU(3) but the result is again negative. Since we know that for the case of a U(1) gauge field coupled to gravity, the $M^4 \times S^2$ vacuum is indeed stable [6], this

suggests to us that a stable G-invariant solution might indeed exist, if the Yang-Mills gauge group is isomorphic to the isotropy group H of the internal space *) $G|H$ [3].

To see why the instability persists in the general case, let us examine the linearized equations of motion for gravity coupled with SU(3) Yang-Mills. The action is

$$S = - \int d^6 x\, g^{1/2} \left[\frac{1}{\kappa^2} R + \frac{1}{4} \vec{F}^2 + \lambda\right] .$$

(For notation see Ref.[6].) It yields the following set of linearized equations:

$$\left(\partial^2 - \frac{1}{a^2} \ell(\ell+1)\right)(h_{aa} + 2h_{+-}) + \frac{2}{a^2} h_{+-} - \frac{\kappa}{a}\sqrt{\frac{\ell(\ell+1)}{2}} \left\langle i \vec{F}_{+-} \right\rangle (\vec{V}_- - \vec{V}_+)$$
$$- \frac{1}{2} T_{aa} - T_{+-} = 0 \qquad \ell \geq 0 \qquad (2a)$$

$$\left(\partial^2 - \frac{1}{a^2} \ell(\ell+1)\right) h_{ab} + \frac{1}{a^2} \eta_{ab} h_{+-} + \frac{1}{2} \frac{\kappa}{a} \eta_{ab} \sqrt{\frac{\ell(\ell+1)}{2}} \left\langle i \vec{F}_{+-} \right\rangle (\vec{V}_- - \vec{V}_+)$$
$$+ (T_{ab} - \frac{1}{4} \eta_{ab} T_{cc}) - \frac{1}{2} \eta_{ab} T_{+-} = 0 \qquad \ell \geq 0 \qquad (2b)$$

$$\left(\partial^2 - \frac{1}{a^2} \ell(\ell+1)\right) h_{a\pm} \pm \kappa \left\langle \vec{F}_{+-} \right\rangle \partial_a \vec{V}_\pm \mp \frac{\kappa}{a} \sqrt{\frac{\ell(\ell+1)}{2}} \left\langle i \vec{F}_{+-} \right\rangle \cdot \vec{V}_a + T_{a\pm} = 0 \qquad \ell \geq 1 \qquad (2c)$$

$$\left(\partial^2 - \frac{1}{a^2} \ell(\ell+1)\right) h_{\pm\pm} + T_{\pm\pm} = 0 \qquad \ell \geq 2 \qquad (2d)$$

$$\left(\partial^2 - \frac{\ell(\ell+1)+1}{a^2}\right) h_{+-} - \frac{3}{2} \frac{\kappa}{a} \sqrt{\frac{\ell(\ell+1)}{2}} \left\langle i \vec{F}_{+-} \right\rangle \cdot (\vec{V}_- - \vec{V}_+) - \frac{1}{4}(T_{aa} - 2T_{+-}) = 0 \qquad \ell \geq 0 \qquad (2e)$$

$$\left(\partial^2 - \frac{\ell(\ell+1) - \lambda^2}{a^2}\right) \vec{V}_a + \frac{\kappa}{a} \sqrt{\frac{\ell(\ell+1)}{2}} \left\langle i \vec{F}_{+-} \right\rangle (h_{a-} - h_{a+}) + \vec{J}_a = 0 \qquad \ell \geq |\lambda| \qquad (2f)$$

$$\left(\partial^2 - \frac{\ell(\ell+1) - (\lambda_\pm \mp 1)^2}{a^2}\right) \vec{V}_\pm \pm \kappa \left\langle \vec{F}_{+-} \right\rangle \partial_a h_{a\pm}$$
$$\pm \frac{\kappa}{a} \sqrt{\frac{\ell(\ell+1)}{2}} \left\langle i \vec{F}_{+-} \right\rangle (h_{+-} - \frac{1}{2} h_{aa}) + \vec{J}_\pm = 0 \qquad \ell \geq |\lambda_\pm| \qquad (2g)$$

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*) This is currently under investigation for the case of 10-dimensional supergravity theory [9] coupled to a supersymmetric Maxwell or Yang-Mills field.

Here h_{ab}, $h_{a\pm}$, $h_{\pm}$ represent the gravitational excitation fields, as defined in [6] and we have introduced source terms T_{AB} and $\vec{J}_A$. In writing Eqs.(2) we have imposed the covariant gauge conditions $D_A\vec{V}_A = 0$ and $\nabla_A(h_{AB} - \frac{1}{2} g_{AB} h_{CC})=0$. Notice the terms in Eq.(2g) that represent the graviton mixing with $V_{\pm}$,

$$\pm \kappa \langle F_{+-} \rangle \left(\partial_a h_{a\pm} + \frac{i}{a}\sqrt{\frac{\ell(\ell+1)}{2}}(h_{+-} - \frac{1}{2} h_{aa})\right) .$$

These terms are proportional to the background field strength,

$$\langle i F_{+-} \rangle = \frac{1}{a^2} H = \frac{1}{a^2} \mathrm{diag}(n_1, n_2, -n_1-n_2)$$

which affects only the diagonal components of $V_{\pm}$. The off-diagonal tachyons found previously for the pure SU(3)-Yang-Mills case are unaffected by the inclusion of gravity. By choosing the gauge group to be as small as possible, i.e. no larger than H, such effects would be minimized. This gives the motive for our conjecture that stability may possibly be achieved for Yang-Mills coupled to gravity, provided that the Yang-Mills gauge group is isomorphic to the isotropy subgroup H of the internal space $G|H$. The solutions of the equations exist as shown in Ref.[3].

We would like to conclude with a related remark. If the Yang-Mills group was not SU(3) but U(1) × SO(10), compactification on S^2 with internal SO(10) unbroken would be a stable configuration. We have shown in Ref.6 that massless spinors will arise. With the inclusion of SO(10) these can be interpreted as families of SO(10). There will be $2\ell+1 = |n|$ families where ℓ is the SU(2) quantum number associated with rotations of S^2 and n characterizes the appropriate vacuum solution. From this point of view, Kaluza-Klein theory is motivating a gauged SU(2) × U(1) family symmetry with coupling strengths $\sim \kappa/a$ which could be $\ll 1$ (so that the scale a^{-1} could even be in the TeV range). We do not know if this possible association of Kaluza-Klein theory with family symmetry has been remarked on before.

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