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CHIRAL ANOMALIES AND DIFFERENTIAL GEOMETRY

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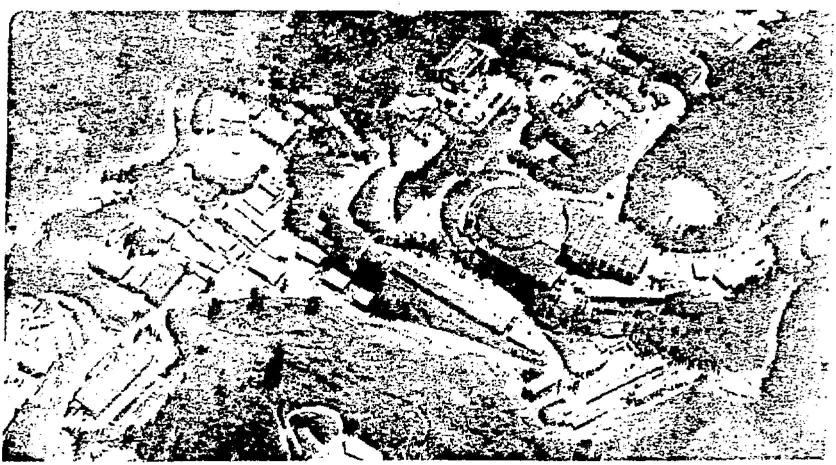

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CHIRAL ANOMALIES AND DIFFERENTIAL GEOMETRY

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1. INTRODUCTION

In these lectures I shall describe a number of properties of chiral anomalies from a geometric point of view. I follow mostly work done in collaboration with Raymond Stora [1]. Some of the results are contained in a recent paper written in collaboration with Wu Yong-Shi and Anthony Zee [2], to which I refer also for an extentive list of old and new references on chiral anomalies. It is possible that the methods and results described in these lectures are fully known in mathematics. On the other hand, several crucial formulas have not been given before (at any rate not explicitly) and their physical relevance is emphasized here.

As an introduction to the main subject let us consider some examples of the relevance of topology to physics:

(1) The Dirac monopole.

The action integral for an electron in the field of a magnetic monopole is given by

$$I = \int_1^2 L_{kin}\, dt + e\int_1^2 \vec{A}(\vec{x},t)\cdot\frac{d\vec{x}}{dt}\, dt \quad . \qquad (1.1)$$

The integral is over a path joining the point 1 to the point 2. Let us consider the second term in the action. If we deform the path of integration keeping the end points fixed, and then come back to the original path, the action returns to its original value, provided the deformation was not too large. However, if we swing the path about the position of the monopole and come back to the original path, we cut all flux lines of the Coulomb-like magnetic field. The action changes by an integer multiple of eg (g is the monopole charge)

$$I' = I + neg \quad . \tag{1.2}$$

One can say that the space of paths is infinitely connected. Classically this fact is not very important, since I′ gives the same equations of motion as I, but quantum mechanically it gives rise to a problem. For instance, the path integral

$$Z = \int \mathcal{D}(\text{path})\, e^{iI} \tag{1.3}$$

is not well defined, unless

$$neg = m\, 2\pi \qquad m, n \text{ integers}, \tag{1.4}$$

which is the Dirac quantization condition. [If this quantization condition is not satisfied, the path integral could be defined as vanishing by destructive interference, when one integrates over the infinitely connected space of paths.] This is a well known example of quantization of classical parameters due to topology. Other examples are

(2). Effective or phenomenological Lagrangions which arise as solutions of the anomalous Ward identities (see Section 4). Witten [3] and Balachandran, Nair and Trahern [4] have observed that a phenomenon similar to that occurring for the Dirac monopole occurs here, except that the 1 dimensional path is replaced by a 4 dimensional sphere.

(3). Non-linear σ-model coupled to supergravity [5]. One finds that Newton's constant has to take quantized values, i.e. multiples of F_π^{-2}.

(4). Three-dimensional Yang-Mills theory [6]. The topological mass of the vector field has a quantized value.

(5). In the Weinberg-Salam model there may exist heavy (unstable) soliton states [7]. The Higgs sector of the model has a global $SU(2)_L \times SU(2)_R$ symmetry. It is a linear σ-model but for large Higgs mass it can be approximated by a non-linear one, hence may have soliten solitions.

A common feature of all of these examples is that they make use of homotopy groups, so a list of the homotopy groups of the classical groups may be useful as a guide for a systematic search. Without going into the details of the definitions let us say roughly that the q^{th}-homotopy group Π_q of a (topological) space X is the set of mappings of S^q (the q-dimensional sphere) into the space X, where two mappings are considered as equivalent when one can be continously deformed into the other. We are interested in homotopy groups of groups i.e. the space X is a classical compact Lie group G. Here is the list of homotopy groups of the classical groups [8].

Π_q : q	U(N)	O(N)	Sp(N)
	N>q/2	N>q+1	N>1/4(q-2)
0	0	Z_2 [P]	0
1	Z [EM]	Z_2 [spin]	0
2	0	0	0
3	Z	Z	Z [Instantions]
4	0	0	Z_2 [Witten]
5	Z [Chir. Lag.]	0	Z_2
6	0	0	0
7	Z	Z	Z
8	0	Z_2	0
period:	2	8	8

(Z : the integers; Z_2 : the group of 2 elements)

The table exhibits the Bott periodicity theorem. Provided the group is sufficiently "large" (the inequalities are indicated) the homotopy groups follow a series of period 2 in the first column (U(N)) and of period 8 in the other two columns (O(N) and Sp(N)). Observe also that the homotopy groups for O(N) and for Sp(N) follow the same pattern, only shifted by 4 (half the period). See Milnor's book [9] last chapter, for a proof of the Bott periodicity theorem.

Remarks:

(1). $\Pi_2 = 0$ for all three classes and also for the exceptional groups (E. Cartan).

(2). Π_0 refers to the connectedness of the group

Π_0 (O(N)) = Z_2 is related to parity

$\Pi_1(O(N \geq 3)) = Z_2$ is related to spin.

(3). The Dirac monople has to do with $\Pi_1(U(1)) = Z$. Note that for the 't Hooft-Polyakov monople the relevant quantity is $\Pi_2(SU(2)/U(1)) = \Pi_1(U(1))$. For homotopy groups of quotients of groups see Hilton [10].

(4). The instanton has to do with $\Pi_3(SU(2)) = Z$. Note that $Sp(1) = SU(2)$.

(5). Witten [11] has pointed out that an SU(2) gauge theory with an odd number of chiral fermion doublets is inconsistent. This is related to $\Pi_4(Sp(1)) = Z_2$.

(6). Chiral Lagrangians (cf. Section 4) are related to $\Pi_5(U(N \geq 3)) = Z$.

(7). $\Pi_3 = Z$. This fact is related to chiral solitons [12], [13].

2. CHIRAL ANOMALIES AND DIFFERENTIAL FORMS

A simple way to introduce the subject of this lecture - the anomalies associatd with chiral fermions - is to consider the Lagrangians

$$L = i\bar{\psi}\gamma^{\mu}(\partial_{\mu} - i\lambda_{k}A_{\mu}^{k})\psi \qquad (2.1)$$

in 4-dimensional space-time, where the ψ's are Dirac spinors, A_{μ}^{k} a set of external vector fields and λ^{k} the generators of a representations of an internal symmetry group (like SU(2), SU(3)). Let us futhermore introduce an axial current operator

$$J_{\mu}^{5} = \bar{\psi}\gamma_{\mu}\gamma_{5}\psi \qquad (2.2)$$

which is a singlet under the internal group, and look at its classical conservation equations

$$\partial^{\mu}J_{\mu}^{5} = 0 \quad . \qquad (2.3)$$

It is well established [14] that in the one-loop approximation of perturbation theory the classical conservation equation breaks down. If one requires vector gauge invariance, the axial vector equation takes the form

$$\partial^{\mu}J_{\mu}^{5} = -\frac{1}{16\pi^{2}}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma} \quad . \qquad (2.4)$$

Here

$$F_{\mu\nu} = -i\,\lambda_{k}F_{\mu\nu}^{k}$$

and $F_{\mu\nu}{}^k$ is the usual Yang-Mills field strength associated with the fields $A_\mu{}^k$ ($\varepsilon_{0123} = 1$). In terms of the latter the "singlet" or "abelian" anomaly, as we shall call the r.h.s. of (2.4), can be written as

$$\partial^\mu J_\mu^5 = -\frac{1}{4\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,\partial_\mu\left(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right). \quad (2.5)$$

Equally well one may consider two currents constructed analogously to the above: fermions are split into left/right one's

$$\psi_{\substack{L\\R}} = \frac{1 \pm \gamma_5}{2}\,\psi \qquad (2.6)$$

and one starts from a Lagrangian in which they are coupled to corresponding left/right vectors fields

$$L = i\bar{\psi}_L\gamma^\mu(\partial_\mu - iA^k_{\mu L}\lambda_k)\psi_L + i\bar{\psi}_R\gamma^\mu(\partial_\mu - iA^k_\mu\lambda_k)\psi_R \quad (2.7)$$

Now, all currents

$$J^H_{\mu i} = \bar{\psi}_H\,\gamma_\mu\,\lambda_i\,\psi_H \qquad H = L, R \qquad (2.8)$$

are covariantly conserved in the classical approximation but lead, upon proper definition [14], to anomalous equations

$$D_H^{\mu} J_{\mu i}^{H} = \eta_H \frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \, Tr\left(\lambda_i \partial_\mu \left(A_\nu^H \partial_\rho A_\sigma^H + \frac{1}{2} A_\nu^H A_\rho^H A_\sigma^H\right)\right)$$

$$H = L, R \quad , \qquad \eta_L = -\eta_R = -1 \quad , \tag{2.9}$$

in higher order. The r.h.s. of (2.9) will be called the non-abelian anomaly. Comparing the factor 1/2 in front of the trilinear A-term with the corresponding factor 2/3 in (2.5) it is clear that the non-abelian anomaly cannot be rewitten in terms of Yang-Mills curls. Nevertheless there is an intricate relation between the two types of anomalies which will be cleared up in the subsequent lectures. To point out that, differential geometric mehods will be used, which are going to be introduced presently.

In terms of <u>differential forms</u> the Yang-Mills fields A_μ^k will be represented by

$$A = -i \, A_\mu^k \, \lambda_k \, dx^\mu \tag{2.10}$$

a matrix of one-forms (i.e. having anti connecting elements), the field strength by

$$F = dA + A^2 \tag{2.11}$$

a matrix of two-forms (wedge symbol suppressed, matrix multiplication understood, elements of F commuting). It is easy to check that the Bianchi-identity

$$DF \equiv dF + [A, F] = 0 \tag{2.12}$$

holds, by making use of the fact that

$$d^2 = 0 \qquad (2.13)$$

and that d anti-commutes with the one-form A. The operation D in (2.12) is the covariant differential. In order to translate Eqs. (2.5)(2.9) into the language of forms we associate with the currents J_μ^5 one-forms $J_\mu^5\, dx^\mu$, go over to their duals $*J^5 = 1/3!\, \varepsilon_{\nu\lambda\mu\rho} J^{5\nu} dx^\lambda dx^\mu dx^\rho$ and then observe that evaluating the divergence (resp. the covariant divergence) is performed with the exterior derivation d (resp. the covariant D):

$$d{*}J^5 = (\partial^\lambda J_\lambda^5)\, \frac{1}{4!}\, \varepsilon_{\mu\nu\rho\sigma}\, dx^\mu dx^\nu dx^\rho dx^\sigma \quad , \qquad (2.14)$$

hence

$$d{*}J^5 \propto \mathrm{Tr}\, F^2 = d\, \mathrm{Tr}\left(A\,dA + \tfrac{2}{3} A^3\right) \quad , \qquad (2.15)$$

$$(\mathcal{D}{*}J^5)_i = -G_i(A) \propto d\, \mathrm{Tr}\, \lambda_i \left(A\,dA + \tfrac{1}{2} A^3\right) . \qquad (2.16)$$

As indicated in the first lecture the fact that the anomalies are d operating on something is crucial for their interrelation, so let us derive this fact. Consider e.g. $\mathrm{Tr}F^2$. Observe first

$$d\, \mathrm{Tr}\, F^2 = \mathrm{Tr}\left(dF\,F + F\,dF\right) = 2\, \mathrm{Tr}\, dF\,F \, . \qquad (2.17)$$

Adding zero in the form $2\mathrm{Tr}[A,F]F$ we obtain

$$d\, \mathrm{Tr}\, F^2 = 2\, \mathrm{Tr}\, \mathcal{D}F\, F = 0 \qquad (2.18)$$

due to the Bianchi identify (2.12). In order to find the form of which $\mathrm{Tr}F^2$ is the derivative we have to perform an integration and therefore look first of all at the variation of $\mathrm{Tr}\,F^2$ induced by varying A into $A + \delta A$.

$$F = dA + A^2 \;,\quad \delta F = d\delta A + \delta A\, A + A\,\delta A = D(\delta A), \tag{2.19}$$

(the signs are correct, A being a one-form.)

$$\begin{aligned}\delta\,\mathrm{Tr}\,F^2 &= 2\,\mathrm{Tr}\,\delta F\,F = 2\,\mathrm{Tr}\,D(\delta A)F \\ &= 2D\,\mathrm{Tr}\,\delta A\,F = 2d\,\mathrm{Tr}\,\delta A\,F \;.\end{aligned} \tag{2.20}$$

We have used the Bianchi-identity and that Tr δAF is a scalar.

Let us now intoduce the variation of A via a parameter t by

$$A_t = tA \quad,\quad F_t = t\,dA + t^2A^2 = tF + (t^2 - t)A^2. \tag{2.21}$$

The equation (2.20) may now be written

$$\delta\,\mathrm{Tr}\,F_t^2 = 2\,d\,\mathrm{Tr}\,\delta A_t\,F_t \tag{2.22}$$

and with $\delta = \delta t\,\partial/\partial t$ we have by integration

$$\int_0^1 \delta t\,\frac{\partial}{\partial t}\mathrm{Tr}\,F_t^2 = 2\,d\int_0^1 \delta t\;\mathrm{Tr}\,A\,F_t \quad, \tag{2.23}$$

hence

$$\begin{aligned}\mathrm{Tr}\,F^2 &= 2d\int_0^1 \delta t\;\mathrm{Tr}\;A\left(tF + (t^2 - t)A^2\right) \\ &= d\;\mathrm{Tr}\left(AF - \tfrac{1}{3}A^3\right).\end{aligned} \tag{2.24}$$

Calling the integral, which is a 3-form, ω_3 we write

$$\omega_3 = 2\int_0^1 \delta t \,\mathrm{Tr}\, A\left(t\,dA + t^2 A\;\right) \qquad (2.25)$$

$$= \mathrm{Tr}\left(A\,dA + \tfrac{2}{3}A^3\right) = \mathrm{Tr}\left(AF - \tfrac{1}{3}A^3\right)$$

and have thus verified that

$$\mathrm{Tr}\, F^2 = d\omega_3 \quad , \qquad (2.26)$$

Analogously one can proceed for higher powers Tr F^n since

$$d\,\mathrm{Tr}\, F^n = n\,\mathrm{Tr}\, dF\,F^{n-1} = n\,\mathrm{Tr}\, \mathcal{D}F\,F^{n-1} = 0 \;. \qquad (2.27)$$

The result is

$$\mathrm{Tr}\, F^n = d\,\omega_{2n-1} \qquad (2.28)$$

$$\omega_{2n-1} = n\,\mathrm{Tr}\int_0^1 \delta t\; A\, F_t^{\,n-1} \quad , \qquad (2.29)$$

Explicitly for n = 3

$$\omega_5 = 3\,\mathrm{Tr}\int_0^1 \delta t\; A\left(t\,dA + t^2 A^2\right)^2$$

$$= \mathrm{Tr}\left(A\,(dA)^2 + \tfrac{3}{5}A^5 + \tfrac{3}{2}A^3 dA\right) , \qquad (2.30)$$

$$\mathrm{Tr}\, F^3 = d\,\omega_5 \quad . \qquad (2.31)$$

The above considerations have to be generalized to the case where d(...) $\neq 0$. The appropriate tool is the so-called homotopy operator k : *it has the properties*

$$dk + kd = 1 \qquad (2.32)$$

$$k^2 = 0 \quad , \quad d^2 = 0 \quad . \qquad (2.33)$$

Let us suppose for the moment that k exists and check on the known case above what k does. Apply (2.32) on Tr F^2 :

$$(dk + kd)\, \mathrm{Tr}\, F^2 = \mathrm{Tr}\, F^2 \quad ; \qquad (2.34)$$

since d Tr $F^2 = 0$ this is simply

$$d(k\, \mathrm{Tr}\, F^2) = \mathrm{Tr}\, F^2 \quad . \qquad (2.35)$$

Hence, if k is known, Tr F^2 is readily expressed as a d(...).

The construction of k proceeds algebraically. Build out of F and A all those formal polynomials that vanish at $F = 0$, $A = 0$. Define an operation d on them by

$$dA = F - A^2 \quad , \qquad (2.36)$$

$$dF = FA - AF \qquad (2.37)$$

and the rule that it acts as anti-derivation (commutes with F, anti-commutes with A, and is linear on sums). Check that

$$d^2 = 0 \quad . \qquad (2.38)$$

Indeed: $d^2A = d(F - A^2) = FA - AF - dAA + AdA = 0$. similarly $d^2F = d(FA - AF) = 0$ (work out!).

Define another operation ℓ by

$$\ell A = 0 \tag{2.39}$$

$$\ell F = \delta A \tag{2.40}$$

and the antiderivation rule. Then verify that

$$\ell d + d \ell = \delta \tag{2.41}$$

on A : $\ell dA + d\ell A = \ell(F - A^2) = \delta A$

on F : $\ell dF + d\ell F = \ell(FA - AF) + d\delta A = \delta AA + A\delta A + \delta dA = \delta F$. Here we have assumed that δ commutes with d.

These definitions of d and ℓ can thus be extended to all formal polynominals (vanishing at F = 0, A = 0) and, in fact, be applied to families A_t, F_t depending on a parameter t:

$$\ell_t A_t = 0 \quad , \quad \ell_t F_t = \delta A_t \equiv \delta t \frac{\partial A_t}{\partial t} \quad , \tag{2.42}$$

with $A_0 = 0, F_0 = 0$.

The anti-commutations relation (2.41) becomes

$$\ell_t d + d \ell_t = \delta = \delta t \frac{\partial}{\partial t} \tag{2.43}$$

and integrating over t from 0 to 1 yields an explicit representation

$$K \equiv \int_0^1 \ell_t \tag{2.44}$$

with

$$kd + dk = 1 \quad . \tag{2.45}$$

Let us illustrate these abstract considerations by an example. Choose as polynomial AF. Then

$$d(AF) = F^2 - AFA \quad , \tag{2.46}$$

$$\ell d(AF) = \delta A\, F + F \delta A + A \delta A\, A \quad . \tag{2.47}$$

Choosing the t-family

$$\begin{aligned} A &\longrightarrow A_t = tA \quad , \\ \delta A &\longrightarrow \delta A_t = \delta t \frac{\partial A_t}{\partial t} = \delta t A \quad , \\ F &\longrightarrow F_t = tF + (t^2 - t) A^2 \quad , \end{aligned} \tag{2.48}$$

we have

$$\begin{aligned} \ell d(AF) \rightarrow \ell_t d(A_t F_t) &= \delta A_t F_t + F_t \delta A_t + A_t \delta A_t A_t \\ &= \delta t \, (A F_t + F_t A + t^2 A^3) \quad . \end{aligned} \tag{2.49}$$

Hence integrating over t from 0 to 1

$$
kd(AF) = \int_0^1 \delta t \left(A F_t + F_t A + t^2 A^3 \right)
$$

$$
= \frac{1}{2}(AF + FA) \ . \qquad (2.50)
$$

On the other hand

$$
\ell(AF) = -A\delta A \longrightarrow \ell_t(A_t F_t) = -\delta t \, t \, A^2, \qquad (2.51)
$$

$$
\int_0^1 \ell_t(A_t F_t) = k(AF) = -\frac{1}{2} A^2 \ , \qquad (2.52)
$$

$$
dk(AF) = -\frac{1}{2} dA^2 = -\frac{1}{2}(FA - AF) \ . \qquad (2.53)
$$

Adding (2.50) and (2.53)

$$
(kd + dk)(AF) = AF \ , \qquad (2.54)
$$

as desired. The lesson we learn therefore is that one must perform first of all the ℓ-operation term by term and then integrate. It is to be noted also that ℓ_t depends on t since $\ell_t F_t = \delta A_t = \delta t \partial A_t / \partial t$ is a variation along the one-parameter family at the point t. On the contrary d is t-independent. Observe that, in the example discussed above, one can verify by direct computation that the square of the operator k vanishes

$$
k^2 = 0
$$

Actually, it is not difficult to show that this is a general fact, when k is defined by means of the family (2.48). We leave the proof as an exercise to the reader.

A word of caution. Equations such as (2.24) and (2.26) are really valid only locally, in some finite neighborhood in x-space. It is however well known that they can be given a global meaning by using a connection on a principal fibre bundle, rather than a vector potential on the base (x-space).

Observe that the forms ω_{2n-1} are local expressions, constructed with the gauge potential and its derivatives up to some finite order, all calculated at a given point (see (2.29)).

Finally we emphasize that in defining the operators d, ℓ and k and in studying their properties (from formula (2.36) on) we have treated A and F as purely algebraic objects from which one can form freely polynomials. No special relations (such as particular commutation relations) have been used and the polynomials were not restricted by any symmetry or invariance property.

3. TRANSFORMATION PROPERTIES OF THE ANOMALIES

The key question whose answer eventually leads to the characterization of the anomalies is: how do they transform under a gauge transformation?

We have seen in the last Section that

$$\mathrm{Tr}\, F^n = d\, \omega_{2n-1}{}^0 \qquad (3.1)$$

with

$$\omega_{2n-1}{}^0 = k\, \mathrm{Tr} F^n = n \int_0^1 \delta t\, \mathrm{Tr}\, A\, F_t^{\,n-1} \quad , \qquad (3.2)$$

(the additional superscript ^{0} is introduced for later convenience), where k was the homotopy operator. Under a finite gauge transformation g(x) the field A transforms into

$$A_g = g^{-1} A g + g^{-1} dg \quad , \qquad (3.3)$$

hence $F = dA + A^2$ into

$$F_g = g^{-1} F g \quad . \qquad (3.4)$$

Under this transformation $\mathrm{Tr}F^n$ is clearly invariant, but how does $\omega_{2n-1}{}^0 = \omega_{2n-1}{}^0(A,F)$, understood as function of A and F, change? Certainly $\omega_{2n-1}{}^0$ may (and will, in general) change by a term $d\alpha$, α being a (2n-2) - form, since this contribution is annihilated by applying the d-operator yielding $\mathrm{Tr}F^n$. But it turns out, that

$$\omega_{2n-1}{}^{0}(A_g, F_g) = \omega_{2n-1}{}^{0}(A,F) + d\,\alpha_{2n-2} + \omega_{2n-1}(g^{-1}dg, 0), \quad (3.5)$$

i.e. the transformed $\omega_{2n-1}{}^{0}$ contains besides $d\alpha$ a term which globally cannot be written as d(...) and nevertheless is annihilated by d, the form $\omega_{2n-1}(g^{-1}dg,0)$ is closed. Let us now derive this result.

Dropping for the moment the indices we write the gauge transformed $\omega_{2n-1}{}^{0}(A_g,F_g)$:

$$\begin{aligned} \omega(A_g, F_g) &= \omega(g^{-1}Ag + g^{-1}dg,\ g^{-1}Fg) \\ &= \omega(A+V, F), \end{aligned} \qquad (3.6)$$

$$V \equiv dg\,g^{-1}, \qquad dV = V^2, \qquad (3.7)$$

since ω is given by a trace. We want to use now the homotopy operator k for obtaining information about $\omega(A + V, F)$, but $\omega(A + V, F) \neq 0$ at $A = 0$ $F = 0$, so we have to substract $\omega(V,0)$. It is convenient to subtract one more term: $\omega(A,F)$. Hence consider

$$\Omega \equiv \omega(A+V,F) - \omega(V,0) - \omega(A,F); \qquad (3.8)$$

observe that

$$d\Omega = 0. \qquad (3.9)$$

$$\begin{aligned} \text{Indeed:} \quad & d\omega(A+V,F) = \mathrm{Tr}\,F^n, \\ & -d\omega(V,F) = 0 \\ & -d\omega(A,F) = -\mathrm{Tr}\,F^n. \end{aligned}$$

Recall (2.32)

$$(dk + kd)\Omega = \Omega \tag{3.10}$$
i.e.
$$d(k\Omega) = \Omega .$$

We have thus indentified the (2n-2) - form α:

$$\alpha_{2n-2} = k\left(\omega_{2n-1}{}^{0}(A+V,F) - \omega_{2n-1}{}^{0}(A,F) - \omega_{2n-1}{}^{0}(V,0)\right). \tag{3.11}$$

This completes the proof of (3.5) if we also use

$$\omega(dg\, g^{-1}, 0) = \omega(g^{-1} dg, 0). \tag{3.12}$$

Actually, $k^2 = 0$ and $\omega_{2n-1}{}^{0}(A,F) = k\mathrm{Tr}F^n$ eliminate the second term in (3.11). Also, $k\omega_{2n-1}{}^{0}(V,0) = 0$, so that

$$\alpha_{2n-2} = k\left(\omega_{2n-1}{}^{0}(A+V,F)\right). \tag{3.13}$$

<u>Exercise</u>: Calculate α_{2n-2} for $n = 2,3$ (Note: in actual calculations it may be simpler to carry along the term $\omega_{2n-1}{}^{0}(A,F)$ in Ω.)

<u>Result:</u>

$$n = 2 \qquad \alpha_2 = -\mathrm{Tr}(VA) \qquad V = dg\, g^{-1} \tag{3.14}$$

$$n = 3 \qquad \alpha_4 = \mathrm{Tr}\left(-\tfrac{1}{2}V(FA+AF) + \tfrac{1}{2}VA^3 + \tfrac{1}{4}VAVA + \tfrac{1}{2}V^3A\right)$$
$$= \mathrm{Tr}\left(-\tfrac{1}{2}V(AdA+dAA) - \tfrac{1}{2}VA^3 + \tfrac{1}{4}VAVA + \tfrac{1}{2}V^3A\right) \tag{3.15}$$

$$\omega_3{}^{0}(A_g, F_g) = \omega_3{}^{0}(A,F) + d\alpha_2 - \tfrac{1}{3}\mathrm{Tr}(g^{-1}dg)^3 \tag{3.16}$$

$$\omega_5{}^{0}(A_g, F_g) = \omega_5{}^{0}(A,F) + d\alpha_4 + \tfrac{1}{10}\mathrm{Tr}(g^{-1}dg)^5 . \tag{3.17}$$

Equation (3.16) has a well known application to instantons. Equation (3.17) will be used in Section 4 and could in principle serve as a definition for the anomaly as well, but a slightly more sophisticated derivation yields the anomaly in a more convenient form, so let us proceed to this one.

We shall distinguish the differentiation in direction of x from that in direction of the group and denote the former by d, the latter by δ:

$$d = dx^\mu \frac{\partial}{\partial x^\mu} \tag{3.18}$$

$$\delta = dt^r \frac{\partial}{\partial t^r} \tag{3.19}$$

(x^μ are coordinates in space-time; t^r any parameters upon which the group elements may depend). In gauge transformation too, we shall separate these variations:

$$A \to g^{-1}Ag + g^{-1}dg + g^{-1}\delta g \ , \tag{3.20}$$

g depends on both x and t, while A is a form in x alone. Clearly

$$\Delta = d + \delta \quad , \qquad \Delta^2 = 0 \tag{3.21}$$

since

$$d^2 = \delta^2 = d\delta + \delta d = 0 \quad . \tag{3.22}$$

For

$$\mathcal{A} \equiv g^{-1}Ag + g^{-1}dg \quad , \tag{3.23}$$

$$v \equiv g^{-1}\delta g \quad , \tag{3.24}$$

one verifies

$$\delta \mathcal{A} = -dv - v\mathcal{A} - \mathcal{A}v = -\mathcal{D}v \qquad (3.25)$$

$$\delta v = -v^2 \qquad . \qquad (3.26)$$

Now $F = dA + A^2$, therefore

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 = g^{-1}Fg \; . \qquad (3.27)$$

Notice that also

$$\mathcal{F} = \Delta(\mathcal{A}+v) + (\mathcal{A}+v)^2 \; , \qquad (3.28)$$

as easily verified. This implies that

$$\Delta\, \omega_{2n-1}^{\,0}(\mathcal{A}+v, \mathcal{F}) = d\,\omega_{2n-1}^{\,0}(\mathcal{A}, \mathcal{F}) \, , \quad (3.29)$$

and both sides equal

$$\mathrm{Tr}\, \mathcal{F}^n = \mathrm{Tr}\, F^n \, . \qquad (3.30)$$

Let us expand $\omega_{2n-2}^{\,0}(\mathcal{A}+v, \mathcal{F})$ in powers of v

$$\omega_{2n-1}^{\,0}(\mathcal{A}+v, \mathcal{F}) = \omega_{2n-1}^{\,0}(\mathcal{A}, \mathcal{F}) + \omega_{2n-2}^{\,1} + \cdots + \omega_0^{\,2n-1} \, , \quad (3.31)$$

where the superscript indicates the power of v. Equation (3.29) implies a set of relations

$$\delta\omega_{2n-1}{}^{0} + d\,\omega_{2n-2}{}^{1} = 0$$
$$\delta\omega_{2n-2}{}^{1} + d\,\omega_{2n-3}{}^{2} = 0$$
$$\cdots\cdots \qquad (3.32)$$
$$\delta\omega_{1}{}^{2n-2} + d\,\omega_{0}{}^{2n-1} = 0$$
$$\delta\omega_{0}{}^{2n-1} = 0$$

We shall see later that $\omega_{2n-2}{}^{1}$ is to be identified with the anomaly. Let us calculate it explicitly. Now, from (2.29), (2.21), we see that

$$\omega_{2n-1}{}^{0}(\mathcal{A}+v, \mathcal{F}) = n\int_0^1 \delta t\ \mathrm{Tr}\left((\mathcal{A}+v)\,\hat{\mathcal{F}}_t^{\,n-1}\right), \quad (3.33)$$

where

$$\begin{aligned}\hat{\mathcal{F}}_t &= t\,\mathcal{F} + (t^2-t)(\mathcal{A}+v)^2 \\ &= \mathcal{F}_t + (t^2-t)\{\mathcal{A}, v\} + (t^2-t)v^2 \quad . \end{aligned} \quad (3.34)$$

It is convenient to replace the trace by the symmetrized trace

$$\mathrm{Str}(B_1, B_2, \cdots, B_n) = \sum_{\mathrm{Perm.}} \frac{1}{n!}\,\mathrm{Tr}\left(B_{P(1)} \cdots B_{P(n)}\right). \quad (3.35)$$

To first order in v, (3.33) gives

$$n\int_0^1 \delta t\, Str\left(v\,\mathcal{F}_t^{n-1} + (t^2-t)\mathcal{A}\left(\mathcal{F}_t^{n-2}\{\mathcal{A},v\} + \mathcal{F}_t^{n-3}\{\mathcal{A},v\}\mathcal{F}_t + \cdots\right)\right.$$

$$= n\int_0^1 \delta t\, Str\left(v\,\mathcal{F}_t^{n-1} + (t^2-t)(n-1)\mathcal{A}\{\mathcal{A},v\}\,\mathcal{F}_t^{n-2}\right)$$

Using the invariance of Str, one can rewrite this as

$$n\int_0^1 \delta t\, Str\left(v\,\mathcal{F}_t^{n-1} + (t^2-t)(n-1)\left(\{\mathcal{A},\mathcal{A}\}v\,\mathcal{F}_t^{n-2} + \mathcal{A}v[\mathcal{A},\mathcal{F}_t^{n-2}]\right)\right)$$

$$= n\int_0^1 \delta t\, Str\left(v\left[\mathcal{F}_t^{n-1} + (t-1)(n-1)\left(t\{\mathcal{A},\mathcal{A}\}\mathcal{F}_t^{n-2} - \mathcal{A}[\mathcal{A}_t,\mathcal{F}_t^{n-2}]\right)\right]\right)$$

Now observe that

$$d\,\mathcal{F}_t^{n-2} = -\left[\mathcal{A}_t\,,\,\mathcal{F}_t^{n-2}\right]$$

and

$$\frac{\partial \mathcal{F}_t}{\partial t} = d\mathcal{A} + t\,\{\mathcal{A},\mathcal{A}\}$$

The above expression becomes

$$n\int_0^1 \delta t\, Str\left(v\left[\mathcal{F}_t^{n-1} + (t-1)(n-1)\left(\left(\frac{\partial \mathcal{F}_t}{\partial t} - d\mathcal{A}\right)\mathcal{F}_t^{n-2} + \mathcal{A}\,d\,\mathcal{F}_t^{n-2}\right)\right]\right)$$

and finally, integrating by parts with respect to t we find the result for anomaly,

$$\omega_{2n-2}{}^{1} = (n-1)\int_0^1 \delta t\,(1-t)\,Str\left(v\,d\left(\mathcal{A}\,\mathcal{F}_t^{n-2}\right)\right). \quad (3.36)$$

Let us give the explicit expressions for

$$n=2\,,\qquad \omega_2{}^{1} = Tr\left(v\,d\mathcal{A}\right)\,, \qquad (3.37)$$

$$n=3\,,\qquad \omega_4{}^{1} = Tr\left(v\,d\left(\mathcal{A}\,d\mathcal{A} + \tfrac{1}{2}\mathcal{A}^3\right)\right). \qquad (3.38)$$

Equation (3.36) is very convenient because it exhibits the anomaly in the canonical form in which the differential operates on a function of $\mathcal{A}$ and $\mathcal{F}$, while v is not differentiated. Equation (3.38) agrees with (2.16).

Equation (3.37) gives a 2-form in x space, which is the non-abelian anomaly in 2 dimensions. Similarly ω_4^1 gives the non-abelian anomaly in 4 dimensions and generally ω_{2n-2}^1 in 2n-2 dimensions. One may wonder wether the other forms, $\omega_{2n-k}^{k-1}(2 \leq k \leq 2n)$, are also relevant to physics. If one is interested in 4 dimensional space time, one must take

$$\begin{aligned} 2n-k &= 4 \\ k-1 &= 2n-5 \end{aligned} \tag{3.39}$$

For any $n \geq 3$ this gives a 4-form in x space, ω_4^{2n-5}, in which the infinitesimal gauge transformation v occurs an odd number of times, 2n-5. There is an infinite number of such forms, as n varies. According to unpublished work by I. Singer, these *generalized anomalies can be identified as obstructions to a definition of the Dirac* propagator in an external potential, globally in the space of all potentials.

4. IDENTIFICATION AND USE OF THE ANOMALIES

In the last Section we have defined the form ω_{2n-2}^{1} to be the non-abelian anomaly. We now wish to justify this definition. To this end we go back to Section 2, generalize appropriately the Lagrangian (2.7) to arbitrary (even) space-time dimensions, renormalize in the one loop approximation and consider the functional of one-particle-irreducible Green's functions W[A] to this order. Gauge transformations are now represented by functional differential operators

$$X_i(x) = -\partial_\mu \frac{\delta}{\delta A_\mu^i} - \left(A_\mu \times \frac{\delta}{\delta A_\mu} \right)_i \qquad (4.1)$$

The cross-product is constructed with the structure constants of the respective simple, compact Lie group under consideration. One may convince oneself that the X_i's form the algebra

$$\left[X_i(x), X_j(y) \right] = f_{ij}{}^{k} X_k(x)\, \delta(x-y), \qquad (4.2)$$

and also that their action on W[A] just yields the current (non-) conservation equation

$$X_i(x)\, W[A] = G_i[A](x) \qquad (4.3)$$

($G_i = 0$ would correspond to the conservation of the respective currents). Now the mere existence of the functional W[A], which we suppose to be ensured by appropriate renormalization, implies a consistency condition for the possible G_i's. Acting twice on W[A] and using (4.2) we derive [15]

$$X_i(x)\, G_j(y) - X_j(y)\, G_i(x) = f_{ij}{}^{k} G_k(x)\, \delta(x-y). \qquad (4.4)$$

Trivial solutions of these equations are, of course, readily found:

$$G_i(x) = X_i(x)\ \hat{G}[A] \qquad (4.5)$$

($\hat{G}[A]$ e.g. local) is a solution. But (4.4) has not been solved in all generality. The anomalies (2.9) arise as solutions of (4.4) which are not variations of a <u>local</u> functional in the basic fields of the theory: This feature we take as definition for the general case: any solution of (4.4) which is not a variation of a <u>local</u> functional in the basic fields (ψ, A_μ) we regard as anomaly.

Before showing that $\omega_{2n-2}{}^1$ does solve just (4.4) we have to reformulate the problem somewhat.

Let us introduce anti-commuting scalar fields (Faddeev-Popov fields) $v_i(x)$ and the notation

$$\begin{aligned} v\cdot X &\equiv \sum_i \int dx\ v_i(x) X_i(x) \\ v\cdot G &\equiv \sum_i \int dx\ v_i(x)\, G_i(x) \end{aligned} \qquad (4.6)$$

Then the consistency conditions (4.4) turn into

$$v\cdot X\ v\cdot G - \frac{1}{2}(v\times v)\cdot G = 0 \qquad (4.7)$$

Similarly the gauge transformation on $A_\mu{}^i$ may be reformulated:

$$\delta A_\mu = v\cdot X\ A_\mu \qquad (4.8)$$

Interpreting (4.7) as invariance manifestation of $v\cdot G$ suggests to transform also v_i:

$$\delta v_i = -\frac{1}{2}(v\times v)_i \qquad (4.9)$$

i.e. (4.7) becomes

$$\delta\,(v \cdot G\,) = 0 \qquad , \qquad (4.10)$$

It should be clear that (4.8), (4.9) are nothing but the BRS-transformations and (4.10) the Slavnov identity for the special case of currents. This statement is confirmed by showing

$$\delta^2 = 0 \quad , \qquad (4.11)$$

i.e. the transformations (4.8), (4.9) are nilpotent. Exercise: check (4.11).

Let us now go over to forms

$$A = -i\, A_\mu^{\ k}\, \lambda_k\, dx^\mu \qquad (4.12)$$

$$v = -i\, v^k\, \lambda_k \quad , \qquad (4.13)$$

where v is a 0-form with values in the Lie-algebra, and re-express (4.8), (4.9) as

$$\delta A = -dv - vA - Av \equiv -\mathcal{D}v \,, \qquad (4.14)$$

$$\delta v = -v^2 \quad . \qquad (4.15)$$

The consistency equation (4.10) becomes

$$\delta \int \mathrm{Tr}\; v \cdot G[A] = 0 \qquad (4.16)$$

or, equivalently,

$$\delta\; \mathrm{Tr}\; v \cdot G\,[A\,](x) = d\chi \quad , \qquad (4.17)$$

(d: exterior x-derivative; χ some quantity). Hence the δ defined in (4.14), (4.15) and the d in (4.17) fulfill the algebra (3.25) (3.26) and will be identified with those operators. What remains to be shown is thus only that (4.17) can be identified with

$$\delta\, \omega_{2n-2}{}^{1} = d(-\omega_{2n-3}{}^{2}) \qquad (4.18)$$

i.e. $\omega_{2n-2}{}^{1}$ with v·G[A](x) and χ with $\tilde{\omega}_{2n-2}{}^{2}$. Indeed let us look at the system (3.32)

$$\begin{aligned}
&\mathrm{Tr}\, F^{n} - d\,\omega_{2n-1}{}^{0} = 0\\
&\delta\, \mathrm{Tr}\, F^{n} = 0\\
&\delta\omega_{2n-1}{}^{0} + d\,\omega_{2n-2}{}^{1} = 0\\
&\delta\omega_{2n-2}{}^{1} + d\,\omega_{2n-3}{}^{2} = 0\\
&\ldots\ldots\ldots
\end{aligned}$$

We see that $\omega_{2n-2}{}^{1}$ is linear in v and satisfies the consistency condition. The problem of finding the most general solution of the consistency condition will not be discussed here [16].

In order to derive physical consequences from the presence of the anomalies we use the approach of phenomenological Largrangians [15]. We permit the presence of another multiplet of fields ξ_i (Lorentz-scalars) and try to adjust its transformation law under the gauge group so that the anomaly can be derived as variatin of a local functional of the gauge fields plus the fields ξ_i.

It turns out [15] that the law of non-linear realization

$$\xi \rightarrow \xi'(\alpha,\xi)\ : \quad e^{\alpha} e^{\xi} = e^{\xi'} \qquad (4.19)$$

is the correct one and that

$$W[A,\xi] = \int_0^1 \delta t\, e^{-t\,\xi\cdot X}\, \xi\cdot G[A] \qquad (4.20)$$

satifies

$$\alpha\cdot(X+Z)\,W[A,\xi] = \alpha\cdot G[A]\,, \qquad (4.21)$$

i.e. fulfills the anomalous Ward-identity. Here

$$Z_i = H_{ij} \frac{\delta}{\delta \xi_j} \qquad (4.22)$$

$$H_{ij} = \frac{\partial \xi'_j}{\partial \alpha_i}\bigg|_{\alpha=0}$$

generates the transformation of ξ (X is given in (4.1)). The identification $\xi_i = 1/F_\pi\ \pi_i$ in the local action $W[A,\xi] + F_\pi^{\ 2}/2\ \mathrm{Tr} \int dx\ \partial^\mu e^\xi\ \partial_\mu e^{-\xi}$ + normal solution shows that the anomaly contributes additional pion-pion and pion-vector interactions in the σ-model-type phenomenological action. Examples where these arguments have been successfully applied are the processes $\pi^0 \to 2\gamma$, $\eta \to \gamma\pi\pi$ etc. [15].

One can show directly [15] that (4.20) gives a solution of the anomalous Ward identity. Here instead we first rewrite it in a more geometric form from which this fact will follow. The factor $e^{-t\xi \cdot X}$ transforms A into $A_{g(t)}$ where abstractly speaking

$$g(t) = e^{-t\xi} \qquad (4.23)$$

which we may understand as a family of group elements parameterized by t. Hence varying this parameter is a variation δ in group space

$$g^{-1}(t)\ \delta g(t) = -\xi\,\delta t \ , \qquad (4.24)$$

$$v = -\xi\,\delta t \ . \qquad (4.25)$$

Thus

$$W[A,\xi] = \int dx \int_0^1 \delta t \; e^{-t\xi\cdot X} \, \xi_i G_i[A](x) = \int dx \int_0^1 \delta t \; \xi_i G_i[A_{g(t)}](x)$$

$$= -\int dx \int_0^1 \delta t \, \mathrm{Tr}\, v\cdot G\,[A_{g(t)}](x) \; . \qquad (4.26)$$

Interchanging the order of integration we can interpret the integral in group space: for any fixed t the x-integral is in fact one over the corresponding configuration $g_t(x)$ in group space, t = 0 parameterizing the identity e and t = 1 the element g(x). We therefore write (up to a sign)

$$W[A; g(x)] = \int_e^{g(x)} \omega_4^{\,1}(\mathcal{A}, v) \quad , \qquad (4.27)$$

Using the expansion

$$\omega_5(\mathcal{A}+v) = \omega_5^{\,0}(\mathcal{A}) + \omega_4^{\,1}(\mathcal{A}, v) + \cdots\cdots$$

we first note that

$$\omega_5^{\,0}(\mathcal{A}) = 0 \qquad , \qquad (4.28)$$

since $\omega_5^{\,0}(\mathcal{A})$ is a 5-form purely in x, but spacetime here is 4-dimensional; next we see that for the special parametrization (4.23)

$$v^2 = v^3 = \cdots = 0 \qquad , \qquad (4.29)$$

since there is only one independent differential δt. So we can write

$$W\left[A\,;g(x)\right]=\int_e^{g(x)}\omega_5\left(\mathcal{A}+v\right)\;. \qquad (4.30)$$

In higher dimensions we would write similarly

$$W\left[A,g(x)\right]=\int_e^{g(x)}\omega_{2n-1}\left(\mathcal{A}+v\right) \qquad (4.31)$$

(2n-2 dimensional space time). Observe that

$$(d+\delta)\,\omega_5\left(\mathcal{A}+v\right)=\mathrm{Tr}\left(g^{-1}Fg\right)=\mathrm{Tr}\,F^3\quad, \qquad (4.32)$$

which is a 6-form purely in x and therefore vanishes in 4 dimensions. Therefore (4.30) is invariant under deformations of the integration manifold provided the limits of integration are kept fixed (similarly for (4.31)).

We now show that W satisfies the anomalous Ward identity. This is a consequence of the second of Eqs. (3.32), slightly reinterpreted. Let us perform a gauge transformation

$$\begin{aligned} A &\longrightarrow h^{-1}Ah+h^{-1}dh=A_h\;, \\ g(x) &\rightarrow h^{-1}g(x)\,. \end{aligned} \qquad (4.33)$$

Observe that

$$\mathcal{A}+v=g^{-1}Ag+g^{-1}(d+\delta)g=\mathcal{a}(A,g)$$

satisfies

$$\mathcal{a}\left(A_h\,,g\right)=\mathcal{a}\left(A\,,hg\right)\;. \qquad (4.34)$$

Therefore

$$W[A_h, h^{-1}g] = \int_e^{h^{-1}g} \omega_5\left(\mathcal{A}(A_h, g')\right) = \int_e^{h^{-1}g} \omega_5\left(\mathcal{A}(A, hg')\right). \tag{4.35}$$

Change the integration variable from g' to g'' = kg', where k = h at the upper limit

and k = e at the lower limit. Then

$$W\left[A_h, h^{-1}g\right] = \int_e^{g} \omega_5\left(\mathcal{A}(A, hk^{-1}g'')\right). \tag{4.36}$$

If h is infinitesimal

$$h(x) = e + m(x) \quad , \tag{4.37}$$

then

$$k = e + m \quad , \tag{4.38}$$

where m = m(x) at the upper limit and m = 0 at the lower limit and

$$hk^{-1} = e + m(x) - m = e + n \quad . \tag{4.39}$$

So we must make an infinitesimal transformation (drop the double-primes)

$$\delta_m g = n g \tag{4.40}$$

and correspondingly

$$\begin{aligned} \delta_m \mathcal{A} &= \mathcal{A}(A, g + ng) - \mathcal{A}(A, g) \\ &= -dV - \mathcal{A}V - V\mathcal{A} \end{aligned} \tag{4.41}$$

where

$$\mathcal{V} = g^{-1}\delta_m g = g^{-1} n g \quad . \qquad (4.42)$$

Now, in analogy with (3.32),

$$\delta_m \omega_5\left(\mathcal{A}(A,g)\right) = -(d+\delta)\,\omega_4^1\left(g^{-1}ng, \mathcal{A}\right) \qquad (4.43)$$

(here δ_m is an even variation), therefore

$$\delta_m W = -\int_e^g (d+\delta)\,\omega_4^1 . \qquad (4.44)$$

The right hand side can be evaluated by Stokes' theorem. Since n vanishes at the upper limit, while $n = m(x)$ at the lower limit, we finally obtain the desired equation

$$\delta_m W = \int_X \omega_4^1\left(m(x), A\right) \quad . \qquad (4.45)$$

The expression (4.30) for the effective Lagrangian W can be simplified if one makes use of (3.5), (3.17), which implies

$$\omega_5(\mathcal{A}+v) = \omega_5(\mathcal{A}) + (d+\delta)\,\alpha_4 + \omega_5\left(g^{-1}(d+\delta)g\right) . \qquad (4.46)$$

Now, the first term $\omega_5(A)$ in the r.h.s. vanishes because it is a 5-form purely in x. The second term can be integrated by Stokes' theorem. Therefore (4.10), using (3.17), gives

$$W[A, g(x)] = \int_x \alpha_4(V, A) + \frac{1}{10}\int_{(5)}^{g(x)} \mathrm{Tr}\left(g^{-1}(d+\delta)g\right)^5. \qquad (4.47)$$

Here α_4 and V are as given in (3.14), (3.15). The last integral is extended over a 5-dimensional manifold in group space having the sphere g(x) as boundary (as x varies over S_4, g(x) describes a sphere in group space). The integral is invariant under deformations of the 5-manifold because the integrand is a closed form (in general

$$d\ \mathrm{Tr}\ V^{2n-1} = (2n-1)\ \mathrm{Tr}\ V^{2n} = 0 \qquad (4.48)$$

for $V = dgg^{-1}, dV = V^2$.) In (4.47) the dependence on the vector fields in explicit, since α_4 is explicitly known. It is polynomical only. One could use the simplified form (4.47) to show that W satisfies the anomalous Ward identity (4.45) (Excercise for the resader).

The last term in (4.47) is an integral in group space. For a group (like SU(3)) with a nontrivial Π_5, there exist 5-cycles C_5 such that the integral (write simply d for $d + \delta$)

$$\int_{C_5} \mathrm{Tr}\ (g^{-1}dg)^5 \neq 0 \qquad (4.49)$$

does not vanish; suitably normalized (see below) it equals an integer. This means that although the integral in (4.47) is unchanged if one performs small deformations of the 5 manifold, it is ambiguous for large deformations. This fact leads to a quantization of the effective action, as mentioned in the Introduction. The normalization to be chosen is 2π times that which gives an integer for (4.49).

5. NORMALIZATION OF THE ANOMALIES

The normalization of the form $\mathrm{Tr}F^n$, which enters in the abelian anomaly (2.5), (2.15) can be related to the formula for the index of the Dirac operator. This gives the correctly normalized abelian (or singlet) anomaly in 2n dimensions. The connection between it and the non-abelain anomaly in 2n-2 dimensions permits then to find that normalization also. So both normalizations can be determined completely from purely geometric arguments.

First the singlet anormaly. In (compactified) Euclidean space-time, one writes

$$\partial^\mu J_{\mu\,reg}^{5} = C(x) - \sum_{zero\ modes} \phi_a^\dagger \gamma_5 \phi_a \quad , (5.1)$$

where C(x) is the anomaly, ϕ_a are normalized zero modes of the Dirac operator with a given external potential, and γ_5 means the analogue of γ_5 in any number of dimensions. $J_{\mu\,reg}^{5}$ *is the (suitably regularized) axial vector current.* The factor 2 comes from carrying out the divergence, which gives the Dirac operator once on the spinor on the right and once on that on the left.

Integrating (5.1) over all space-time, the left hand side gives zero. Therefore

$$\int C(x)dx = 2\int \sum \phi_a^\dagger \gamma_5 \phi_a = 2(n_+ - n_-) \quad , (5.2)$$

where $n_+(n_-)$ is the number of zero modes of positive (negative) chirality. Their difference is the index. Now it is known (see eg. Ref. [17], Eq. (7.22)) that the index is given by the integral of the Chern character

$$ch(V) = \mathrm{Tr}\, e^{\frac{i}{2\pi}F} \qquad (5.3)$$

More precisely, in 2n dimensions

$$n_+ - n_- = \int \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n \mathrm{Tr}\, F^n \quad . \qquad (5.4)$$

There for (F = 1/2 $F_{\mu\nu} dx^\mu dx^\nu$)

$$C(x) = \frac{2}{n!}\frac{i^n}{(2\pi)^n 2^n} \mathrm{Tr}\, F_{\mu_1\mu_2} F_{\mu_3\mu_4} \cdots F_{\mu_{2n-1}\mu_{2n}} \varepsilon^{\mu_1\mu_2\cdots\mu_{2n}} . \qquad (5.5)$$

(This is real because $F_{\mu\nu} = -i\, F_{\mu\nu}{}^i \lambda_i$). This requires, of course, that one knows somehow the correct formula for the index. A nice derivation (for physicists) based on quantum mechanical supersymmety, has been given recently by Alvarez-Gaumé [18] and by Friedan and Windey [19].

In order to determine the normalization of the non-abelian anomaly, we shall proceed as follows. Since the non-abelian anomaly determines the phenomenological Lagrangian (see Sect. 4), we shall require that it be normalized so that the Lagrangian satisfies the quantization condition. As we shall see, the normalization of the non-abelian anomaly is then related directly to that of the index formula, without the extra factor 2 necessary for the singlet anomaly. Of course this procedure is, in a sense, like going backwards, and the normalization of the non-abelian anomaly can be computed directly in perturbation theory. What we are saying is that the perturbation theory result agrees with the correct normalization for the phenomenological Lagrangian, as required by geometric considerations.

Remember that

$$\mathrm{Tr}\, F^n = d\,\omega_{2n-1}(A, F) \qquad (5.6)$$

where

$$\omega_{2n-1}(A,F) = n\int_0^1 dt\ \mathrm{Tr}\, A\, F_t^{\,n-1}, \qquad (5.7)$$

$$F_t = tF + (t^2 - t)A^2. \qquad (5.8)$$

One finds

$$\omega_{2n-1}(V,0) = n\int_0^1 \delta t\ (t^2-t)^{n-1}\,\mathrm{Tr}\, V^{2n-1}. \qquad (5.9)$$

The integral is easily carried out (successive integrations by parts) with the result

$$n\int_0^1 \delta t\ (t^2-t)^{n-1} = (-1)^{n-1}\frac{(n-1)!\ n!}{(2n-1)!}. \qquad (5.10)$$

Multiplying (5.10) by the factor in front of the index formula, $1/n!\,(i/2\pi)^n$, (without the extra factor 2) gives

$$\frac{1}{n!}\left(\frac{i}{2\pi}\right)^n(-1)^{n-1}\frac{(n-1)!\,n!}{(2n-1)!} = (-1)^{n-1}\left(\frac{i}{2\pi}\right)^n\frac{(n-1)!}{(2n-1)!} \qquad (5.11)$$

From the fact that the index is an integer one can then deduce that the form

$$\left(\frac{i}{2\pi}\right)^n\frac{(n-1)!}{(2n-1)!}\,\mathrm{Tr}\, V^{2n-1}, \quad V = dg\,g^{-1}, \qquad (5.12)$$

also integrates to a integer, the integral being performed over a (2n-1)-cycle in group space (see [20]). Now we know that the phenomenological Lagrangian must be normalized with an additional factor 2π. This means that the non-abelian anomaly in 2n-2 dimensions is given by (up to a sign)

$$\frac{1}{n!}\frac{i^{n}}{(2\pi)^{n-1}}\;\omega_{2n-2}{}^{1}\;, \qquad (5.13)$$

with $\omega_{2n-2}{}^1$ given by the expansion (3.31). For $n = 3$, formula (5.13) with (3.38) agrees with (2.9).

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APPENDIX. A SIMPLE FORMULA FOR a_{2n-2}

It is often useful to have a simple explicit formula, (A.16) below, for the differential form a_{2n-2} occurring in Eqs. (3.5), (3.14) to (3.17). The formulas (3.11) or (3.13) are sufficient, but they still require some work to evaluate a_{2n-2}.

We define a connection depending upon two parameters λ and μ

$$\mathcal{A}_{\lambda,\mu} = \lambda A - \mu V \quad , \qquad (A.1)$$

where, as in the text,

$$V = dg\, g^{-1} \quad , \quad dV = V^2 \, . \qquad (A.2)$$

The corresponding field strength is

$$\mathcal{F}_{\lambda,\mu} = d\,\mathcal{A}_{\lambda,\mu} + \mathcal{A}_{\lambda,\mu}^2 \qquad (A.3)$$

and it satisfies the Bianchi identities

$$d\,\mathcal{F}_{\lambda,\mu} = -\left[\mathcal{A}_{\lambda,\mu}, \mathcal{F}_{\lambda,\mu}\right] \, . \qquad (A.4)$$

Differentiating (A.3) one finds

$$\frac{\partial \mathcal{F}_{\lambda,\mu}}{\partial \lambda} = dA + \left\{\mathcal{A}_{\lambda,\mu}, A\right\} \qquad (A.5)$$

and

$$\frac{\partial \mathcal{F}_{\lambda,\mu}}{\partial \mu} = -dV - \left\{\mathcal{A}_{\lambda,\mu}, V\right\} \, . \qquad (A.6)$$

We consider the integral

$$n \int \mathrm{Tr}\left(\left(\delta\lambda\, A - \delta\mu\, V\right) \mathcal{F}_{\lambda,\mu}^{\,n-1}\right) \qquad (A.7)$$

over a one-dimensional path, which is a clockwise triangle in the λ, μ plane going from the origin to the point (0, 1) to (1, 0) back to thr origin. On the segment from (1, 0) to (0, 0), $\mu = 0$, $\mathcal{A}_{\lambda,\mu} = \lambda A$, $\mathcal{F}_{\lambda,\mu} = \lambda dA + \lambda^2 A^2 = F_\lambda$ (as defined in 2.48), and therefore (A.7) equals, by (2.29)

$$n \int_1^0 \mathrm{Tr}\ \delta\lambda\ A\ F_\lambda^{\,n-1} = -\,\omega_{2n-1}(A, F). \qquad (A.8)$$

On the segment from (0, 0) to (0, 1), $\lambda = 0$, $\mathcal{A}_{\lambda,\mu} = -\mu V$, $\mathcal{F}_{\lambda,\mu} = -\mu dV + \mu^2 V^2$ $= (\mu^2 - \mu)V^2$. Therefore (A.7) equals

$$-n \int_0^1 \mathrm{Tr}\left(\delta\mu\, V\left((\mu^2-\mu)V^2\right)^{n-1}\right) = -\,\omega_{2n-1}(V, 0). \qquad (A.9)$$

On the segment from (0, 1) to (1, 0), $\lambda + \mu = 1$, $\mathcal{A}_{\lambda,\mu} = \lambda A + (\lambda - 1)V$, $\mathcal{F}_{\lambda,\mu} = F_\lambda + (\lambda^2 - \lambda)(V^2 + \{A, V\})$. Therefore (A.7) equals

$$n \int_0^1 \delta\lambda\ \mathrm{Tr}(A+V)\left(F_\lambda + (\lambda^2-\lambda)(V^2+\{A,V\})\right)^{n-1}$$
$$= \omega_{2n-1}(A+V, F) \qquad (A.10)$$

Finally, (A.7) integrated over the clockwise triangle equals

$$\omega_{2n-1}(A+V,F)-\omega_{2n-1}(A,F)-\omega_{2n-1}(V,0)\ , \qquad (A.11)$$

which is the expression we would like to equate to $d\alpha_{2n-2}$. If we consider $Tr(A\,\mathcal{F}_{\lambda,\mu}^{\ n-1})$ and $-Tr(V\,\mathcal{F}_{\lambda,\mu}^{\ n-1})$ as the two components of a 2-vector in the plane, we can apply Stokes' theorem to (A.7) and transform it into an integral over the <u>inside</u> of the triangle

$$n\iint Tr\left(\left(A\frac{\partial}{\partial\mu}+V\frac{\partial}{\partial\lambda}\right)\mathcal{F}_{\lambda,\mu}^{\ n-1}\right). \qquad (A.12)$$

Using (A.5) and (A.6), (A.12) becomes

$$n(n-1)\iint Str\left(\left(-A\,dV+V\,dA\right)\mathcal{F}_{\lambda,\mu}^{\ n-2} \qquad (A.13)\right.$$

$$\left.-A\{\mathcal{A}_{\lambda,\mu},V\}\mathcal{F}_{\lambda,\mu}^{\ n-2}+V\{\mathcal{A}_{\lambda,\mu},A\}\mathcal{F}_{\lambda,\mu}^{\ n-2}\right)$$

Using the invariance of Str, the last two terms can be rewritten

$$Str\left(-\{\mathcal{A}_{\lambda,\mu},V\}A\,\mathcal{F}_{\lambda,\mu}^{\ n-2}+V\{\mathcal{A}_{\lambda,\mu},A\}\mathcal{F}_{\lambda,\mu}^{\ n-2}\right)$$

$$=Str\left(VA\left[\mathcal{A}_{\lambda,\mu},\mathcal{F}_{\lambda,\mu}^{\ n-2}\right]\right) \qquad (A.14)$$

$$=-Str\left(VA\,d\,\mathcal{F}_{\lambda,\mu}^{\ n-2}\right)$$

where we have also used (A.4). Therefore (A.13) becomes

$$-n(n-1)\, d \iint \mathrm{Str}\left(VA\, \mathcal{F}_{\lambda,\mu}^{\,n-2}\right) \quad , \qquad (A.15)$$

whichfinally gives the desired formula

$$\alpha_{2n-2} = -n(n-1) \iint \mathrm{Str}\left(VA\, \mathcal{F}_{\lambda,\mu}^{\,n-2}\right) \,, \qquad (A.16)$$

as a two dimentional integral over the interior of the triangle: in (A.16)

$$\iint = \int_0^1 \delta\lambda \int_0^{1-\lambda} \delta\mu \qquad (A.17)$$

One may have preferred a one-dimensional integral formula for a_{2n-2} like that for ω_{2n-1}, but (A.16) is just as easy to evaluate. In the expansion of $\mathcal{F}_{\lambda\mu}^{\,n-2}$ one encounters only the integrals

$$\int_0^1 \delta\lambda \int_0^{1-\lambda} \delta\mu\; \lambda^h \mu^k = \frac{h!\; k!}{(h+k+2)!} \qquad (A.18)$$

As an exercise, the reader may check that (A.16) agrees with (3.14) (obvious) and (3.15) and then go on to the next case n = 4.

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Added Note

A very interesting (and rather mathematical) paper relevant to the subject of these lectures is: L. Bonora and P. Cotta-Ramusino, *Commun. Math. Phys.* 87, 589 (1983); See also L. Bonora, P. Cotta-Ramusino and C. Reina, *Phys. Lett.* 126B, 305 (1983).