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QUANTUM CHROMODYNAMICS,
CHIRAL SYMMETRY
AND BAG MODELS

by

Madeleine Soyeur

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Madeleine Soyeur
CEN de Saclay
Service de Physique Théorique
F-91191 - Gif-sur-Yvette Cédex

C O N T E N T S

1. Quarks in the sixties

Quarks were introduced in 1964 by Murray Gell-Mann[1] and George Zweig[2] as the building blocks of the hadronic SU(3) flavour symmetry. In this scheme, the low-lying baryons are made of three quarks and the low-lying mesons of one quark and one antiquark.

The fundamental triplet of (light) quarks consists of an isospin doublet (the u and d quarks) and of an isospin singlet (the s quark) of spin $\frac{1}{2}$. Their charge Q, baryon number B, isospin I, isospin projection I_3 and strangeness S are summarized in Table 1.

Table 1 : Light quark quantum numbers

Flavour	Q	B	I	I_3	S
u	$+\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0
d	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
s	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	- 1

The $SU(3)_F$ symmetry scheme provides a classification of the low-lying baryonic and mesonic states in SU(3) multiplets of degenerate masses. The observed hadrons fit quite well in this classification. For future reference, we show in Figs. 1 and 2 the baryon decuplet ($J^\pi = \frac{3}{2}^+$) and octet ($J^\pi = \frac{1}{2}^+$) and in Figs. 3 and 4 the meson pseudo-scalar ($J^\pi = 0^-$) and pseudo-vector ($J^\pi = 1^-$) octets. It is interesting to remark that the mass degeneracy in the meson pseudo-scalar octet is rather badly broken by the very small mass of the pion.

Following the assumption of Gell-Mann and Zweig, the spectroscopy and the properties of baryons and mesons have been studied extensively in the non-relativistic quark model[3], a description of hadrons analogous to the independent particle model of the nucleus. The dynamics of quarks and antiquarks is approximated by a non-relativistic Hamiltonian with

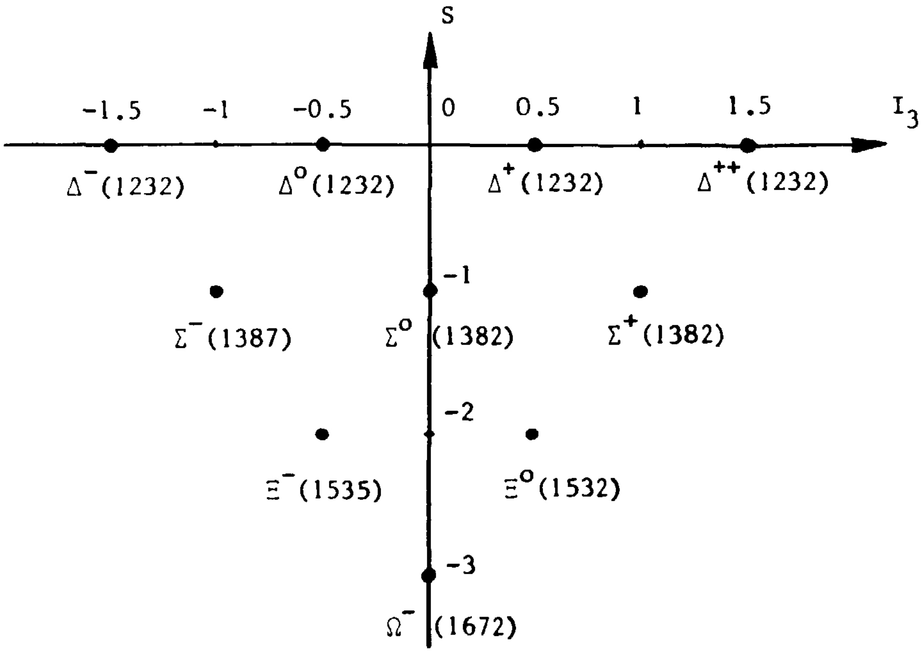

Figure 1 : Baryon decuplet

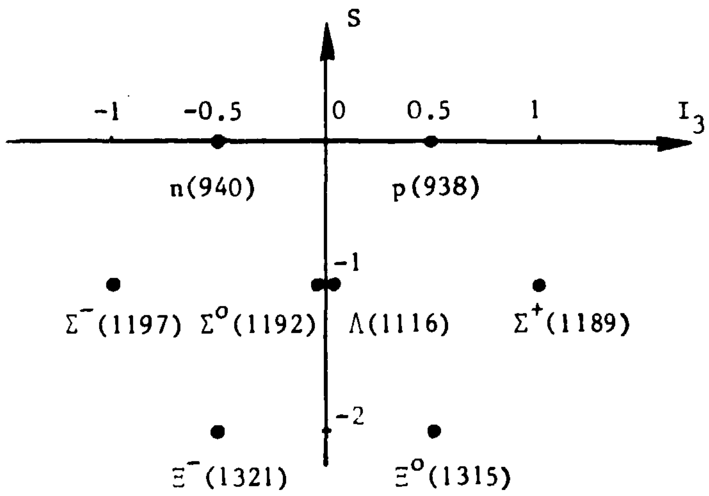

Figure 2 : Baryon octet

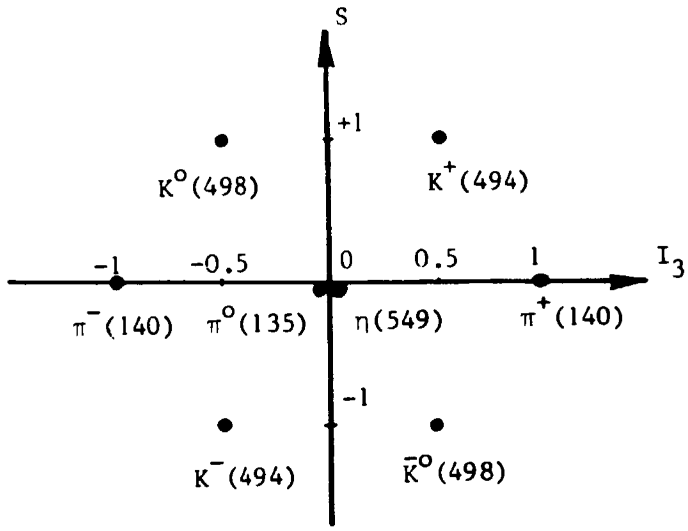

Figure 3 : Meson pseudo-scalar octet

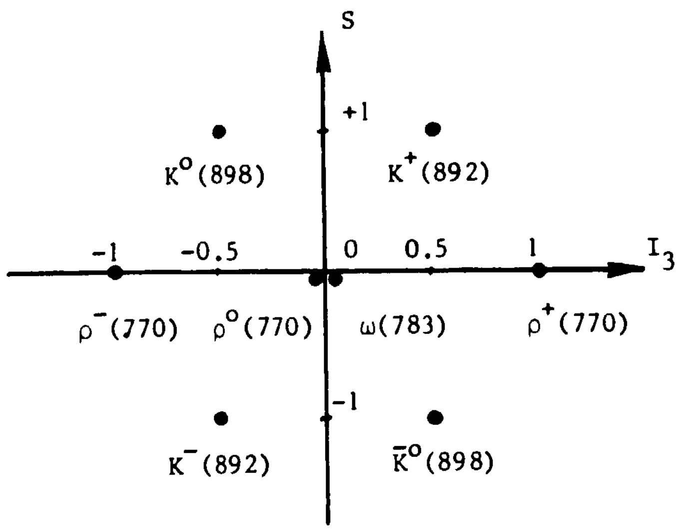

Figure 4 : Meson pseudo-vector octet

two-body binding forces ; the properties of hadrons are obtained by adding up the properties of quarks. The non-relativistic quark model provided a successful scheme to understand the spectrum and gross properties of hadrons but had a few unsatisfactory features. The most striking of these was the symmetry of the hadronic wavefunctions. Quarks are fermions and should therefore obey Fermi statistics ; however, in order to reproduce the experimental spectrum, baryons must have completely symmetric wavefunctions in the combined flavour and angular momentum space.

Beside the symmetry of the quark model wavefunctions, a few observations indicated the need for a new degree of freedom to describe strong interaction dynamics.

As noted by Jaffe and Johnson[4], the universality of the slope of Regge trajectories for baryons and mesons is very hard to understand if quarks are the only elementary constituents of hadrons. Regge trajectories are families of hadrons having the same isospin and strangeness quantum numbers whose spin J and mass M are related by

$$J = \alpha' M^2 + \alpha_o \qquad . \qquad (1.1)$$

For the known baryons and mesons, the slope α' is of the order of 0.9 Gev^{-2} [except for the pion for which $\alpha' \simeq 0.75$ Gev^{-2}]. Assuming the energy and the angular momentum to be carried by quarks only, the difference in the quark content of baryons and mesons should produce different values for the slopes of baryonic and mesonic trajectories, in contradiction with what is observed.

The need for a new quark quantum number taking three values is strongly suggested by the discrepancy between observed and calculated values for two quantities. These are the decay rate of the neutral pion into 2 photons and the ratio,

$$R = \frac{e^+e^- \rightarrow \mu^+\mu^-}{e^+e^- \rightarrow \text{hadrons}} \quad ,$$

of the high energy electron-positron annihilation into muon-antimuon pairs to the electron-positron annihilation into hadrons[5].

These problems were solved by the introduction of the color quantum number[6] and the advent of quantum chromodynamics[7]. Quarks are assumed to come in three colors and it is postulated that the physical hadrons are color singlets. The baryonic wavefunctions are therefore totally antisymmetric in color space and obey Fermi statistics.

The color charge of quarks turns out to be more than just an additional quantum number. It is thought to be the degree of freedom governing the dynamics of strong interactions. The corresponding quantum field theory is quantum chromodynamics (QCD).

2. Quantum chromodynamics[8)]

2.1. The classical Lagrangian of QCD

The classical Lagrangian of QCD reads[(*)]

$$\begin{aligned} L^{QCD}(x) = {} & \bar{\psi}^f_a(x)\; i\, \gamma^\mu\, \partial_\mu\; \psi^f_a(x) \\ & + \frac{g}{2}\, \bar{\psi}^f_a(x)\gamma^\mu[\lambda_i]_{ab}\; \psi^f_b(x)\; A^i_\mu(x) - \bar{\psi}^f_a(x) m_f\; \psi^f_a(x) \\ & - \frac{1}{4}\, F^i_{\mu\nu}(x)\; F^{i\mu\nu}(x) \quad , \end{aligned} \tag{2.1}$$

$$F^i_{\mu\nu}(x) \equiv \partial_\mu\, A^i_\nu(x) - \partial_\nu\, A^i_\mu(x) + g\, f_{ijk}\, A^j_\mu(x)\; A^k_\nu(x) \quad , \tag{2.2}$$

in which $\psi^f_a(x)$ is the quark field labelled by a three value colour index a and a flavour index f (≡ u,d,s,c,b...), A^i_μ is the gluon field labelled by an eight value color index i, g is the strong interaction coupling constant, λ_i and f_{ijk} are the hermitian generators and the structure constants of the SU(3) colour group and m_f are the flavour dependent quark masses. A sum over all flavour and colour indices is understood.

The meaning of the four terms of $L^{QCD}(x)$ is the following : the first term gives the quark kinetic energy, the second term describes the quark-gluon interaction, the third term accounts for the quark masses

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(*) We use the metric and conventions of J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields, Mc Graw-Hill, 1965.

and the fourth term contains the dynamics of gluons.

Quantum chromodynamics is a gauge theory. We shall now briefly explain what this means by showing explicitly how the Lagrangian (2.1) is derived.

Let us consider first the free massless Dirac Lagrangian for coloured quarks,

$$\begin{aligned} L_D(x) &= \bar{\psi}^f_a(x)\ i\ \gamma_\mu \partial_\mu\ \psi^f_a(x) \\ &= \bar{\psi}^f_a(x)\ i\ \gamma_\mu\ [\delta_{ab}\ \partial_\mu]\ \psi^f_b(x) \quad . \end{aligned} \tag{2.3}$$

We define a *global colour gauge transformation* on the quark fields by

$$\psi^f_a(x) \quad \rightarrow \quad U_{ab}\ \ \psi^f_b(x) \quad , \tag{2.4}$$

in which U is an SU(3) transformation acting in colour space that can be written as

$$U = e^{ig\theta^i \frac{\lambda_i}{2}} \quad , \tag{2.5}$$

$\{\theta^i\}$ being a set of eight arbitrary real parameters. Because the adjoint spinor $\bar{\psi}^f_a$ transforms like

$$\bar{\psi}^f_a(x) = \psi^{+f}_a(x)\gamma_o \rightarrow \bar{\psi}^f_b(x) \left[e^{-ig\theta^i \frac{\lambda_i}{2}}\right]_{ba} \quad , \tag{2.6}$$

the Lagrangian $L_D(x)$ is invariant for the global colour gauge transformation (2.4).

By Noether's theorem, to this global gauge invariance, there is an associated (eight component) conserved current

$$j^\mu_i(x) = g\ \bar{\psi}^f_a(x)\ \gamma^\mu \left[\frac{\lambda_i}{2}\right]_{ab}\ \psi^f_b(x) \quad , \tag{2.7}$$

$$\partial_\mu\ j^\mu_i(x) = 0 \quad , \tag{2.8}$$

and the colour charge,

$$Q_i = \int d^3\bar{x} \;\; j_i^o(x)$$

$$= -\int d^3\bar{x} \;\; g \, \psi_a^{+f}(x) \left[\frac{\lambda_i}{2}\right]_{ab} \psi_b(x) \quad , \qquad (2.9)$$

is a constant of the motion,

$$\frac{d\, Q_i}{dt} = 0 \quad . \qquad (2.10)$$

Now, we define *a local colour gauge transformation* on the quark fields by

$$\psi_a^f(x) \rightarrow U_{ab}(x)\, \psi_b^f(x)$$
$$= \left[e^{ig\theta^i(x)\frac{\lambda_i}{2}}\right]_{ab} \psi_b^f(x) \quad , \qquad (2.11)$$

$\{\theta^i(x)\}$ being this time a set of eight arbitrary real functions of x. The transformation of the derivative of the field, $\partial_\mu \psi_a^f(x)$, is given by

$$\partial_\mu \psi_a^f(x) \rightarrow \left[e^{ig\theta^i(x)\frac{\lambda_i}{2}}\right]_{ab} \partial_\mu \psi_b^f(x)$$
$$+ \left[ig \frac{\lambda_i}{2} \partial_\mu \theta^i(x)\, e^{ig\theta^i(x)\frac{\lambda_i}{2}}\right]_{ab} \psi_b^f(x) \quad . \qquad (2.12)$$

Clearly, because of the second term in (2.12), the Lagrangian (2.3) is not invariant under the local gauge transformation (2.11).

The simplest way to modify the Lagrangian (2.3) to make it invariant for local colour gauge transformations is to replace the ordinary derivative, ∂_μ, by a covariant derivative, D_μ, which transforms like the field,

$$[D_\mu]_{ab} \rightarrow \left[e^{ig\theta^i\frac{\lambda_i}{2}}\right]_{ac} [D_\mu]_{cb} \quad . \qquad (2.13)$$

This is achieved by choosing

$$[D_\mu]_{ab} = \delta_{ab}\,\partial_\mu - ig\left[\frac{\lambda_i}{2}\right]_{ab} A^i_\mu(x) \quad , \tag{2.14}$$

in which $A^i_\mu(x)$ are eight gauge fields which transform as

$$[A_\mu]_{ab} \rightarrow U_{ac}(x)[A_\mu]_{cd}(x)U^{-1}_{db}(x) + \frac{i}{g}\,U_{ac}(x)\,\partial_\mu\,[U^{-1}_{cb}(x)] \quad , \tag{2.15}$$

A_μ being a compact notation defined by

$$A_\mu \equiv A^i_\mu\,\frac{\lambda_i}{2} \quad . \tag{2.16}$$

Consequently, the Lagrangian

$$L'(x) = \bar{\psi}^f_a(x)\; i\gamma^\mu\; [D_\mu]_{ab}\;\; \psi^f_b(x) \quad , \tag{2.17}$$

is invariant for the local colour gauge transformation defined by (2.11) and (2.15). Adding a gauge invariant term of the form

$$-\frac{1}{4}\,F^i_{\mu\nu}(x)\;F^{i\mu\nu}(x) \quad , \tag{2.18}$$

to describe the free gauge field dynamics [$F_{\mu\nu}$ is defined by (2.2)], we obtain the QCD Lagrangian for massless quarks

$$\begin{aligned} L^{QCD}_{massless}(x) = {} & \bar{\psi}^f_a(x) i\,\gamma^\mu\,\partial_\mu\,\psi^f_a(x) + \frac{g}{2}\;\bar{\psi}^f_a(x)\gamma^\mu[\lambda_i]_{ab}\;\psi^f_b(x)A^i_\mu(x) \\ & - \frac{1}{2}\,\partial_\mu\;A^i_\nu(x)\;\partial^\mu\;A^{i\nu}(x) + \frac{1}{2}\,\partial_\mu\;A^i_\nu(x)\;\partial^\nu\;A^{i\mu}(x) \\ & - g\;f_{ijk}\;\partial^\mu\;A^{i\nu}(x)\;A^j_\mu(x)\;A^k_\nu(x) \\ & - \frac{1}{4}\,g^2\;f_{ijk}\;f_{i\ell m}\;A^j_\mu(x)\;A^k_\nu(x)\;A^{\ell\mu}(x)\;A^{m\nu}(x) \quad . \end{aligned} \tag{2.19}$$

We have written (2.18) in terms of the $A^i_\mu(x)$ fields to make the gluon dynamics more explicit. The first two terms of (2.18) give the gluon kinetic energy while the last two terms account for the gluon self-interactions. These self-interactions are cubic or quartic in the gauge field and responsible for the non-linearity of QCD.

The above derivation illustrates the dynamical character of local gauge invariance which dictates the nature of the bosons mediating the interaction and the way the fundamental and the gauge fields couple (at least in a minimal way).

2.2. Quark masses

The Lagrangian $\mathcal{L}^{QCD}_{massless}(x)$ derived in the previous paragraph is invariant for global flavour SU(3) gauge transformations,

$$\psi_a^{f_1}(x) \rightarrow \left[e^{ig\theta^i \frac{\lambda_i}{2}}\right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \qquad (2.20)$$

f_1 and f_2 being this time flavour indices.

As can be inferred for example from the masses of the hadrons belonging to the SU(3) flavour multiplets displayed in the first chapter, the SU(3) flavour symmetry is only an approximate symmetry of strong interactions. This is taken into account in QCD by adding to $\mathcal{L}^{QCD}_{massless}(x)$ a flavour symmetry breaking term. To be compatible with Dirac equation and to preserve the renormalizability of the theory, this term is of the form

$$\mathcal{L}_{SB}(x) = - m_f \bar{\psi}_a^f(x) \psi_a^f(x) \quad , \qquad (2.21)$$

in which m_f is the flavour dependent quark mass. Consequently, the complete Lagrangian of QCD is

$$\mathcal{L}^{QCD}(x) = \mathcal{L}^{QCD}_{massless}(x) + \mathcal{L}_{SB}(x) \quad . \qquad (2.22)$$

In bag models, the values obtained for the quark masses are

$$\begin{cases} m_u = m_d = 0 & \text{or a few Mev} \\ m_s \simeq 300 \text{ Mev} \end{cases} \qquad (2.23)$$

It is clear from the derivation of chapter 1 that the masses of quarks are not related to the colour or strong interaction dynamics. Our present understanding is that the origin of the flavour symmetry breaking term lies in the domain of weak and electromagnetic interactions[8].

2.3. The classical equations of motion of QCD

To derive the classical equations of motion of QCD, we rewrite the Lagrangian (2.1) in the symmetrized form

$$\begin{aligned} L^{QCD}(x) = {} & \frac{i}{2}\left\{\bar{\psi}_a^f(x)\gamma^\mu\partial_\mu\,\psi_a^f(x) - \partial_\mu\,\bar{\psi}_a^f(x)\gamma^\mu\,\psi_a^f(x)\right\} \\ & + \frac{g}{2}\,\bar{\psi}_a^f(x)\gamma^\mu\,[\lambda_i]_{ab}\,\psi_b^f(x)\,A_\mu^i(x) \\ & - \frac{1}{4}\,F_{\mu\nu}^i(x)\,F^{i\mu\nu}(x) - m_f\,\bar{\psi}_a^f(x)\,\psi_a^f(x) \quad . \end{aligned} \tag{2.24}$$

The equation of motion for the quark field is then

$$\frac{\partial L^{QCD}(x)}{\partial\bar{\psi}_a^f(x)} - \partial_\mu\,\frac{\partial L^{QCD}(x)}{\partial(\partial_\mu\,\bar{\psi}_a^f(x))} = 0 \quad , \tag{2.25}$$

i.e.,

$$(i\,\gamma^\mu\,\partial_\mu - m_f)\,\psi_a^f(x) + \frac{g}{2}\,\gamma^\mu\,[\lambda_i]_{ab}\,\psi_b^f(x)\,A_\mu^i(x) = 0 \quad . \tag{2.26}$$

The adjoint equation is given by

$$\frac{\partial L^{QCD}(x)}{\partial\psi_a^f(x)} - \partial_\mu\,\frac{\partial L^{QCD}(x)}{\partial(\partial_\mu\,\psi_a^f(x))} = 0 \quad , \tag{2.27}$$

i.e.,

$$\bar{\psi}_a^f(x)(-i\,\overleftarrow{\partial}_\mu\,\gamma^\mu - m_f) + \frac{g}{2}\,\bar{\psi}_b^f(x)\,\gamma^\mu\,[\lambda_i]_{ba}\,A_\mu^i(x) = 0 \quad . \tag{2.28}$$

Finally, the equations of motion for the gluon fields,

$$\frac{\partial L^{QCD}(x)}{\partial A_\mu^i(x)} - \partial_\nu\,\frac{\partial L^{QCD}(x)}{\partial(\partial_\nu\,A_\mu^i(x))} = 0 \quad , \tag{2.29}$$

become after a little bit of straightforward algebra, using the antisymmetry of the SU(3) structure constants under the permutation of two indices,

$$\partial_\mu F^{j\mu\nu}(x) + g\, f_{jk\ell}\, A^k_\lambda(x) F^{\ell\lambda\nu}(x) + g\, \bar{\psi}^f_a(x)\, \gamma^\nu \left[\frac{\lambda_j}{2}\right]_{ab} \psi^f_b = 0 \ . \qquad (2.30)$$

The covariant derivative acting on the strength tensor $F^{j\mu\nu}(x)$ is defined by[9]

$$[D_\mu]_{ab}\, F^b_{\nu\rho}(x) \equiv \partial_\mu F^a_{\nu\rho}(x) + g\, f_{abc}\, A^b_\mu(x)\, F^c_{\nu\rho}(x) \qquad , \qquad (2.31)$$

and therefore eqs.(2.30) can be rewritten as

$$[D_\mu]_{jk}\, F^k(x)^{\mu\nu} = -\, g\, \bar{\psi}^f_a(x)\, \gamma^\nu \left[\frac{\lambda_j}{2}\right]_{ab} \psi^f_b \ . \qquad (2.32)$$

These last equations are the generalization of Maxwell's equations to the SU(3) colour gauge fields. The source term in the right-hand side of eqs. (2.32) is the colour current. As can be seen most easily from (2.30), the equations of motion for the gluon fields are non-linear and therefore very hard to solve.

2.4. General Properties of QCD

For completeness, we mention briefly a few important properties of quantum chromodynamics. We refer to the original papers on QCD[7] or to the review article of Marciano and Pagels[10] for more details.

First, QCD can be made a consistent quantum field theory. The quantization procedure requires the addition of a gauge fixing term and of Faddeev-Popov ghost terms to the classical Lagrangian (2.1). In bag models, the dynamics of quarks and gluons is treated to first order in the coupling constant ; therefore, these terms do not come in and will not be discussed here.

Second, QCD is a renormalizable field theory. This implies that there are well-defined prescriptions to calculate the amplitudes of processes involving quarks and gluons to all orders in the coupling constant g.

Third, QCD is an asymptotically free field theory. This means that the quark-gluon coupling constant tends to zero at short distances or large momentum transfers. This property is a consequence of the gluon self-interactions. The deep inelastic lepton-nucleon scattering experiments[11)] have confirmed that at very short distances quarks behave as if they were essentially free. On the contrary at large distances (typically larger than hadronic sizes), quarks and gluons interact strongly and perturbative calculations in the coupling constant are no longer valid. As a consequence of this, it has not yet been possible to show that QCD provides a dynamical explanation for the observed confinement of quarks into hadrons and to calculate low-energy processes from the QCD Lagrangian. An important progress in this direction has been achieved by studying the theory on a lattice on the one hand and in the large N_c (N_c = number of colours) limit on the other hand.

2.5. Lattice QCD

The formulation of quantum chromodynamics on a discrete four-dimensional lattice is originally due to Ken Wilson[12)]. The essential interest of this formulation is that lattice QCD has a computable strong coupling limit and provides in this limit a mechanism for the absolute confinement of quarks[12)]. Lattice QCD is therefore an appropriate model to study phenomena associated with the strong coupling regime.

The Monte-Carlo method[13)] is now being used extensively to obtain informations on the phase structure of lattice QCD and to calculate on lattices the values of physical observables such as the masses of the lowest lying hadrons[14)]. At present, this approach suffers from important approximations. The two main problems are the lattice size and the lack of proper treatment of the quark degrees of freedom[13)]. Much work is in progress to remedy those mostly technical difficulties.

2.6. QCD in large N_c limit

As we have seen, a major problem to perform actual calculations in QCD is the absence of a small free parameter that could be used in perturbative expansions.

To generate such parameter, t'Hooft[15] has proposed a generalization of QCD in which the number of colours is changed from three to a large number N_c, the gauge group SU(3) being accordingly replaced by SU(N_c). In the large N_c limit, there exists a systematic expansion in the parameter $\frac{1}{N_c}$ [16]. The hope is that this expansion makes sense for $N_c = 3$ and that the large N_c limit will help understanding the non-perturbative (in the quark-gluon coupling constant) aspects of QCD.

It is rather easy to understand why the theory simplifies in the large N_c limit. The quark field in this limit has N_c components while the gluon field has N_c^2 components [for large N_c, $(N_c^2 - 1) \simeq N_c^2$]. The coupling constant scales like $\frac{g}{\sqrt{N_c}}$ [16]. Therefore, the only Feynman diagrams which remain in the large N_c limit are those for which the number of intermediate states compensates for the $\frac{1}{\sqrt{N_c}}$ factors at the interaction vertices. It can be shown that the following simple topological selection rule applies in the large N_c limit : non-planar Feynman diagrams are suppressed by factors of at least $\frac{1}{N_c^2}$ [16]. To leading order, only *planar diagrams* need to be evaluated. A second selection rule reduces further the number of planar diagrams to calculate : because there are N_c times more gluon states than quark states, *the internal quark loops are suppressed by factors of* $\frac{1}{N_c}$.

Despite these simplifications, the summation of the remaining diagrams is still out of reach ; in particular, one has not shown that QCD in the large N_c limit confines quarks. Assuming it does, interesting properties of mesons and baryons result from the selection rules mentioned above[16].

In the large N_c limit, QCD appears as a weakly coupled field theory of mesons. To zeroeth order ($N_c = \infty$), the mesons are free and stable. To leading order in $\frac{1}{N_c}$, they interact by exchanging mesons and their dynamics is described by an effective local Lagrangian with local vertices and local fields. Baryons in this weakly coupled field theory are interpreted as soliton states[16]. The characteristic property of baryons which supports this interpretation is that for large N_c, their mass diverges like N_c, i.e. like the inverse of the coupling (which goes as $\frac{1}{N_c}$ to leading order). We shall come back to this picture of baryons in chapter 5.

3. Chiral symmetry

3.1. Massless free Dirac theory

To become familiar with the concept of chiral symmetry, we shall first discuss the massless free Dirac theory described by the Lagrangian

$$L_D(x) = \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) = \psi^+(x)\gamma^o i\gamma^\mu \partial_\mu \psi(x) \quad , \qquad (3.1)$$

for which the equations of motion are

$$i\gamma^\mu \partial_\mu \psi(x) = 0 \quad , \qquad (3.2)$$

and

$$-i\partial_\mu \bar{\psi}(x)\gamma^\mu = 0 \quad . \qquad (3.3)$$

The *helicity* operator whose eigenvalues characterize the alignment of the spin and the 3-momentum $\bar{p}$ of a Dirac particle is given by

$$\begin{cases} \dfrac{\bar{\Sigma}.\bar{p}}{|\bar{p}|} & \text{for positive energy states,} \\ -\dfrac{\bar{\Sigma}.\bar{p}}{|\bar{p}|} & \text{for negative energy states,} \end{cases} \qquad (3.4)$$

in which $\bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$ is the spin operator.

The *chirality* operator is defined by

$$\gamma_5 = i\,\gamma_o\gamma^1\gamma^2\gamma^3 \quad , \qquad (3.5)$$

and it is easy to check , using the properties of the γ matrices, that

$$\bar{\Sigma} = \gamma^5\gamma^o\,\bar{\gamma} \quad . \qquad (3.6)$$

Then, multiplying the massless Dirac equation,

$$(\gamma_o p^o - \bar{\gamma}.\bar{p})\ \psi(x) = 0 \quad , \tag{3.7}$$

by $\gamma^5\gamma^o$, we find

$$\gamma^5\ \psi(x) = \frac{\bar{\Sigma}.\bar{p}}{|\bar{p}|}\,\psi \quad . \tag{3.8}$$

From (3.8), we see that the chirality of a positive energy state (or particle) is identical to its helicity while the chirality of a negative energy state (or antiparticle) is opposite to its helicity.

A particle is called *right-handed* if it has positive chirality (or helicity),

$$\gamma_5\ \psi_R(x) = \psi_R(x) \quad , \tag{3.9}$$

and *left-handed* if it has negative chirality (or helicity),

$$\gamma_5\ \psi_L(x) = -\psi_L(x) \quad . \tag{3.10}$$

A positive energy solution can be decomposed into right-handed and left-handed components,

$$\psi(x) = \psi_R(x) + \psi_L(x) \quad , \tag{3.11}$$

using the projection operators R and L such that

$$\psi_R(x) = R\ \psi(x) = \left(\frac{1+\gamma_5}{2}\right)\psi(x) \quad , \tag{3.12}$$

$$\psi_L(x) = L\ \psi(x) = \left(\frac{1-\gamma_5}{2}\right)\psi(x) \quad . \tag{3.13}$$

The Lagrangian (3.1) is invariant for the two global gauge transformations :

$$\psi(x) \rightarrow e^{-i\theta_V}\ \psi(x) \quad , \tag{3.14}$$

and

$$\psi(x) \rightarrow e^{-i\theta_A\gamma_5}\ \psi(x) \quad , \tag{3.15}$$

in which θ_V and θ_A are real arbitrary parameters. The invariance of (3.1) for the transformation (3.14) is obvious. Its invariance for (3.15) can be shown easily by making an infinitesimal transformation and using the property that γ^μ and γ_5 anticommute. The gauge group associated to each of the phase transformations (3.14) and (3.15) is the U(1) group. Therefore, the invariance property of (3.1) for the transformations (3.14) and (3.15) is called the *chiral* $U(1) \times U(1)$ *symmetry* of the free massless Dirac theory.

The Noether currents associated to the invariance of (3.1) for the transformations (3.14) and (3.15) respectively are the vector current,

$$V^\mu(x) = \psi^+(x)\gamma^o\gamma^\mu\ \psi(x) = \bar{\psi}(x)\gamma^\mu\ \psi(x) \quad , \tag{3.16}$$

and the axial vector current,

$$A^\mu(x) = \psi^+(x)\gamma^o\gamma^\mu\gamma^5\ \psi(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\ \psi(x) \quad . \tag{3.17}$$

It is easy to check that they are conserved as a consequence of the equations of motion (3.2) and (3.3). The corresponding charges,

$$Q = \int d^3\bar{x}\ \ V^o(x) = \int d^3\bar{x}\ \ \psi^+(x)\psi(x) \quad , \tag{3.18}$$

$$Q_5 = \int d^3\bar{x}\ \ A^o(x) = \int d^3\bar{x}\ \ \psi^+(x)\gamma^5\psi(x) \quad , \tag{3.19}$$

are constants of the motion,

$$\frac{dQ}{dt} = 0 \quad , \tag{3.20}$$

$$\frac{dQ_5}{dt} = 0 \quad . \tag{3.21}$$

Eq. (3.20) expresses the *fermion number conservation.*

The chiral $U(1) \times U(1)$ symmetry of the massless Dirac theory can be formulated equivalently in terms of the left and right components of the field defined in (3.12) and (3.13). To do this, we note that (3.1) can be rewritten as

$$L_D(x) = i\ \bar{\psi}_L(x)\gamma^\mu\partial_\mu\ \psi_L(x) + i\ \bar{\psi}_R(x)\gamma^\mu\partial_\mu\ \psi_R(x)\ . \tag{3.22}$$

It is then easy to see that (3.22) is invariant for the two phase transformations,

$$\psi_L(x) \rightarrow e^{-i\theta_L}\ \psi_L(x) \quad , \tag{3.23}$$

and

$$\psi_R(x) \rightarrow e^{-i\theta_R}\ \psi_R(x) \quad , \tag{3.24}$$

in which θ_L and θ_R are real, arbitrary parameters. The associated conserved Noether currents are

$$j^\mu_L(x) = \bar{\psi}_L(x)\gamma^\mu\ \psi_L(x) \quad , \tag{3.25}$$

$$j^\mu_R(x) = \bar{\psi}_R(x)\gamma^\mu\ \psi_R(x) \quad . \tag{3.26}$$

The left and right charges defined by

$$Q_L = \int d^3\bar{x}\ \ j^o_L(x) = \int d^3\bar{x}\ \psi^+_L(x)\ \psi_L(x) \quad , \tag{3.27}$$

$$Q_R = \int d^3\bar{x}\ \ j^o_R(x) = \int d^3\bar{x}\ \psi^+_R(x)\ \psi_R(x) \quad ,$$

are constants of the motion,

$$\frac{dQ_L}{dt} = 0 \quad , \tag{3.28}$$

$$\frac{dQ_R}{dt} = 0 \quad . \tag{3.29}$$

Eqs. (3.28) and (3.29) express the *separate conservation of the left and right fermion numbers.*

The invariance of the Lagrangian (3.22) for the U(1) gauge transformations (3.23) and (3.24) is the chiral $U(1)_L \times U(1)_R$ symmetry. We caution that the properties derived above hold only for free massless Dirac particles.

In the presence of an electromagnetic field, (3.21) is no longer true due to the axial anomaly[9].

3.2. Chiral symmetry of QCD

Consider the QCD Lagrangian introduced in chapter 2,

$$L^{QCD}(x) = \bar{\psi}_a^f(x) i\gamma^\mu \partial_\mu \psi_a^f(x) + \frac{g}{2} \bar{\psi}_a^f(x)\gamma^\mu [\lambda_i]_{ab} \psi_b^f(x) A_\mu^i(x)$$
$$- \frac{1}{4} F_{\mu\nu}^i(x) F^i(x)^{\mu\nu} - \bar{\psi}_a^f(x) m_f \psi_a^f(x) \quad . \tag{3.30}$$

We remark first that $L^{QCD}(x)$ is invariant for the phase transformation

$$\psi_a^f(x) \to e^{-i\theta_B} \psi_a^f(x) \quad , \tag{3.31}$$

in which θ_B is an arbitrary real parameter. We call $U(1)_B$ the corresponding gauge group. The corresponding conserved Noether current is the baryonic current,

$$j_B^\mu = \bar{\psi}_a^f(x)\gamma^\mu \psi_a^f(x) \quad , \tag{3.32}$$

where as usual, the summation over flavour and colour indices is understood. The baryonic charge,

$$Q_B = \int d^3\vec{x} \; j_B^o(x) = \int d^3\vec{x} \; \psi_a^{+f}(x) \psi_a^f(x) \quad , \tag{3.33}$$

is a constant of the motion,

$$\frac{dB}{dt} = 0 \quad . \tag{3.34}$$

Eq. (3.34) expresses the *baryon number conservation*. We do not discuss here the $U(1)_A$ transformations analogous to (3.15) for which classical derivations are incorrect due to the axial anomaly mentioned above[9].

Let us study now the invariances of $L^{QCD}(x)$ for flavour dependent gauge transformations.

First, it is clear that $L^{QCD}(x)$ is invariant for phase transformations involving only one flavour state,

$$\psi_a^f(x) \rightarrow e^{-i\theta_f} \psi_a^f(x) \quad , \tag{3.35}$$

in which the states $\psi_a^{f'}(x)$ with $f' \neq f$ are unaffected by the transformation. If we consider only the light quarks of Table 1, there are three such transformations. The corresponding gauge group is therefore the product $U(1)_u \times U(1)_d \times U(1)_s$.

The conserved Noether currents are

$$j_f^\mu(x) = \bar{\psi}_a^f(x)\gamma^\mu \psi_a^f(x) \quad , \tag{3.36}$$

in which only colour indices are summed over. The flavour charges,

$$Q^f = \int d^3\bar{x}\ \psi_a^{+f}(x)\ \psi_a^f \quad , \tag{3.37}$$

are constants of the motion,

$$\frac{dQ^f}{dt} = 0 \quad . \tag{3.38}$$

Eq. (3.38) express the *baryon number conservation for each flavour independently.*

Let us now assume that the up and down quark masses are equal, $m_u = m_d$, as suggested by isospin symmetry. In this case, $L^{QCD}(x)$ is invariant for global SU(2) gauge transformations acting in the 2-dimensional (u,d) isospin space,

$$\psi_a^{f_1}(x) \rightarrow \left[e^{-i\ \theta^i\ \frac{\tau^{(i)}}{2}} \right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \tag{3.39}$$

in which f_1 and f_2 are restricted to the u and d flavours, $\tau^{(i)}$ are the infinitesimal generators of SU(2) and $\{\theta^i\}$ is a set of three arbitrary real parameters.

The conserved Noether currents associated to this invariance are the three isospin components of the vector current

$$V^{(i)\mu}(x) = \bar{\psi}_a^{f_1}(x)\gamma^\mu \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \qquad (3.40)$$

with

$$\partial_\mu V^{(i)\mu}(x) = 0 \quad . \qquad (3.41)$$

The vector charge

$$Q^{(i)} = \int d^3\bar{x}\, \psi_a^{+f_1}(x) \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \qquad (3.42)$$

is a constant of the motion

$$\frac{dQ^{(i)}}{dt} = 0 \quad . \qquad (3.43)$$

Let us now discuss the invariance properties of $L^{QCD}(x)$ for transformations involving the chirality operator.

The mass term in (3.30) is clearly not invariant for such transformations. However, if we assume that $m_u = m_d = 0$, then $L^{QCD}(x)$ is invariant for the transformation

$$\psi_a^{f_1}(x) \rightarrow \left[e^{-i\,\phi^i\,\frac{\tau^{(i)}}{2}\,\gamma_5}\right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \qquad (3.44)$$

in which f_1 and f_2 are restricted to the u and d flavours and $\{\phi^i\}$ is a set of three arbitrary real parameters.

The conserved Noether currents associated to this invariance are the three isospin components of the axial vector current

$$A^{(i)\mu}(x) = \bar{\psi}_a^{f_1}(x)\gamma^\mu\gamma_5 \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \qquad (3.45)$$

with

$$\partial_\mu A^{(i)\mu}(x) = 0 \quad . \qquad (3.46)$$

The axial vector charge,

$$Q_5^{(i)} = \int d^3\bar{x}\, \psi_a^{+f_1}(x)\gamma_5 \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x) \quad , \qquad (3.47)$$

is a constant of the motion

$$\frac{dQ_5^{(i)}}{dt} = 0 \quad . \tag{3.48}$$

By quantizing the field $\psi_a^f(x)$ and using the equal time commutation relations between local currents involving γ matrices, one can show[17] that the vector and axial vector charges defined in (3.42) and (3.47) satisfy the commutation relations of Gell-Mann's current algebra[18],

$$[Q^{(i)}, Q^{(j)}] = i\epsilon_{ijk} Q^{(k)} \quad , \tag{3.49}$$

$$[Q^{(i)}, Q_5^{(j)}] = i\epsilon_{ijk} Q_5^{(k)} \quad , \tag{3.50}$$

$$[Q_5^{(i)}, Q_5^{(j)}] = i\epsilon_{ijk} Q^{(k)} \quad , \tag{3.51}$$

in which ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol. This symmetry of $L^{QCD}(x)$ for $m_u = m_d = 0$ is the *chiral SU(2) × SU(2) symmetry*.

A simple generalization of this result is that massless QCD with n flavours possesses the chiral SU(n) × SU(n) symmetry.

Again, the chiral SU(2) × SU(2) symmetry of QCD with $m_u = m_d = 0$ can be formulated in terms of the left and right components of the fields.

The Lagrangian (3.30) can be rewritten as

$$\begin{aligned} L^{QCD} = & \bar{\psi}^f_{aL}\, i\gamma_\mu \partial^\mu\, \psi^f_{aL} + \bar{\psi}^f_{aR}\, i\gamma^\mu \partial_\mu\, \psi^f_{aR} \\ & + \frac{g}{2}\, \bar{\psi}^f_{aL}\, \gamma^\mu\, [\lambda_i]_{ab}\, \psi^f_{bL}\, A^i_\mu(x) \\ & + \frac{g}{2}\, \bar{\psi}^f_{aR}\, \gamma^\mu\, [\lambda_i]_{ab}\, \psi^f_{bR}\, A^i_\mu(x) \\ & - \frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \bar{\psi}^f_{aL}\, m_f\, \psi^f_{aL} - \bar{\psi}^f_{aR}\, m_f\, \psi^f_{aR} \quad . \end{aligned} \tag{3.52}$$

Assuming $m_u = m_d = 0$, (3.52) is invariant for the gauge transformations

$$\psi^{f_1}_{aL} \rightarrow \left[e^{-i\theta^i_L \frac{\tau^{(i)}}{2}} \right]_{f_1 f_2} \psi^{f_2}_{aL} \quad , \tag{3.53}$$

$$\psi^{f_1}_{aR} \rightarrow \left[e^{-i\theta^i_R \frac{\tau^{(i)}}{2}} \right]_{f_1 f_2} \psi^{f_2}_{aR} \quad , \tag{3.54}$$

in which f_1 and f_2 are restricted to the u and d flavours. The time independent charges associated to the conserved Noether currents are

$$Q^{(i)}_L = \frac{1}{2} [Q^{(i)} - Q^{(i)}_5] \quad , \tag{3.55}$$

$$Q^{(i)}_R = \frac{1}{2} [Q^{(i)} + Q^{(i)}_5] \quad ; \tag{3.56}$$

they satisfy the algebra

$$[Q^{(i)}_R \; , \; Q^{(j)}_R] = i\varepsilon_{ijk} \, Q^{(k)}_R \quad , \tag{3.57}$$

$$[Q^{(i)}_L \; , \; Q^{(j)}_L] = i\varepsilon_{ijk} \, Q^{(k)}_L \quad , \tag{3.58}$$

$$[Q^{(i)}_L \; , \; Q^{(j)}_R] = 0 \quad . \tag{3.59}$$

This symmetry of $\mathcal{L}^{QCD}(x)$ for $m_u = m_d = 0$ is the *chiral* $SU(2)_L \times SU(2)_R$ *symmetry.*

If $m_u = m_d = m \neq 0$, the vector current is conserved [eq.(3.41)]. What happens to the divergence of the axial vector current ? Using the definition (3.45) and the equations of motion of QCD (2.26) and (2.28), we find

$$\begin{aligned} \partial_\mu A^{(i)\mu}(x) &= \left[\partial_\mu \bar{\psi}^{f_1}_a(x) \right] \gamma^\mu \gamma_5 \left[\frac{\tau^{(i)}}{2} \right]_{f_1 f_2} \psi^{f_2}_a(x) \\ &+ \bar{\psi}^{f_1}_a(x) \, \gamma^\mu \gamma_5 \left[\frac{\tau^{(i)}}{2} \right]_{f_1 f_2} \partial_\mu \, \psi^{f_2}_a(x) \\ &= im \, \bar{\psi}^{f_1}_a(x) \, \gamma_5 [\tau^{(i)}]_{f_1 f_2} \psi^{f_2}_a(x) \quad . \end{aligned} \tag{3.60}$$

The right-hand side of eq.(3.60) is non-zero and proportional to the quark mass. Recalling the basic equation of PCAC[19),

$$\partial^{\mu} A_{\mu}^{(i)}(x) = f_{\pi} m_{\pi}^{2} \phi_{\pi}^{(i)}(x) \quad , \tag{3.61}$$

in which f_{π}, m_{π} and $\phi_{\pi}^{(i)}$ are respectively the pion decay constant, the pion mass and an interpolating field for the pion, we can identify

$$\phi_{\pi}^{(i)}(x) = i \frac{m}{f_{\pi} m_{\pi}^{2}} \bar{\psi}_{a}^{f_1}(x)\gamma_5 [\tau^{(i)}]_{f_1 f_2} \psi_{a}^{f_2}(x) \ . \tag{3.62}$$

3.3. Realizations of chiral symmetry

How do we observe chiral symmetry in nature ? We have seen in chapter 1 that the SU(3) flavour symmetry manifests itself by the classification of the low-lying hadrons in SU(3) multiplets of approximately degenerate mass. What could be an analogous signature of the chiral SU(2) × SU(2) symmetry ?

The answer to this question requires a brief discussion of the two modes in which chiral symmetry can be realized. Chiral symmetry can be realized in two ways :

(i) the *Wigner-Weyl mode* in which the chiral charges annihilate the vacuum

$$Q^{(i)}|0\rangle = 0 \quad \text{and} \quad Q_5^{(i)}|0\rangle = 0 \quad ; \tag{3.63}$$

(ii) the *Nambu-Goldstone mode* in which only part of the chiral charges annihilate the vacuum.

Let us first explain briefly the consequences of the realization of chiral symmetry in the *Wigner-Weyl mode*[20]. To do this, we shall use the property that the parity operator P commutes with the vector charges and anticommutes with the axial charges,

$$P\, Q^{(i)} = Q^{(i)}\, P \quad , \tag{3.64}$$

$$P\, Q_5^{(i)} = -\, Q_5^{(i)}\, P \quad . \tag{3.65}$$

Then, consider a physical proton at rest. It is an eigenstate of the strong interaction Hamiltonian H_{st} with an eigenvalue equal to the proton mass M_p,

$$H_{st}|p\rangle = M_p|p\rangle \quad ; \tag{3.66}$$

we choose it to be of positive parity,

$$P|p\rangle = |p\rangle \quad . \tag{3.67}$$

We operate on the proton state with the vector charges and calculate the energy and parity of the states $Q^{(i)}|p\rangle$. In the limit of exact chiral symmetry [eq. (3.43)],

$$H_{st}\, Q^{(i)}|p\rangle = Q^{(i)}\, H_{st}|p\rangle = M_p\, Q^{(i)}|p\rangle \ , \tag{3.68}$$

and

$$P\, Q^{(i)}|p\rangle = Q^{(i)}\, P|p\rangle = Q^{(i)}|p\rangle \quad . \tag{3.69}$$

The states $Q^{(i)}|p\rangle$ are linear superpositions of states belonging to the proton isospin multiplet which are degenerate in mass. Similarly, if we calculate the energy and parity of the states $Q_5^{(i)}|p\rangle$, we find, in the limit of exact chiral symmetry [eq. (3.48)],

$$H_{st}\, Q_5^{(i)}|p\rangle = Q_5^{(i)} H_{st}|p\rangle = M_p\, Q_5^{(i)}|p\rangle \ , \tag{3.70}$$

and

$$P\, Q_5^{(i)}|p\rangle = -Q_5^{(i)}\, P|p\rangle = -\, Q_5^{(i)}|p\rangle \quad . \tag{3.71}$$

Therefore, the realization of the chiral $SU(2) \times SU(2)$ symmetry in the Wigner-Weyl mode implies that massive hadrons should appear in parity doublets of (approximately) degenerate mass. Such parity doublets are not observed.

The implications of the realization of chiral symmetry in the *Nambu-Goldstone mode* are contained in Goldstone's theorem[21)]. This theorem says that to each generator (chiral charge) which does not annihilate the vacuum, there is an associated pseudo-scalar boson. This seems to be the realization of chiral symmetry chosen by nature with

$$Q^{(i)}|0\rangle = 0 \qquad \text{and} \qquad Q_5^{(i)}|0\rangle \neq 0 \tag{3.72}$$

Indeed, in this case, there are three generators which do not annihilate the vacuum and by Goldstone's theorem, a triplet of pseudo-scalar bosons of zero mass. This triplet of pseudo-scalar bosons is identified with the three isospin components of the pion (π^-,π^o,π^+) whose anomalously small mass has been underlined in chapter 1. The fact that the pion is not a zero mass particle measures the breaking of the chiral SU(2)×SU(2) symmetry.

3.4. The σ-model

The σ-model[22)] is a phenomenological model of strong interactions in which chiral symmetry can be realized in both the Wigner-Weyl and the Nambu-Goldstone modes. The elementary fields of the model are an isodoublet of massless fermions (ψ_n and ψ_p if we deal with nucleons, ψ_u and ψ_d if we deal with quarks), an isotriplet of pseudo-scalar bosons ($\bar{\pi}$) and an isosinglet scalar meson (σ).

The Lagrangian of the σ-model consists of a chiral symmetric part and of a symmetry breaking term,

$$L^\sigma(x) = L^\sigma_{CS}(x) + L^\sigma_{SB}(x) \quad , \tag{3.73}$$

given by

$$\begin{aligned} L^\sigma_{CS}(x) = {} & \bar{\psi}^f(x) i\gamma^\mu\partial_\mu \psi^f(x) + g\, \bar{\psi}^{f_1}(x)\Big[\sigma(x)+i\bar{\pi}(x).\bar{\tau}\gamma_5\Big]_{f_1f_2} \psi^{f_2}(x) \\ & + \frac{1}{2}\Big\{[\partial_\mu\sigma(x)]^2 + [\partial_\mu\bar{\pi}(x)]^2\Big\} \\ & - \frac{\lambda}{4}\,[\sigma(x)^2 + \bar{\pi}(x)^2 - f^2]^2 \quad , \end{aligned} \tag{3.74}$$

$$L^\sigma_{SB}(x) = c\,\sigma(x) \quad , \tag{3.75}$$

in which g,λ,f and c are constants.

The Lagrangian $\mathcal{L}_{CB}^{\sigma}(x)$ is invariant for the (infinitesimal) chiral $SU(2) \times SU(2)$ transformations[19)]

$$\begin{cases} \psi^{f_1}(x) \rightarrow \left(1 - i\,\frac{\beta_i \tau^{(i)}}{2}\right)_{f_1 f_2} \psi^{f_2}(x) \quad , \\ \sigma(x) \rightarrow \sigma(x) \\ \bar{\pi}(x) \rightarrow \bar{\pi}(x) + \bar{\beta} \times \bar{\pi}(x) \quad , \end{cases} \tag{3.76}$$

$$\begin{cases} \psi^{f_1}(x) \rightarrow \left(1 - i\,\frac{\alpha_i \tau^{(i)}}{2}\,\gamma_5\right)_{f_1 f_2} \psi^{f_2} \quad , \\ \sigma(x) \rightarrow \sigma(x) - \bar{\alpha}.\bar{\pi}(x) \\ \bar{\pi}(x) \rightarrow \bar{\pi}(x) + \bar{\alpha}\,\sigma(x) \quad , \end{cases} \tag{3.77}$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are two sets of three real infinitesimal parameters.

The conserved Noether currents associated to this invariance are the three isospin components of the vector current,

$$V^{(i)\mu}(x) = \bar{\psi}^{f_1}(x)\gamma^{\mu}\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi^{f_2}(x) + \varepsilon^{ijk}\,\pi^j(x)\partial^{\mu}\pi^k(x) \tag{3.78}$$

and of the axial vector current,

$$A^{(i)\mu}(x) = \bar{\psi}^{f_1}(x)\gamma^{\mu}\gamma_5\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi^{f_2}(x) + \sigma(x)\partial_{\mu}\pi^i(x) - \pi^i\partial_{\mu}\sigma \quad . \tag{3.79}$$

In the presence of the symmetry breaking term $\mathcal{L}_{SB}^{\sigma}(x)$, the vector current is still divergenceless but the axial current is not :

$$\partial_{\mu} V^{(i)\mu} = 0 \qquad\qquad \partial_{\mu} A^{(i)\mu} = -c\,\pi^i(x) \quad . \tag{3.80}$$

To discuss the two possible realizations of chiral symmetry in the σ-model, we regard the fields $\sigma(x)$ and $\bar{\pi}(x)$ in (3.74) as classical and study the effective potential,

$$V(\sigma^2,\bar{\pi}^2) = \frac{\lambda}{4} [(\sigma^2+\bar{\pi}^2)^2 - 2f^2 (\sigma^2+\bar{\pi}^2) + f^4] \quad . \tag{3.81}$$

For the system to be stable, we must choose $\lambda > 0$.

The vacuum expectation values of the fields, $\langle\sigma\rangle$ and $\langle\bar{\pi}\rangle$, are obtained by minimizing the potential energy (3.81).

For $f^2 \leq 0$, $V(\sigma^2, \bar{\pi}^2)$ has a minimum at $\sigma = 0$ and $\bar{\pi} = 0$. The vacuum is defined by

$$\begin{aligned} \langle\psi^f\rangle &= 0 \quad , \\ \langle\sigma\rangle &= 0 \quad , \\ \langle\bar{\pi}\rangle &= 0 \quad ; \end{aligned} \tag{3.82}$$

it is invariant for chiral transformations and the symmetry is realized in the Wigner-Weyl mode. In this mode, the fermions are massless and the σ and $\bar{\pi}$ fields have degenerate masses, $m_\sigma = m_{\bar{\pi}} = -\frac{\lambda f^2}{2} > 0$.

For $f^2 > 0$, $V(\sigma^2,\bar{\pi}^2)$ has a minimum at $\sigma^2 + \bar{\pi}^2 = f^2$. If we ask that the vacuum has positive parity, then it is defined by

$$\begin{aligned} \langle\psi^f\rangle &= 0 \quad , \\ \langle\sigma\rangle &= \pm f \quad , \\ \langle\bar{\pi}\rangle &= 0 \quad ; \end{aligned} \tag{3.83}$$

it is not invariant for chiral transformations and the symmetry is realized in the Nambu-Goldstone mode. To study this mode, let us define the translated σ field,

$$\sigma' = \sigma - f \quad , \tag{3.84}$$

in such a way that $\langle\sigma'\rangle = 0$ [we have chosen the + sign in (3.83)]. The Lagrangian (3.74) becomes

$$\begin{aligned} L^\sigma_{CS} &= \bar{\psi}^f i\gamma^\mu\partial_\mu \psi^f + g\,\bar{\psi}^{f_1} [\sigma' + i\bar{\pi}.\bar{\tau}\,\gamma_5]_{f_1 f_2} \psi^{f_2} \\ &+ gf\,\bar{\psi}^f \psi^f + \frac{1}{2} [(\partial_\mu\bar{\pi})^2 + (\partial_\mu\sigma')^2] \\ &- \frac{\lambda}{4} (\sigma'^2 + \bar{\pi}^2)^2 - \lambda f\sigma' (\sigma'^2 + \pi^2) - \sigma'^2 \lambda f^2 \quad . \end{aligned} \tag{3.85}$$

The fermion field ψ has picked up a mass,

$$m_\psi = - gf \quad , \tag{3.86}$$

the pion field has become massless and the σ' mass is given by

$$m_{\sigma'} = \lambda f^2 \quad . \tag{3.87}$$

The chiral symmetry of the theory is no longer explicit. Its realization in the Nambu-Goldstone mode manifests itself by a triplet of massless pseudo-scalar bosons.

4. The M.I.T. Bag model

4.1. Basic assumptions and equations of motion

The M.I.T. bag model[23)] is a phenomenological model of hadrons formulated in terms of quark and gluon fields which incorporates the short distance (perturbative) dynamics of QCD and contains a prescription to ensure the confinement of quarks and gluons.

The quark and gluon fields are defined only in a small region of space, the bag, in which there exists a constant positive energy density (called B). Inside the bag, quarks and gluons interact weakly (as suggested by the asymptotic freedom of QCD) and the vacuum in this region is the perturbative vacuum. The region outside the bag is the true QCD vacuum.

Inside the bag, the dynamics of quarks and gluons is determined by a Lagrangian of the general form

$$\mathcal{L}^{\text{MIT bag}}(x) = [\mathcal{L}^{QCD}(x) - B] \quad \theta_B(x) \quad , \tag{4.1}$$

in which B is a positive constant and $\theta_B(x)$ is the step function,

$$\theta_B(x) \quad \begin{cases} = + 1 & \text{inside the bag} \quad , \\ = 0 & \text{outside the bag} \quad . \end{cases} \tag{4.2}$$

To simplify the derivation of the M.I.T. bag equations, let us first assume that inside the bag, the quarks are *free and massless*. Then (4.1) becomes

$$L^{\text{MIT bag}}(x) = \left\{ \frac{i}{2} \left[\bar{\psi}^f_a(x)\gamma^\mu \partial_\mu \psi^f_a(x) - \partial_\mu \bar{\psi}^f_a(x)\gamma^\mu \psi^f_a(x) \right] - B \right\} \theta_B(x) \quad , \tag{4.3}$$

and the equation of motion for the quarks inside the bag is the free Dirac equation,

$$\forall\ x \in B \qquad i\gamma^\mu \partial_\mu \psi^f_a(x) = 0 \quad . \tag{4.4}$$

The boundary conditions on the surface of the bag are chosen to confine the quark fields inside the bag.

The first requirement is that *no colour-flavour flux should leave the bag* or that the normal component of the colour-flavour flux should vanish on the surface,

$$\forall\ x \in S \qquad n_\mu\, j^\mu_{(ab)(f_1 f_2)}(x) = \bar{\psi}^{f_1}_a(x)\gamma.n\ \psi^{f_2}_b(x) = 0 \quad ; \tag{4.5}$$

the outward normal n_μ is a space-like vector ($n_\mu n^\mu = -1$).

The condition (4.5) is fulfilled if, on the surface, the quark field satisfies the equation

$$\forall\ x \in S \qquad i\gamma.n\ \psi^f_a(x) = \psi^f_a(x) \quad , \tag{4.6}$$

with the adjoint field satisfying

$$\forall\ x \in S \qquad -i\bar{\psi}^f_a(x)\ \gamma.n = \bar{\psi}^f_a(x) \quad . \tag{4.7}$$

Indeed, using (4.6) and (4.7), we find

$$\begin{aligned} \forall\ x \in S \qquad n_\mu\, j^\mu_{(ab)(f_1 f_2)}(x) &= -i\ \bar{\psi}^{f_1}_a(x)\ \psi^{f_2}_b(x) \quad , \\ &= i\ \bar{\psi}^{f_1}_a(x)\ \psi^{f_2}_b(x) \quad , \\ &= 0 \quad . \end{aligned} \tag{4.8}$$

The second requirement is that *no energy-momentum flux should leave the bag* or that the energy-momentum tensor inside the bag,

$$T^{\mu\nu}_{MIT\ bag} = \left\{\frac{i}{2}\,[\bar{\psi}^f_a\,\gamma^\mu\partial^\nu\,\psi^f_a - \partial^\nu\,\bar{\psi}^f_a\,\gamma^\mu\,\psi^f_a] + Bg^{\mu\nu}\right\}\theta_B\ , \qquad (4.9)$$

should be conserved,

$$\partial_\mu\,T^{\mu\nu}_{MIT\ bag} = 0 \quad . \qquad (4.10)$$

If $T^{\mu\nu}_D$ is the energy-momentum tensor of the free massless Dirac theory, we have

$$\begin{aligned}\partial_\mu\,T^{\mu\nu}_{MIT\ bag} &= \partial_\mu\,T^{\mu\nu}_D\,\theta_B(x) + (T^{\mu\nu}_D + B\,g^{\mu\nu})\,\partial_\mu\,\theta_B(x)\ ,\\ &= n_\mu\,(T^{\mu\nu}_D + B\,g^{\mu\nu})\,\delta_S(x) \quad . \qquad (4.11)\end{aligned}$$

where $\delta_S(x)$ is a surface delta function. The condition (4.10) becomes

$$\forall\ x\in S \qquad n_\mu\,T^{\mu\nu}_D + B\,n^\nu = 0 \quad , \qquad (4.12)$$

or

$$\forall\ x\in S \qquad B = -\frac{1}{2}\,n_\nu\,\partial^\nu\,[\bar{\psi}^f_a(x)\,\psi^f_a(x)] \quad . \qquad (4.13)$$

Eq. (4.13) means that the surface of the bag is a set of points of constant Dirac pressure. It is easy to see that B is an energy density inside the bag. Indeed, using (4.9), we find

$$E = \int T^{oo}_{MIT\ bag}(x)\,d^3\bar{x} = \int d^3\bar{x}\,[T^{oo}_D(x) + B]\,\theta_B(x) \quad . \qquad (4.14)$$

The equation of motion of the quarks inside the bag (4.4), the linear boundary condition (4.6) and the quadratic boundary condition (4.13) are the MIT bag equations for free and massless quarks.

These equations can be generalized to include massive quarks and quark-gluon interactions to first order in the coupling constant g. The equations of motion for the quarks and gluons inside the bag are those derived for QCD, i.e. eqs. (2.26) and (2.30). The linear boundary conditions for the quark and gluon fields are

$$\forall\ x \in S \qquad \begin{cases} i\gamma . n\ \psi_a^f(x) = \psi_a^f(x) , \\ n_\mu\ F^{i\mu\nu}(x) = 0 . \end{cases} \tag{4.15}$$

The quadratic boundary condition becomes

$$\forall\ x \in S \qquad B = -\frac{1}{2} n^\nu \partial_\nu\ [\bar{\psi}_a^f(x)\psi_a^f(x)] - \frac{1}{4} F^{i\mu\nu}(x)F_{\mu\nu}^i(x) . \tag{4.16}$$

4.2. The M.I.T. bag model and chiral symmetry

Is the chiral symmetry of massless QCD preserved by the M.I.T. bag boundary conditions ? As we have seen in chapter 3, this would imply the conservation of the vector and axial vector currents in the M.I.T. bag model.

The divergence of the vector current is given by

$$\begin{aligned} \partial_\mu\ V^{(i)\mu}(x) &\equiv \partial_\mu \left\{ \bar{\psi}_a^{f_1}(x)\gamma^\mu \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x)\ \theta_B(x) \right\} \\ &= \left\{ \partial_\mu\ \bar{\psi}_a^{f_1}(x)\gamma^\mu \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x) + \bar{\psi}_a^{f_1}(x)\gamma^\mu \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \partial_\mu \psi_a^{f_2}(x) \right\} \\ &\qquad \times\ \theta_B(x) \\ &+ \bar{\psi}_a^{f_1}(x) \left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} n.\gamma\ \psi_a^{f_2}(x)\ \delta_S(x) . \end{aligned} \tag{4.17}$$

The term proportional to $\theta_B(x)$ is zero as a consequence of the equations of motion inside the bag. The term proportional to $\delta_S(x)$ vanishes because of the linear boundary condition on the quark field. Therefore, the vector current is conserved,

$$\partial_\mu\ V^{(i)\mu}(x) = 0 . \tag{4.18}$$

The divergence of the axial vector current is given by

$$\partial_\mu A^{(i)\mu}(x) \equiv \partial_\mu \left\{\bar{\psi}_a^{f_1}(x)\gamma^\mu\gamma_5\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x)\theta_B(x)\right\}$$

$$= \left\{\partial_\mu\bar{\psi}_a^{f_1}(x)\gamma^\mu\gamma_5\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2}(x) + \bar{\psi}_a^{f_1}(x)\gamma^\mu\gamma_5\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \partial_\mu \psi_a^{f_2}(x)\right\} \times \theta_B(x)$$

$$+ \bar{\psi}_a^{f_1}(x)(n.\gamma)\gamma_5\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2} \delta_S(x) \quad . \qquad (4.19)$$

The term proportional to $\theta_B(x)$ is again zero as a consequence of the equations of motion inside the bag. The term proportional to $\delta_S(x)$ does not vanish. Using the linear boundary condition on the quark field, we find

$$\forall\ x \in S \qquad \partial_\mu A^{(i)\mu}(x) = i\ \bar{\psi}_a^{f_1}(x)\gamma_5\left[\frac{\tau^{(i)}}{2}\right]_{f_1 f_2} \psi_a^{f_2} \neq 0 \quad . \qquad (4.20)$$

Eq. (4.20) shows that the chiral $SU(2) \times SU(2)$ symmetry is explicitly violated by the boundary conditions of the M.I.T. bag model.

4.3. Spherical cavity approximation and M.I.T. bag parameters

The M.I.T. bag equations derived in paragraph 4.2. cannot be solved exactly, even for free massless quarks.

Therefore, one makes the additional assumption that, for the hadrons of lowest mass, the Dirac pressure is spherically symmetric. The bag surface is a sphere of fixed radius, classically at rest.[23)]

In the spherical cavity approximation, the bag equations (4.4), (4.6) and (4.13) become

$$i\ \gamma^\mu\partial_\mu\ \psi_a^f(r) = 0\ , \qquad r < R \qquad (4.21)$$

$$-i\ \vec{\gamma}.\hat{r}\ \psi_a^f(r) = \psi_a^f(r)\ , \qquad r = R \qquad (4.22)$$

$$B = -\frac{1}{2}\frac{\partial}{\partial r}\left[\bar{\psi}_a^f(r)\ \psi_a^f(r)\right] . \qquad r = R \qquad (4.23)$$

Eqs. (4.21) and (4.22) can be solved exactly. The lowest single quark mode (the "s $\frac{1}{2}$" mode) is given by

$$\psi^f_{K=-1,j=\frac{1}{2},m} = \frac{N(-1)}{\sqrt{4\pi}} \begin{pmatrix} i\, j_o\left(\frac{x-1}{R}\, r\right) u_m \\ -j_1\left(\frac{x-1}{R}\, r\right) \bar{\sigma}.\hat{r}\, u_m \end{pmatrix} e^{-i\omega_{-1} t} \quad , \tag{4.24}$$

in which j_o and j_1 are spherical Bessel functions, R is the bag radius, ω_{-1} is the frequency of the mode defined by

$$\omega_{-1} = \frac{x-1}{R} \quad , \tag{4.25}$$

with x_{-1} solution of

$$tg\ x_{-1} = \frac{x_{-1}}{1 - x_1} \quad , \tag{4.26}$$

N(-1) is the norm given by

$$N(-1)^{-2} = R^3\, j_o(x_{-1})\ \frac{2(x_{-1} - 1)}{x_{-1}} \quad , \tag{4.27}$$

and u_m are the Pauli spinors,

$$u_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad u_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

This mode is consistent with the quadratic boundary condition (4.23).

To see how the model works in its simplest version, let us consider a hadron made of n "s $\frac{1}{2}$" massless quarks. The mass of such hadron is given by

$$M = \int_{Bag} d^3\bar{x}\ \ [T^{oo}_D + B] = n\, \omega_{-1} + \frac{4\pi}{3}\ B\, R^3 \quad . \tag{4.28}$$

Minimizing M with respect to R, we have

$$\frac{\partial M}{\partial R} = 0 = -\frac{n x_{-1}}{R^2} + 4\pi\ B\, R^2 \quad , \tag{4.29}$$

i.e.,

$$B = \frac{n x_{-1}}{4\pi R^4} \tag{4.30}$$

and

$$M = \frac{4}{3} \frac{n x_{-1}}{R} \quad . \tag{4.31}$$

The solution of (4.26) is $x_{-1} = 2.04$. For a baryon made of three massless quarks in the lowest mode, the model predicts

$$R = 1.7 \text{ fm} \quad ,$$
$$B \simeq 11 \text{ Mev fm}^{-3} \quad . \tag{4.32}$$

In this simple model, strange and nonstrange baryons or baryons with the same flavour content but different spins have the same mass.

This degeneracy is resolved by allowing for a finite strange quark mass and by calculating, to lowest order in g, the residual interaction between the quarks resulting from their coupling to the gluon field[23]. This generates spin-dependent forces between the quarks. The colour magnetic interaction energy contributes significantly to the mass of hadrons ; the colour electric interaction energy is negligible.

Finally, the M.I.T. bag model accounts for the zero point motion of the fields inside the bag by a phenomenological correction to the mass formula of hadrons of the form $-\frac{Z_o}{R}$, Z_o being a dimensionless parameter. Consequently, the M.I.T. bag model depends on four parameters : the bag constant B, the strange quark mass m_s, the quark-gluon coupling constant $\alpha_c = \frac{g^2}{4\pi}$ and the zero point motion parameter Z_o.

4.4. Properties of hadrons in the M.I.T. bag model

The values of the M.I.T. bag model parameters B, Z_o and α_c are chosen to fit the proton, $\Delta(1232)$ and $\omega(783)$ masses ; the strange quark mass is determined from the $\Omega^-(1672)$ mass. The numerical values obtained this way are

$$B = 58 \text{ Mev fm}^{-3} \quad ,$$

$$Z_o = 1.84 \quad ,$$

$$\alpha_c = 2.2 \quad , \qquad (4.33)$$

$$m_s = 279 \text{ Mev} .$$

With these values, the masses, charge radii and magnetic moments of hadrons have been calculated[23]. The results shown in Table 2 indicate that there is a good quantitative agreement between the experimental masses of the low lying hadrons and those predicted by the M.I.T. bag model except for the pion which is twice too heavy. Somewhat disturbing is the very large contribution of the zero point energy term whose parametrization is not too well understood. The average baryon radius is of the order of 1 fm.

Table 2 : Experimental (M_{exp}) and predicted (M_{bag}) masses of hadrons in the M.I.T. bag model. The radius of the bag R, the zero point energy $-Z_o/R$, the volume energy BV, the quark kinetic energy E_{kin} and the colour magnetic and electric interaction energies, ΔE_M and ΔE_E, are also shown.

Particle	M_{exp} (Mev)	M_{bag} (Mev)	R (fm)	$-\frac{Z_o}{R}$ (Mev)	BV (Mev)	E_{kin} (Mev)	ΔE_M (Mev)	ΔE_E (Mev)
P	938	938	1.00	-367	234	1226	-155	0
Λ	1116	1105	0.99	-371	227	1400	-156	5
Σ^+	1189	1144	0.99	-371	227	1400	-116	5
Ξ^o	1321	1289	0.98	-374	222	1572	-136	5
Δ	1232	1233	1.10	-336	308	1119	141	0
Σ	1385	1382	1.09	-338	301	1292	122	5
Ξ	1533	1529	1.08	-341	293	1465	106	5
Ω^-	1672	1672	1.07	-343	287	1636	92	0
π	139	280	0.67	-549	70	1222	-462	0
K	495	497	0.65	-564	65	1407	-415	3
ρ	770	783	0.94	-390	196	868	110	0
ω	783	783	0.94	-390	196	868	110	0
K	892	928	0.93	-395	189	1039	91	4
ϕ	1020	1068	0.92	-399	183	1207	76	0

Charge radii are shown in Table 3 and magnetic moments in Table 4. A new feature of the M.I.T. bag model compared to previous SU(6) models is that the proton magnetic moment can be calculated. It is significantly too small. The other quantities shown in Tables 3 and 4 are in reasonable agreement with the data considering the crudeness of the model.

Table 3 : Experimental and predicted charge radii in the M.I.T. bag model.

Particle	Experiment	M.I.T. bag
Proton	$\langle r_p^2\rangle^{1/2} = 0.862 \pm 0.012$ fm [24)]	0.73 fm
Neutron	$\langle r_n^2\rangle = -0.12 \pm 0.01$ fm^2 [25)]	0
π	$\langle r_\pi^2\rangle^{1/2} = 0.74 \pm^{0.11}_{0.13}$ fm [26)]	0.49 fm
K^o	$\langle r_{K^o}^2\rangle = -0.054 \pm 0.026$ fm [27)]	-0.012 fm^2

Table 4 : Experimental and predicted baryon magnetic moments in the M.I.T. bag model.

	Experiment	M.I.T. bag
$\mu_p\left(\frac{e\hbar}{2m_p c}\right)$	2.79 [28)]	1.9
μ_n/μ_p	$-$ 0.685 [28)]	$-$ 2/3
μ_Λ/μ_p	$-$ 0.22 $\pm$ 0.01 [28)]	$-$ 0.26
μ_{Σ^+}/μ_p	+ 0.84 $\pm$ 0.05 [28)]	0.97
μ_{Σ^-}/μ_p	$-$ 0.32 $\pm$ 0.05 [29)]	$-$ 0.36
μ_{Ξ^-}/μ_p	$-$ 0.66 $\pm$ 0.27 [28)]	$-$ 0.23
μ_{Ξ^o}/μ_p	$-$ 0.45 $\pm$ 0.05 [28)]	$-$ 0.56

Finally, the value obtained for the ratio of the axial to the vector coupling constant, $\frac{g_A}{g_V}$, in the M.I.T. bag model is 1.09 compared to the experimental value of 1.25 [23)].

5. Chiral bag models

5.1. Basic assumptions

Chiral bag models[30-32] have been introduced to restore the chiral symmetry explicitly broken by the linear boundary condition of the M.I.T. bag model [eq. (4.20)].

The general idea of these models is to implement chiral symmetry by introducing the σ and $\vec{\pi}$ fields of the σ-model as elementary fields in the bag Lagrangian and coupling them to the quarks with the chirally invariant fermion-meson coupling of the σ-model. The σ field is then eliminated from the model by assuming the non-linear realization of chiral symmetry,

$$\sigma^2 + \vec{\pi}^2 = f^2 \quad . \tag{5.1}$$

The two approaches discussed in these lectures, the little bag model (LBM)[31] and the cloudy bag model (CBM)[32], differ originally by the region of space in which the σ and $\vec{\pi}$ fields are present. In the little bag model, the σ and $\vec{\pi}$ fields exist only outside the bag (the $\bar{B}$ region) while in the cloudy bag model, the σ and $\vec{\pi}$ fields are present everywhere.

The original Lagrangians corresponding to these assumptions are

$$\begin{aligned} L_o^{LBM}(x) = {} & [i\bar{\psi}_a^f \gamma^\mu \partial_\mu \psi_a^f - B]\, \theta_B(x) \\ & - \frac{1}{2f} \bar{\psi}_a^{f_1} (\sigma + i\, \vec{\tau}.\vec{\pi}\, \gamma_5)_{f_1 f_2} \psi_a^{f_2}\, \delta_S(x) \\ & + \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2]\, \theta_{\bar{B}}(x) \quad , \end{aligned} \tag{5.2}$$

$$\begin{aligned} L_o^{CBM}(x) = {} & [i\bar{\psi}_a^f \gamma^\mu \partial_\mu \psi_a^f - B]\, \theta_B(x) \\ & - \frac{1}{2f} \bar{\psi}_a^{f_1} (\sigma + i\, \vec{\tau}.\vec{\pi}\, \gamma_5)_{f_1 f_2} \psi_a^{f_2}\, \delta_S(x) \\ & + \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2] \quad . \end{aligned} \tag{5.3}$$

5.2. The cloudy bag model

We discuss first the cloudy bag model[32)] in which pion effects are treated perturbatively.

To derive the actual Lagrangian of the cloudy bag model from the original Lagrangian (5.3), one proceeds in four steps :

a) The pion field is replaced by a new field $\bar{\phi}(x)$ defined by the relations[33)]

$$\left\{ \begin{array}{lll} \bar{\pi} & = & f\,\hat{\phi}\,\sin\left(\frac{\phi}{f}\right) \quad , \\ \sigma & = & f\,\cos\left(\frac{\phi}{f}\right) \quad , \end{array} \right. \tag{5.4}$$

in which ϕ is the magnitude of the field and $\hat{\phi}$ a unit vector giving its direction in isospin space. The relations (5.4) satisfy the condition (5.1).

b) The chiral symmetry of the Lagrangian is explicitly broken by giving the pion a mass. A term,

$$L_{SB} = -\frac{1}{2}\,m_\pi^2\,\bar{\phi}^2 \quad , \tag{5.5}$$

is added to the Lagrangian (5.3). Consequently, the axial current will not be conserved in the cloudy bag model.

c) The terms involving derivatives of the $\bar{\phi}$ field are rewritten using the covariant derivative defined by

$$D_\mu\bar{\phi} \equiv (\partial_\mu\phi)\hat{\phi} + f\,\sin\left(\frac{\phi}{f}\right)\,\partial_\mu\hat{\phi} \quad . \tag{5.6}$$

The new Lagrangian obtained from (5.3) using (5.4) and (5.6) and adding the symmetry breaking term (5.5) reads

$$\begin{aligned} L'^{CBM} = {} & [i\bar{\psi}_a^f\,\gamma^\mu\partial_\mu\,\psi_a^f - B]\,\theta_B(x) + \frac{1}{2}\,(D_\mu\bar{\phi})^2 \\ & - \frac{1}{2}\,\bar{\psi}_a^{f_1}\left(e^{i\bar{\tau}.\bar{\phi}\gamma_5/f}\right)_{f_1 f_2}\,\psi_a^{f_2}\,\delta_S(x) - \frac{1}{2}\,m_\pi^2\,\bar{\phi}^2 \; . \end{aligned} \tag{5.7}$$

d) The model is linearized, i.e. expanded around $\bar{\phi}=0$. The underlying assumption here is that the parameter,

$$\varepsilon = \frac{g_A}{8\pi f^2 R^2} \simeq \frac{0.22}{R^2(\mathrm{fm}^2)} \quad , \tag{5.8}$$

which measures the strength of the pion field at the bag surface is small[33]. This implies that the bag radius R shouldn't be much smaller than 1 fm.

The actual Lagrangian of the cloudy bag model is therefore

$$L^{CBM} = [i\bar{\psi}_a^f \gamma^\mu \partial_\mu \psi_a^f - B]\,\theta_B(x) + \frac{1}{2}(\partial_\mu \bar{\phi})^2$$
$$- \frac{1}{2}\bar{\psi}_a^{f_1}\left(1 + \frac{i}{f}\bar{\tau}.\bar{\phi}\gamma_5\right)_{f_1 f_2} \psi_a^{f_2}\,\delta_S(x) - \frac{1}{2} m_\pi^2 \bar{\phi}^2 \quad . \tag{5.9}$$

The next step is to derive a hamiltonian formulation of the cloudy bag model. This Hamiltonian is written in terms of three quark M.I.T. bag states (with quantum numbers of physical baryons) and of pion states. In its second quantized version, the cloudy bag Hamiltonian reads[34]

$$H^{CBM} = H_{MIT} + H_\pi + H_{int} \quad , \tag{5.10}$$

with

$$H_{MIT} = \sum_\alpha M_\alpha^{\text{MIT bag}}\, \alpha^+ \alpha \quad , \tag{5.11}$$

$$H_\pi = \sum_i \int d^3\bar{k}\; \omega_{\bar{k}}\, a^+_{\bar{k}_i}\, a_{\bar{k}_i} \quad , \tag{5.12}$$

$$H_{int} = (2\pi)^{-\frac{3}{2}} \sum_{\substack{i \\ \alpha\beta}} \int d^3\bar{k} \left(v^{\beta\alpha}_{\bar{k}_i}\, \beta^+ \alpha\, a_{\bar{k}_i} + hc \right) \quad , \tag{5.13}$$

in which α^+ and β^+ are three quark bag creation operators, $a^+_{\bar{k}_i}$ is the creation operator for a pion of momentum $\bar{k}$ and isospin i, $M_\alpha^{\text{MIT bag}}$ is the mass of the three quark bag α, $\omega_{\bar{k}}$ is the energy of the pion of momentum $\bar{k}$ and $v^{\beta\alpha}_{\bar{k}_i}$ is defined by

$$v^{\beta\alpha}_{\bar{k}_i} = \frac{i}{2f}\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \int d^3\bar{x}\; e^{i\bar{k}.\bar{x}}\, \delta(x-R) \langle\beta|\bar{q}(x)\tau^{(i)}\gamma_5 q(\bar{x})|\alpha\rangle \quad , \tag{5.14}$$

where q(x) is the quark field. The Fock space for baryon-like states consists of three quark states, three quark states + one pion, three quark states + two pions and so on.

Let us now mention a few results obtained in the cloudy bag model[34].

First, to have some idea of the importance of the pion field, it is interesting to mention that the average number of pions in the nucleon cloud is $\leq$ 0.9 for a bag radius larger than 0.8 fm. Therefore, for most purposes, the Fock space can be limited to states containing at most one pion.

The pionic contribution to the nucleon mass is of the order of 200 Mev for a bag radius of 1 fm. By changing the bag parameters, the spectrum of baryons can be refitted to include this contribution[35].

We show in Table 5 the charge radii of the proton and the neutron obtained in the cloudy bag model for a bag radius of 0.8 fm. It is interesting that the $|p\pi^-\rangle$ component of the neutron produces a negative value for $\langle r_n^2\rangle$, in good agreement with the experimental data.

Table 5 : Charge radii of the proton and the neutron calculated in the cloudy bag model compared to the experimental and M.I.T. values.

Particle	Experiment	M.I.T.	C B M R=0.8fm
Proton	$\langle r_p^2\rangle^{1/2} = 0.862 \pm 0.012$ fm[24]	0.73 fm	0.73 fm
Neutron	$\langle r_n^2\rangle = -0.12 \pm 0.01$ fm^2 [25]	0	-0.15 fm

The magnetic moments of the baryons belonging to the nucleon octet are shown in Table 6[36]. The pion contribution improves significantly the proton magnetic moment.

The value found for $\frac{g_A}{g_V}$ in the cloudy bag model is similar to the M.I.T. value of 1.09.

Table 6 : Magnetic moments of the baryons belonging to the nucleon octet calculated in the cloudy bag model compared to the experimental and M.I.T. values.

	Experiment	M.I.T.	C B M R=1 fm
$\mu_p\left(\frac{e\hbar}{2m_pc}\right)$	2.79[28)]	1.9	2.65
μ_n/μ_p	-0.685[28)]	-0.67	-0.73
μ_Λ/μ_p	-0.22 ± 0.01[28)]	-0.26	-0.22
μ_{Σ^+}/μ_p	+0.84 ± 0.05[28)]	0.97	0.84
μ_{Σ^-}/μ_p	-0.32 ± 0.05[29)]	-0.36	-0.39
μ_{Ξ^-}/μ_p	-0.66 ± 0.27[28)]	-0.23	-0.19
μ_{Ξ^o}/μ_p	-0.45 ± 0.05[28)]	-0.56	-0.46

The calculations done in the cloudy bag model indicate that perturbative pion field corrections may improve somewhat on the M.I.T. bag results. Whether a perturbative treatment of the pion field is appropriate remains an open question.

5.3. The little bag model

In contrast to the perturbative approach chosen in the cloudy bag model, the little bag model approach is to derive and discuss a particular non-perturbative solution of the non-linear chiral bag Lagrangian.

As previously, the pion field in (5.2) is replaced by a new field $\bar{\phi}(x)$ defined by the relations,

$$\vec{\pi} = \frac{\vec{\phi}}{\sqrt{1 + \frac{\vec{\phi}^2}{f^2}}} \quad ,$$

$$\sigma = \frac{f}{\sqrt{1 + \frac{\vec{\phi}^2}{f^2}}} \quad , \tag{5.15}$$

which satisfy the condition (5.1). A covariant derivative of the pion field is defined by

$$D_\mu \vec{\phi} \equiv \frac{1}{1 + \frac{\vec{\phi}^2}{f^2}} \partial_\mu \vec{\phi} \quad . \tag{5.16}$$

Using (5.15) and (5.16), the Lagrangian (5.2) becomes

$$L^{LBM} = [i\bar{\psi}^f_a \gamma_\mu \partial_\mu \psi^f_a - B] \theta_B(x)$$

$$- \frac{1}{2} \frac{1}{\sqrt{1 + \frac{\vec{\phi}^2}{f^2}}} \bar{\psi}^{f_1}_a \left(1 + i \frac{\vec{\tau}.\vec{\phi}}{f} \gamma_5\right)_{f_1 f_2} \psi^{f_2}_a \delta_S(x)$$

$$+ \frac{1}{2} \left\{ \left(1 + \frac{\vec{\phi}^2}{f^2}\right) (D_\mu \vec{\phi})^2 - \frac{1}{4f^2} (D_\mu \vec{\phi}^2)^2 \right\} \theta_{\bar{B}}(x) \quad . \tag{5.17}$$

The basic assumption of the model is that chiral symmetry is realized in a different mode inside and outside the bag. Inside the bag, chiral symmetry is realized in the Wigner-Weyl mode with massless quarks. Outside the bag, chiral symmetry is realized in the Nambu-Goldstone mode with a triplet of massless pions.

The equations of motion of the little bag model are given by[37)]

$$\forall\ x \in B \qquad i\gamma^\mu \partial_\mu \psi^f_a = 0 \tag{5.18}$$

$$\forall\ x \in \bar{B} \qquad D^2_\mu \vec{\phi} = 0 \tag{5.19}$$

$$\forall\ x \in S \qquad i\bar{\gamma}.\bar{n}\ \psi_a^{f_1} = \frac{1}{\sqrt{1+\frac{\phi^2}{f^2}}} \left(1 + i\,\frac{\bar{\tau}.\bar{\phi}}{f}\,\gamma_5\right)_{f_1 f_2} \psi_a^{f_2} \tag{5.20}$$

$$\forall\ x \in S \qquad B = -\frac{1}{2}\,\frac{1}{\left(1+\frac{\phi^2}{f^2}\right)}\, n_\nu \partial^\nu \left[\bar{\psi}_a^{f_1} \left(1 + i\,\frac{\bar{\tau}.\bar{\phi}}{f}\,\gamma_5\right)_{f_1 f_2} \psi_a^{f_2}\right]$$

$$-\frac{1}{2}\left\{\left(1+\frac{\phi^2}{f^2}\right)\ (D_\mu \phi)^2 - \frac{1}{4f^2}\,(D_\mu \bar{\phi}^2)^2\right\} \quad , \tag{5.21}$$

$$\forall\ x \in S \qquad f\, n^\mu\, D_\mu\, \bar{\phi} = \bar{\psi}_a^{f_1}\ \gamma_5\ n.\gamma \left[\frac{\bar{\tau}}{2}\right]_{f_1 f_2} \psi_a^{f_2} \quad . \tag{5.22}$$

These equations look hard to solve. It turns out however that a solution can be calculated if one assumes that the system consisting of the bag and the pion field is spherically symmetric ; one uses the "hedgehog" ansatz[30,38),

$$\bar{\phi} = \hat{r}\ G(r) \quad , \tag{5.23}$$

which implies that the isospin points in the radial direction.

Because of the boundary condition (5.20), the Dirac equation cannot be solved in the basis of good angular momentum J and good isospin I. A good quantum number is instead

$$\bar{K} = \bar{J} + \bar{I} \quad . \tag{5.24}$$

To build the lowest-lying three quark state which has $K = 0$, the quarks are put in the lowest mode, analogous to (4.24)[38),

$$\psi = \frac{N}{\sqrt{4\pi}} \begin{pmatrix} i\ j_o\left(\frac{\Omega}{R}\,r\right) v \\ \\ -\ j_1\left(\frac{\Omega}{R}\,r\right) \bar{\sigma}.\hat{r}\ v \end{pmatrix} e^{-i\omega t} \quad , \tag{5.25}$$

in which $\omega = \frac{\Omega}{R}$ is the mode frequency, N is a normalization factor

$$N^{-2} = R^3\, j_o^2(\Omega) \left\{1 + y^2 - \frac{2y}{\Omega}\right\} \quad , \tag{5.26}$$

with

$$y = \frac{j_1(\Omega)}{j_o(\Omega)} \quad , \tag{5.27}$$

and v is given by

$$v = \frac{1}{\sqrt{2}} \quad (|\uparrow ; -> - |\downarrow ; +>) \quad , \tag{5.28}$$

where arrows refer to spin and ± to isospin projections.

Rather than discussing in detail how one solves the equations (5.18)-(5.22) (see ref.38), we shall briefly mention a few results.

An interesting property of the $K = 0$ three quark bag is that for small radii, the wave function (5.25) becomes non-relativistic[38]. Fig.5 shows the behaviour of the frequency Ω as a function of R. For large radii, Ω tends to the M.I.T. value ($\Omega_{MIT} = x_{-1}$). When R approaches R_c ($\simeq 0.3$ fm),

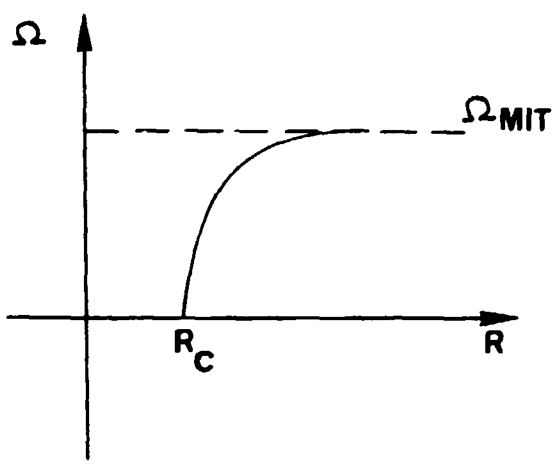

Figure 5 : Behaviour of the quark frequency Ω versus the bag radius R.

Ω tends to zero and the small components of the Dirac spinor (5.25) vanish.

The importance of the non perturbative pion effects are illustrated in Figs 6 and 7 in which the exact solution is compared to the perturbative approach for the mass E and bag constant B. Non-perturbative

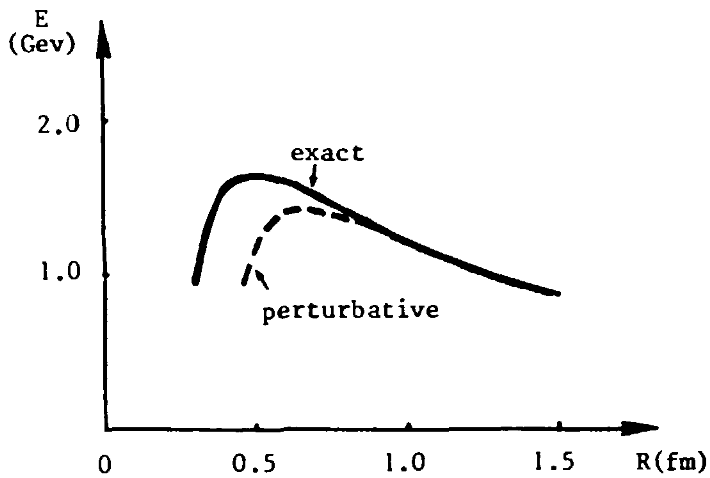

Figure 6 : Mass of the hedgehog bag versus its radius. The solid line is the exact solution ; the dashed line is a perturbative approach.

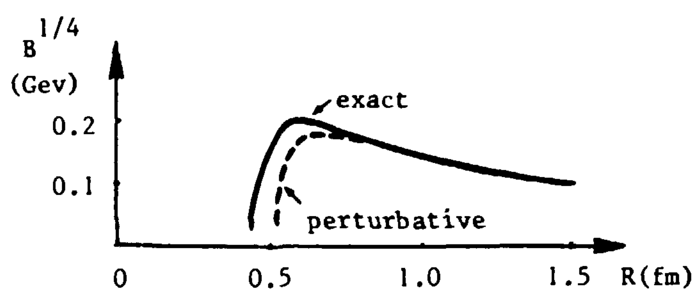

Figure 7 : Bag constant behaviour as a function of the bag radius (same notation as Fig.6).

effects are clearly very important for small radii. We should remark that in the absence of additional repulsive effects at short radii[39) the hedgehog bag is unstable and collapses. An other problem arising for chiral bags of small radii is that the value of $\frac{g_A}{g_V}$ becomes substantially too large[38].

A very interesting development concerning the hedgehog non-perturbative solution is its recent interpretation as a topological soliton[40]. This development is based on the work of Skyrme[41] who suggested that baryons are topological solitons in the non-linear σ-model. As discussed in paragraph 2.6, QCD becomes equivalent to an effective field theory of mesons in the large N_c limit and baryons appear as topological solitons in this theory. At low energy, the theory reduces to a non-linear σ-model so that the solitons of the σ-model can be related to the baryons of QCD[42].

The idea then is to make the hedgehog ansatz to calculate the soliton configuration of the Skyrme Lagrangian and to insert as a "defect" a quark bag in the soliton configuration. The details of this procedure are given in ref.40. The baryon number in this scheme is shared between the interior fermion bag and the exterior boson cloud. It is interesting that this model predicts the correct value of $\frac{g_A}{g_V}$ and gives a quantitatively correct estimate of the nucleon-nucleon repulsion[40].

6. Non-topological soliton bag models

In the Friedberg-Lee non-topological soliton bag model[43], the quark fields interact with a confining scalar field σ. The σ field can be viewed as a phenomenological representation of the color singlet excitations of the gluon fields.

The Lagrangian of the model is

$$L^{FL} = \bar{\psi} \; (i\gamma^\mu \partial_\mu - g\sigma) \; \psi \; + \frac{1}{2} (\partial_\mu \sigma)^2 - U(\sigma) \quad , \tag{6.1}$$

in which $U(\sigma)$ is a non-linear potential given by

$$U(\sigma) = \frac{a}{2} \sigma^2 + \frac{b}{6} \sigma^3 + \frac{c}{24} \sigma^4 + p \quad , \tag{6.2}$$

where a,b,c and p are real constants.

The equations of motion read

$$(i\gamma^{\mu}\partial_{\mu} - g\sigma)\,\psi = 0 \quad , \tag{6.3}$$

$$\partial_{\mu}^{2}\sigma + U'(\sigma) = -g\,\bar{\psi}\,\psi \quad , \tag{6.4}$$

with

$$U'(\sigma) = \frac{dU(\sigma)}{d\sigma} \quad . \tag{6.5}$$

Let us discuss briefly how this model produces confinement. Consider classical fields and let us call σ_V the vacuum expectation value of the σ-field. Then, choose $U(\sigma)$ in such a way that it has an absolute minimum at $\sigma = \sigma_V$ and a local minimum at $\sigma = 0$ (Fig.8) .

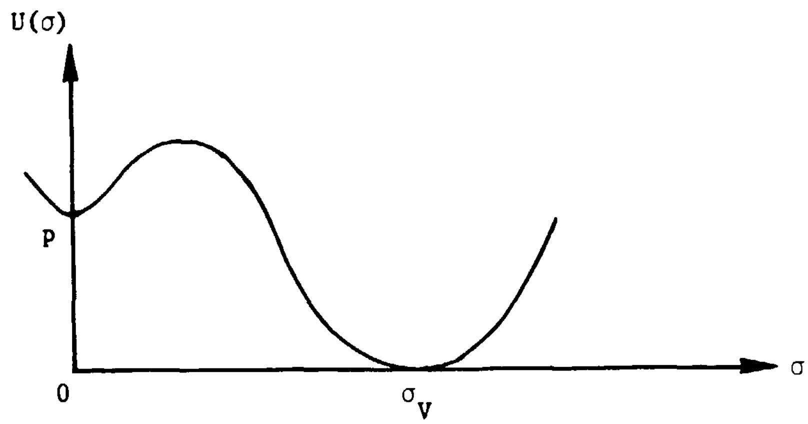

Figure 8 : The potential U versus σ.

In the absence of quarks, the ground state is a constant classical field $\sigma(x) = \sigma_V$.

Assume now that there is a positive quark density at the origin. The term $-g\sigma\bar{\psi}\psi$ in the Lagrangian becomes non-zero and generates a quark mass linear in the σ field. The new "potential" defined by

$$V(\sigma) = U(\sigma) + g\sigma\bar{\psi}\psi \quad , \tag{6.6}$$

is plotted in Fig.9.

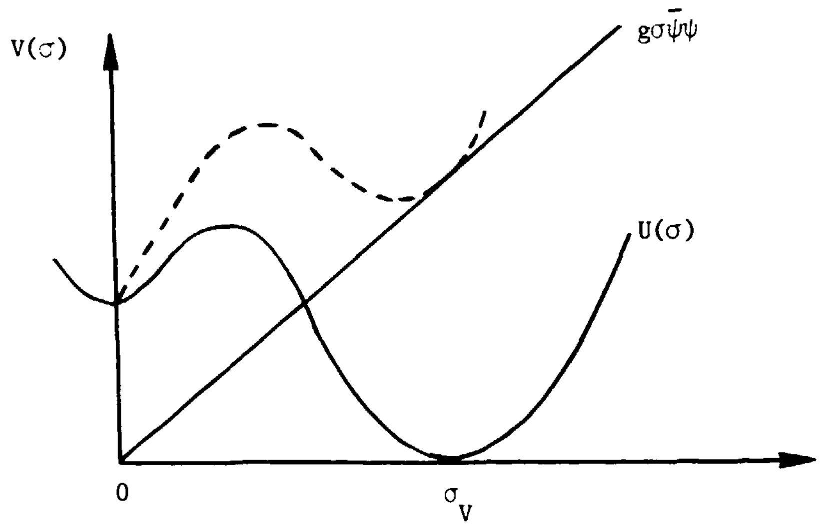

Figure 9 : V versus σ (dashed line).

If g or $\bar{\psi}\psi$ are large enough, the minimum of V can occur at $\sigma = 0$. In this region, quarks will be bound. If σ_o is the time-independent mean-field and $\{\psi_k\}$ a complete orthonormal basis, we find

$$(\bar{\alpha}.\bar{p} + g\,\gamma^o\sigma_o)\psi_k = \varepsilon_k\,\psi_k \quad , \tag{6.7}$$

$$-\bar{\nabla}^2\sigma_o + U'(\sigma_o) = -g\sum_k \bar{\psi}_k\,\psi_k \quad . \tag{6.8}$$

These equations have been solved numerically

and quantum corrections to this approximation have been discussed[43]. By appropriate choices of the parameters, the M.I.T. bag limit[23] or the SLAC bag limit[45] can be recovered. In this sense the model is a very useful tool.

Recently[46], non-topological soliton bags (bound states of valence quarks) have been studied in the σ-model. The additional pion degree of freedom is shown to have very important dynamical consequences. They are discussed in detail in Ref. 46.

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