

QUANTIZATION OF DYNAMICAL SYSTEMS

AND STOCHASTIC CONTROL THEORY *

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ABSTRACT

In the general framework of stochastic control theory we introduce a suitable form of stochastic action associated to the controlled process. Then a variational principle gives all main features of Nelson's stochastic mechanics. In particular we derive the expression of the current velocity field as the gradient of the phase action. Moreover the stochastic corrections to the Hamilton -Jacobi equation are in agreement with the quantum mechanical form of the Madelung fluid (equivalent to the Schrödinger equation).

Therefore stochastic control theory can provide a very simple model simulating quantum mechanical behavior.

1 - INTRODUCTION

Methods based on the theory of stochastic processes have been very useful for the investigation of the physical properties of quantum mechanical models and quantum field theory (see for example [1-7] and references quoted there).

In fact in the last years there have been continuous efforts to enlarge the overlapping area between quantum mechanics and probability theory, not only from the point of view of methods and techniques but also with reference to physical interpretation (see for example [8,9]).

Moreover the merging of quantum mechanics and probability theory stimulated research aiming at the foundations of a new " quantum probability " theory (see for example [10] and previous references quoted there).

It is very well known that there is a close relation between classical mechanics and deterministic control theory [11] . In particular the Hamilton-Jacobi equation can be understood as programming equation for an optimal control problem, as it will be also recalled in Section 2.

Therefore, on the basis of the previous considerations, it is very natural to investigate about possible connections between quantum mechanics and stochastic control theory. Here a typical difficulty arises. In fact the programming equations of the standard versions of stochastic control theory are *not time reversal invariant in general* and describe a typical parabolic behavior. On the other hand quantum mechanics

is time reversal invariant and the resulting candidate for a programming equation is the Madelung form of Hamilton-Jacobi equation with quantum mechanical corrections. In this case the programming equation describes more some kind of hyperbolic behavior, with no essential dissipation involved.

But it is easy to see that a proper choice of the stochastic action allows a generalization of the classical case. Then it is possible to reach the objective of connecting stochastic control theory and quantum mechanical behavior.

The purpose of this paper is to give a complete and concise account of the emerging general structure.

For related work on the connections between quantum mechanics and stochastic control theory we refer to [12,13,14], where different ideas and techniques are developed. The analogy between non-linear filtering and quantum physics is investigated in [15]. The relevance of stochastic control theory also for classical mechanics is stressed in [16].

The paper is organized as follows :

Section 2 is dedicated to a brief review of the classical case, showing the connections between classical mechanics and deterministic optimal control theory. For a more detailed and precise treatment we refer to [11]. Our considerations are introduced in order to familiarize the reader with the general structure of the problems in control

theory and prepare the grounds for a natural generalization to the stochastic case. In particular we find of some interest to recall that, even in classical mechanics, the Hamilton-Jacobi principal function S can be introduced as a field of Lagrangian multipliers, connected with a constrained variational problem, where initial and final densities of the classical fluid in configuration space are kept constant.

In Section 3 we introduce the general frame of the stochastic control theory employed in the following. In particular we recall some of the useful formulae connected with the solutions of stochastic differential equations.

Section 4 contains the definition of the stochastic action associated to the general controlled process. Various different action functionals are introduced and their connections are pointed out. We find also convenient to give the explicit expression for the variations of the action functionals as a consequence of a change in the controlling drift field.

The main results of this paper are found in Section 5. In fact we establish two equivalent versions of the stochastic variational principle. In one version the variations of the controlling drift field are constrained in such a way that the density of the controlled process is kept fixed at initial and final times. In the other version only the initial density is fixed and no constraints are put on the control field, but additional terms are introduced in the action functional. The equivalence of the two principles allows the interpretation of the phase

function S as a field of Lagrangian multiplier connected with the constraint of fixed density.

The stochastic variational principle allows then to derive the whole structure of stochastic mechanics [17,18]. In particular we show that the expression of the mean velocity field as the gradient of the phase action follows easily as a consequence of the variational principle, as in the classical case. Moreover the stochastic corrections to the classical Hamilton-Jacobi programming equation are shown to reproduce correctly the quantum corrections, provided the strength of the disturbing noise in the control equation is properly adjusted as a function of the Planck constant. Therefore the Schrödinger equation can be interpreted as a consequence of the stochastic variational principle.

Finally Section 6 deals with possible further development and applications of the theory.

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2 - CLASSICAL MECHANIC AND DETERMINISTIC CONTROL THEORY

We only recall some basic facts. For a more complete treatment we refer to [11]. Our account will be very simple minded and pedagogical. In fact we consider this Section as a preparation to the more complicated case of stochastic control theory developed later, where all essential features are more or less direct generalization of the classical case.

Let us consider a dynamical system with configuration space \mathcal{M} . For the sake of simplicity we assume $\mathcal{M} \in \mathbb{R}^n$, but our considerations can be easily generalized to the case where \mathcal{M} is a general Riemannian manifold, with metric given by the kinetic energy, as it will be shown elsewhere [19].

We consider trajectories of the type :

$$1) [t_0, t_1] \ni t \rightarrow q(t) \in \mathcal{M}$$

evolving in time according to a feedback control law enforced by the differential equation

$$2) \dot{q}(t) = v(q(t), t).$$

the control field v is some given time dependent vector field on \mathcal{M} .

Here $v(\cdot, t) \in \mathbb{R}^n$, $t_0 \leq t \leq t_1$.

Standard conditions of regularity on v (see for example [20,21])

assure existence and uniqueness for the solution of 2) for some given condition at time t' , $t_0 \leq t' \leq t_1$.

$$3) q(t') = x'.$$

In some cases it is relevant to explicitly show in the notation the condition 3). Then we write $q(t; x', t')$ for the solution of 2) satisfying 3).

The following properties hold

$$4) \quad q(t; x, t) = x \quad ; \quad q(t; q(t'; x, t), t') = x \quad ; \\ q(t; q(t'; x'', t''), t') = q(t; x'', t'') \quad .$$

For sufficiently regular fields $F(\cdot, t)$ on \mathbb{R}^n we define the substantial time derivative through

$$5) \quad (DF)(x, t) = \lim_{\Delta t \rightarrow 0} (\Delta t)^{-1} \left[F(q(t+\Delta t; x, t), t+\Delta t) - F(x, t) \right] ;$$

clearly it is enough to know $q(t')$ for t' in a small interval containing t in order to evaluate 5).

By taking into account that

$$6) \quad F(q(t+\Delta t; x, t), t+\Delta t) - F(x, t) = \\ = (\partial_t F)(x, t) \Delta t + (\nabla F)(x, t) \cdot (q(t+\Delta t; x, t) - x) + O((\Delta t)^2), \\ q(t+\Delta t; x, t) - x = v(x, t) \Delta t + O((\Delta t)^2),$$

we immediately have

$$7) \quad (DF)(x, t) = (\partial_t F)(x, t) + v(x, t) \cdot (\nabla F)(x, t) \quad .$$

In particular for the specification field

$$8) \quad \hat{q}(x, t) \equiv q(t; x, t) = x$$

we have

$$9) \quad (D\hat{q})(x, t) = v(x, t) \quad ,$$

while for the generic trajectory $q(t'; x, t)$ we have

$$10) \quad (Dq)(t'; x, t) = 0 ,$$

as a consequence of the control equation 2). In fact

$$11) \quad (Dq)(t'; x, t) = \lim_{\Delta t \rightarrow 0} (\Delta t)^{-1} \left[q(t'; q(t+\Delta t; x, t), t+\Delta t) - q(t'; x, t) \right],$$

and 10) follows from the last of 4).

From the definition 5) we also have the transport equation

$$12) \quad F(q(t_2), t_2) - F(q(t_0), t_0) = \int_{t_0}^{t_2} (DF)(q(t), t) dt ,$$

where $q(t)$ is a general trajectory controlled by v with end points $q(t_0)$ at t_0 and $q(t_2)$ at t_2 .

We introduce the canonical Lagrangian

$$13) \quad \mathcal{L} : T\mathcal{M} \rightarrow \mathbb{R} \\ (x, v) \rightarrow \mathcal{L}(x, v) = \frac{1}{2} m v^2 - V(x) ,$$

where $T\mathcal{M}$ is the tangent bundle on \mathcal{M} , $x \in \mathcal{M}$, $v \in T\mathcal{M}_x$ (the tangent space to \mathcal{M} in x). Here we have simply $\mathcal{M} = \mathbb{R}^n$, $T\mathcal{M} = \mathbb{R}^n \times \mathbb{R}^n$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$.

Then the action $-I$ spent by the controller, while moving the point from time t , starting at x , to time t_2 , is defined as

$$14) \quad -I(x, t; t_2; v) = \int_t^{t_2} \mathcal{L}(q(t'; x, t), v(q(t'; x, t), t')) dt' ,$$

for $t_0 \leq t \leq t_2$. Notice that I in 14) depends only on the specification of v in the time interval $[t, t_2]$. The following additivity property holds

$$15) \quad I(x_0, t_0; t_1, v) = I(x_0, t_0; t, v) + I(q(t; x_0, t_0), t; t_1; v),$$

for $t_0 \leq t \leq t_1$.

It is very ^{well} known that action functionals of this type find applications in the classical calculus of variations, where usually some additional conditions are enforced for the end points of trajectories. It is therefore also convenient to introduce action functionals with additional end point contributions, which act as a kind of Lagrangian multipliers, as it will be clear later.

For some given smooth field $S_1(\cdot)$ on \mathcal{M} we define

$$16) \quad J(x, t; t_1, S_1; v) = I(x, t; t_1; v) + S_1(q(t_1; x, t)),$$

for $t_0 \leq t \leq t_1$. Then we have the following property

$$17) \quad J(x_0, t_0; t_1, S_1; v) = J(x_0, t_0; t, S_t; v),$$

where $t_0 \leq t \leq t_1$ and the field S_t is given by

$$18) \quad S_t(x) \equiv J(x, t; t_1, S_1; v).$$

When there is no danger of confusion we keep only the specification $(x, t; v)$ in 14) and 16) and suppress the dependence on t_1, S_1 in the notations. As a consequence of 7) and 10) we have that I and J satisfy the following transport equations with the appropriate final boundary conditions

$$19) \quad (DI)(x, t; v) = \mathcal{L}(x, v(x, t)), \quad I(\cdot, t_1) = 0,$$

$$20) \quad (DJ)(x, t; v) = \mathcal{L}(x, v(x, t)), \quad J(\cdot, t_1) = S_1(\cdot).$$

The basic laws of classical mechanics, i.e. the Newton second principle of dynamics, or the equivalent Hamilton-Jacobi equation, can be easily established through variational principles, by exploiting the functionals \mathcal{I} or \mathcal{J} .

Consider for example \mathcal{J} . In addition to the control $v(\cdot, t)$, $t_0 \leq t \leq t_1$ appearing in 14), 16), we introduce also another control $v'(\cdot, t)$, $t_0 \leq t \leq t_1$, and call \mathcal{D}' , \mathcal{J}' , $q'(t; x_0, t_0)$ the corresponding substantial derivative, action functional and controlled trajectory.

Then we have

$$\begin{aligned} 21) \quad \mathcal{D}'(\mathcal{J}' - \mathcal{J}) &= \mathcal{L}(x, v'(x, t)) - \mathcal{L}(x, v(x, t)) + (\mathcal{D} - \mathcal{D}')\mathcal{J} = \\ &= \frac{1}{2} m (v'^2 - v^2) + (v' - v) \cdot \nabla \mathcal{J}. \end{aligned}$$

Let us now integrate along $q'(t; x_0, t_0)$, $t_0 \leq t \leq t_1$; by exploiting the transport equation 12) and the common boundary condition 20)₂ for \mathcal{J} and \mathcal{J}' , we have

$$\begin{aligned} 22) \quad \mathcal{J}(x_0, t_0; v') - \mathcal{J}(x_0, t_0; v) &= \\ &= \frac{1}{2} m \int_{t_0}^{t_1} \left[(v' - v) \cdot \left(\frac{2 \nabla \mathcal{J}}{m} - v' - v \right) \right] (q'(t; x_0, t_0), t) dt. \end{aligned}$$

Now we exploit this basic formula along two lines. First of all let us define

$$23) \quad \delta v(x, t) = v'(x, t) - v(x, t)$$

and consider in 22) only first order variations in δv . Then we have

$$24) \quad (\delta \mathcal{J})(x_0, t_0; v) = m \int_{t_0}^{t_1} \left[\delta v \cdot \left(\frac{\nabla \mathcal{J}}{m} - v \right) \right] (q(t; x_0, t_0), t) dt.$$

Therefore, if we assume that under any small variation of the control $\nu \rightarrow \nu + \delta \nu$ the variation of J is zero, then we must necessarily have

$$25) \quad \nu(x, t) = \frac{1}{m} (\nabla J)(x, t; \nu) ,$$

for any point x reached by the controlled trajectory at time t , $t_0 \leq t \leq t_1$. Therefore if we impose $\delta J = 0, \forall x_0$, then 25) must hold everywhere.

Formula 24) shows the typical structure of the classical variational problem. In fact we can impose $\delta J(x_0, t_0; \nu) = 0$ for $x_0 \in \mathcal{O}$, where \mathcal{O} is some open set of the configuration space \mathcal{M} . Then the condition 25) will hold in the causal shadow of \mathcal{O} , i.e. for all points x which can be reached at time t by trajectories controlled by ν and starting in \mathcal{O} at time t_0 . In the next Sections it will be apparent that this typical "fibration" of the classical variational problem does not carry through the stochastic case, where some average procedure will be necessary in order to introduce action functionals.

It is clear that the gradient condition 25) implies the Hamilton-Jacobi equation. In fact if we call S the functional J corresponding to the stationarizing control 25)

$$26) \quad S(x, t) \equiv J(x, t; \frac{\nabla J}{m})$$

then we must necessarily have

$$27) \quad (\partial_t S)(x, t) + \frac{1}{2m} (\nabla S)^2(x, t) + V(x) = 0 ,$$

with the boundary condition

$$28) \quad \hat{S}(\cdot, t_2) = \hat{S}_2(\cdot) ,$$

following from 20)₂ .

It is important to recall that 27) is equivalent to the second principle of dynamics in the following hydrodynamical form. Define the acceleration field through

$$29) \quad a(x, t) = (Dv)(x, t) = (\partial_t v)(x, t) + (v \cdot \nabla v)(x, t) .$$

Notice that for $v = \nabla S / m$ 29) reduces to

$$30) \quad a(x, t) = \frac{1}{m} \nabla \left(\partial_t S + \frac{1}{2m} (\nabla S)^2 \right) ,$$

therefore 27) implies

$$31) \quad a(x, t) = - \frac{1}{m} (\nabla V)(x) ,$$

and this is the Newton equation in local form. Vice versa if 30) and

31) hold then

$$32) \quad \nabla \left(\partial_t S + \frac{1}{2m} (\nabla S)^2 + V \right) = 0 .$$

Therefore for some $\alpha(t)$ we must have

$$33) \quad \partial_t S + \frac{1}{2m} (\nabla S)^2 + V = \alpha(t) .$$

Putting

$$34) \quad \hat{S}(x, t) = S(x, t) - \int_t^{t_2} \alpha(t') dt'$$

we have that \hat{S} satisfies 27) and 28), while still $v = \nabla \hat{S} / m$.

The second way of exploiting formula 22) leads to considerations of optimal control theory. In fact assume that a solution S exists for the Hamilton-Jacobi equation 27) with the boundary condition 28). Consider 22) with v given by the Hamilton-Jacobi control $v = \nabla S/m$, so that $J = S$, and for a generic v' . Writing v in place of v' for convenience, we have

$$35) \quad J(x_0, t_0; v) - S(x_0, t_0) = \\ = -m \int_{t_0}^{t_1} (v - \nabla S/m)^2 (q(t; x_0, t_0), t) dt .$$

Since the r.h.s. is negative unless v satisfies $v = \nabla S/m$, then we have that the Hamilton-Jacobi control is optimal in the sense that

$$36) \quad J(x_0, t_0; v) \leq S(x_0, t_0) \quad , \quad \forall v \quad , \quad x_0 \in \mathcal{M} \quad ,$$

while

$$37) \quad J(x_0, t_0; \nabla S/m) = S(x_0, t_0) .$$

It is also possible to show that the field S_1 , appearing in the definition of the action functional J in 16), can be interpreted as a field of Lagrangian multipliers, corresponding to a new variational problem where the action functional now is \bar{J} defined in 14) but the variations of the control field are not free but subject to suitable constraints. We give some details about this rather remarkable fact, which holds also in the stochastic case as will be shown in Section 5.

First of all let us introduce a kind of hydrodynamic picture in classical mechanics, also in reference to the control problem. Instead of considering single trajectories we consider clouds of particles moving

under the action of the control field. Therefore we assume that some density $\rho_0(\cdot)$ is given at the initial time t_0 on the configuration space. By acting on each single particle the control field will let the density $\rho(\cdot, t)$ evolve in time according to the continuity equation

$$38) \quad (\partial_t \rho)(x, t) = - \nabla \cdot (\rho(x, t) v(x, t)) ,$$

with initial condition

$$39) \quad \rho(\cdot, t_0) = \rho_0(\cdot) .$$

The connection between the fluid picture and the particle picture is easily established by introducing the transition " probability density " from time t' to time t defined by

$$40) \quad p(x, t; x', t') = \delta(x - q(t; x', t')) .$$

The δ function on the r.h.s. assures the deterministic motion of the particles forming the fluid. Notice that 40) is meaningful also for $t \leq t'$.

The deterministic transition probability 40) satisfies the forwards equation

$$41) \quad (\partial_t p)(x, t; x', t') + \nabla \cdot (p(x, t; x', t') v(x, t)) = 0 ,$$

with boundary condition

$$42) \quad \lim_{t \rightarrow t'} p(x, t; x', t') = \delta(x - x') ,$$

and the backward equation

$$43) \quad (D' p)(x, t; x', t') = 0 ,$$

where D^1 , defined in 7), acts on the (x', t') variables.

The following identity of the Kolmogorov type is also easily checked as a consequence of 4)

$$44) \quad p(x, t; x'', t'') = \int p(x, t; x', t') p(x', t'; x'', t'') dx',$$

with no restriction on the ordering of t, t', t'' .

Finally the evolution of the density can be easily expressed in the form

$$45) \quad \rho(x, t) = \int p(x, t; x', t') \rho(x', t') dx'.$$

Variations of p and ρ under a change of the control variable

$v \rightarrow v' = v + \delta v$ can be easily found starting from 43). In fact

by using the same method as given by 21), 22), we find

$$46) \quad p'(x_2, t_2; x_0, t_0) - p(x_2, t_2; x_0, t_0) = \int_{t_0}^{t_2} \nabla p'(x_2, t_2; x, t) \cdot (v' - v)(x, t) p(x, t; x_0, t_0) dx dt,$$

where p and p' refer to the controls v and v' respectively.

By exploiting 45) we have also the variation of the density in the form

$$47) \quad \rho'(x_2, t_2) - \rho(x_2, t_2) = \int_{t_0}^{t_2} \nabla p'(x_2, t_2; x, t) \cdot (v' - v)(x, t) \rho(x, t) dx dt.$$

Now we are ready to establish a constrained variational principle. We

consider $I(x_0, t_0; t_2; v)$ and its first order variation

$$48) \quad (\delta I)(x_0, t_0; t_2; v) = m \int_{t_0}^{t_2} [\delta v \cdot (\nabla I/m - v)](q(t; x_0, t_0), t) dt = m \int_{t_0}^{t_2} \int [\delta v \cdot (\nabla I/m - v)](x, t) p(x, t; x_0, t_0) dx dt,$$

which is found by following the same method leading to 24). Notice that \bar{I} does not contain the boundary term with S_1 , as J in 16).

It is also convenient to introduce the average

$$49) \quad A(t_0, t_1; \rho_0; v) = - \int I(x_0, t_0; t_1; v) \rho_0(x_0) dx_0;$$

then 48) and 45) give

$$50) \quad (\delta A)(t_0, t_1; \rho_0; v) = -m \int_{t_0}^{t_1} \int [\rho \delta v \cdot (\nabla I/m - v)](x, t) dx dt.$$

Now we explore the consequences of the variational principle

$$51) \quad (\delta A)(t_0, t_1; \rho_0; v) = 0,$$

under the following constraint for δv

$$52) \quad (\delta \rho)(x_1, t_1) = 0, \quad \forall x_1.$$

From 47) we have

$$53) \quad \delta \rho(x_1, t_1) = \int_{t_0}^{t_1} \int \nabla \rho(x_1, t_1; x, t) \cdot (\rho \delta v)(x, t) dx dt;$$

then the constraint 52) obliges $\rho \delta v$ to be "orthogonal" in

$L^2(R^n \times [t_0, t_1], dx dt)$ to all functions of the type

$$54) \quad \nabla \rho(x_1, t_1; x, t), \quad \forall x_1.$$

Therefore from the condition 51) and the identity 50) we can not conclude

that $v = \nabla I/m$ must be true but only that the difference $v - \nabla I/m$

must be in the linear hull spanned by functions of the type 54) for

all values of x_1 . Therefore there must be some $S_1(\cdot)$ such that

$$55) \quad m v(x, t) = \nabla I(x, t; v) + \nabla \int S_1(x_1) \rho(x_1, t_1; x, t) dx_1.$$

Therefore if we define, for v given by 55)

$$56) \quad S(x, t) = I(x, t; v) + \int S_2(x_2) p(x_2, t_2; x, t) dx_2$$

then the constrained variational problem 51), 52) enforces the Hamilton-Jacobi form of the control

$$57) \quad v(x, t) = (\nabla S)(x, t) / m ,$$

while it is simple to check that the equation 27) holds for S , with the boundary condition 28). In conclusion we have seen that the constrained variational problem 51), 52) is equivalent to the variational problem based on J . They both lead to the Hamilton-Jacobi structure. The principal function S can be interpreted as a field of Lagrangian multipliers associated to the constraints of fixed density. From this point of view S is the field conjugated to ρ , in agreement with the fluid picture of classical mechanics as expressed by the equation 27), 38), 57) (the Hamilton-Jacobi fluid).

Let us remark that the constrained problem corresponds to making stationary the average action A of 49), under variations of v , while keeping fixed the boundary densities $\rho(\cdot, t_0)$ and $\rho(\cdot, t_2)$. Since it is always possible to smoothly interpolate many $\rho(\cdot, t)$ between fixed end point densities, there will be many possible choices of v compatible with the constraints, as follows from 38) interpreted now as a condition on v in correspondence of some interpolating ρ .

Let us conclude this Section with few remarks, related to obvious extensions of the methods here recalled. Clearly it is possible to consider more general Lagrangians with respect to those of the type 13). Of particular physical interest is the introduction of terms corresponding to generalized Lorentz forces

$$58) \quad \mathcal{L} \rightarrow \mathcal{L} + A(x, t) \cdot v(x, t) .$$

All considerations developed in this Section can be extended to 58) with obvious modifications. For example now the Hamilton-Jacobi condition 57) would read

$$59) \quad w(x, t) = (A(x, t) + (VS)(x, t)) / m .$$

Moreover it is possible, by time reversal invariance, to consider variational problems where for example J is modified through the introduction of terms related to the initial time t_0 .

In the following Sections the whole scheme will be generalized to the stochastic case.

3 - KINEMATICS OF THE CONTROLLED STOCHASTIC THEORY

In the stochastic theory we promote the classical variable $q(t)$ to a diffusion Markov stochastic process, still denoted by $q(t)$, taking values on the configuration space $\mathcal{M} = \mathbb{R}^n$, with a given initial distribution $\rho_0(\cdot)$ at time t_0 . The control equation 2) is replaced by the Ito stochastic differential equation [22]

$$60) \quad dq(t) = \mathcal{V}_{(t)}(q(t), t) dt + d\omega(t),$$

where now $\mathcal{V}_{(t)}(\cdot, t)$ plays the role of feedback control field and $\omega(t)$ is a Brownian motion on \mathbb{R}^n with

$$61) \quad E(d\omega(t) | q(t) = x) = 0,$$

$$62) \quad E(d\omega(t) d\omega(t) | q(t) = x) = 2\nu I dt, \quad (dt > 0).$$

In 62) I is the unit matrix corresponding to the Euclidean metric on \mathbb{R}^n and ν is some fixed diffusion coefficient. We refer to [23,19] for the generalization to non flat Riemann manifolds.

In 61), 62) $E(A|B)$ are conditional expectations denoting the average of the random variable A under the condition that the event B happens. In general we denote by $E(A)$ the (unconditional) expectation of the random variable A .

For the density $\rho(\cdot, t)$ of the process we have

$$63) \quad E(F(q(t), t)) = \int F(x, t) \rho(x, t) dx,$$

while 60) implies the forward diffusion equation

$$64) \quad (\partial_t \rho)(x, t) = -\nabla \cdot (\rho(x, t) \underline{v}_{\leftrightarrow}(x, t)) + \nu (\Delta \rho)(x, t),$$

with initial condition

$$65) \quad \rho(\cdot, t_0) = \rho_0(\cdot).$$

For the transition probability density $p(x, t; x', t')$,

$$t \geq t',$$

we have the forward Fokker-Planck equation

$$66) \quad (\partial_t p)(x, t; x', t') = -\nabla \cdot (p \underline{v}_{\leftrightarrow}) + \nu \Delta p,$$

with initial condition

$$67) \quad \lim_{\substack{t \downarrow t' \\ t' \uparrow t'}} p(x, t; x', t') = \delta(x - x').$$

Notice that p depends only on $\underline{v}_{\leftrightarrow}$ (not on ρ_0 !). The density is propagated in time through

$$68) \quad \rho(x, t) = \int p(x, t; x', t') \rho(x', t') dx', \quad t \geq t'.$$

It is also convenient to introduce

$$69) \quad \underline{v}_{\leftarrow}(x, t) = \underline{v}_{\leftrightarrow} - 2\nu \nabla \rho / \rho,$$

and define the osmotic velocity [17]

$$70) \quad u(x, t) = \frac{1}{2} (\underline{v}_{\leftrightarrow} - \underline{v}_{\leftarrow})(x, t) = \nu \nabla \rho / \rho = 2\nu \nabla R,$$

where

$$71) \quad \rho(x, t) = \exp [2R(x, t)].$$

The current velocity is defined as

$$72) \quad v(x, t) = \frac{1}{2} (v_{(+)} + v_{(-)})(x, t)$$

Then the diffusion equation 64) can be equivalently written either as a continuity equation

$$73) \quad (\partial_t \rho)(x, t) = -\nabla \cdot (\rho v) \quad ,$$

or in the backward form

$$74) \quad (\partial_t \rho)(x, t) = -\nabla \cdot (\rho v_{(-)}) - v \Delta \rho \quad .$$

It is also useful to introduce the linear diffusion operators

$$75) \quad (\mathcal{D}_{(\pm)}, F)(x, t) = -\nabla \cdot (F v_{(\pm)}) \pm v \Delta F \quad ,$$

$$(\mathcal{D} F)(x, t) = -\nabla \cdot (F v) \quad ; \quad (\delta \mathcal{D} F)(x, t) = -\nabla \cdot (F u) + v \Delta F,$$

$$\mathcal{D} = \frac{1}{2} (\mathcal{D}_{(+)} + \mathcal{D}_{(-)}) \quad , \quad \delta \mathcal{D} = \frac{1}{2} (\mathcal{D}_{(+)} - \mathcal{D}_{(-)}) \quad .$$

Then 64), 74), 73), 70) can be written as

$$76) \quad (\partial_t \rho)(x, t) = (\mathcal{D}_{(\pm)} \rho)(x, t) = (\mathcal{D} \rho)(x, t) \quad , \quad (\delta \mathcal{D} \rho)(x, t) = 0 \quad .$$

Notice that we could also assume either v or $v_{(-)}$ as control fields.

In the first case ρ would be defined through 73) with initial condition 65), then 70), 72) would give $v_{(+)}$ and $v_{(-)}$ in the form

$$77) \quad v_{(\pm)}(x, t) = v \pm u \quad .$$

In the second case ρ would be defined through 74), 65) and 70), 72) would give v and $v_{(+)}$ as functions of $v_{(-)}$ and ρ . We prefer to keep $v_{(+)}$ in the central position as a control field because it appears

in the basic control law 60) and in the following Ito-Girsanov formula.

Standard theorems [22,11] give the sufficient regularity conditions on $\mathcal{V}_{(\pm)}$ which assure existence and uniqueness for the solutions of 60). More general control fields $\mathcal{V}_{(\pm)}$ can be introduced, if the solution of 60) is defined through the Ito-Girsanov formula (see for example theorem V.10.2 of [1]).

It is useful to introduce mean forward and backward derivatives $D_{(\pm)}$, which generalize the substantial derivative D in 7), according to the standard definitions [17,18]

$$78) \quad (D_{(\pm)}, F)(x, t) = \\ = \pm \lim_{\Delta t \rightarrow 0^+} (\Delta t)^{-1} E(F(q(t \pm \Delta t), t \pm \Delta t) - F(q(t), t) \mid q(t) = x).$$

It is immediate to verify

$$79) \quad (D_{(\pm)}, \hat{q})(x, t) = \mathcal{V}_{(\pm)}(x, t),$$

where \hat{q} is the specification field $\hat{q}(x, t) = x$, already introduced in 8). For a generic field $F(\cdot, t)$, under appropriate conditions of smoothness, we have

$$80) \quad (D_{(\pm)}, F)(x, t) = (\partial_t F)(x, t) + \mathcal{V}_{(\pm)}(x, t) \cdot (\nabla F)(x, t) \pm \nu(\Delta F)(x, t).$$

We can also introduce the transport operators

$$81) \quad (A_{(\pm)}, F)(x, t) = \mathcal{V}_{(\pm)} \cdot \nabla F \pm \nu \Delta F, \\ (A F)(x, t) = \nu \cdot \nabla F, \quad (\delta A F)(x, t) = \mu \cdot \nabla F + \nu \Delta F, \\ A = \frac{1}{2} (A_{(+)} + A_{(-)}), \quad \delta A = \frac{1}{2} (A_{(+)} - A_{(-)}),$$

so that

$$82) \mathcal{D}_{(\pm)} = \partial_t + \mathcal{A}_{(\pm)} \quad , \quad \mathcal{D} = \partial_t + \mathcal{A} \quad , \quad \delta \mathcal{D} = \delta \mathcal{A} \quad ,$$

$$\mathcal{D} = \frac{1}{2} (\mathcal{D}_{(+)} + \mathcal{D}_{(-)}) \quad , \quad \delta \mathcal{D} = \frac{1}{2} (\delta \mathcal{D}_{(+)} - \delta \mathcal{D}_{(-)}) \quad .$$

Notice the following conjugation properties in $L^2(\mathbb{R}^n, dx)$

$$83) \mathcal{D}_{(\pm)}^\dagger = \mathcal{A}_{(\pm)} \quad , \quad \mathcal{D}^\dagger = \mathcal{A} \quad , \quad (\delta \mathcal{D})^\dagger = \delta \mathcal{D} \quad .$$

While on the Hilbert space $L^2(\mathbb{R}^n \times \mathbb{R}, \rho dx dt)$ we have [17]

$$84) \mathcal{D}_{(\pm)}^\dagger = -\mathcal{D}_{(\mp)}$$

It is convenient to recall that conditional expectations can be easily expressed in terms of transition probability densities. In particular we have, for $t \geq t'$,

$$85) E(F(q(t), t) | q(t') = x') = \int F(x, t) p(x, t; x', t') dx \quad ,$$

while, in the case of forward conditioning,

$$86) E(F(q(t'), t') | q(t) = x) = \rho(x, t)^{-1} \int p(x, t; x', t') F(x', t') \rho(x', t') dx' \quad .$$

Clearly $\rho(x, t)$ must be different from zero if 86) has a meaning, in fact only in this case it is possible to impose the conditioning $q(t) = x$.

Now we list some useful expressions involving the forward and backward derivatives defined in 80).

First of all let us recall that the backward Fokker-Planck equation for the transition probability density can be written as

$$87) \quad (D'_{t+}, p)(x, t; x', t') = 0, \quad t \geq t',$$

where D'_{t+} is defined in 80) and acts on the (x', t') variables.

We also have [17]

$$88) \quad \frac{d}{dt} E(F(q(t), t) G(q(t), t)) = \\ = E((D_x, F)(q(t), t) G(q(t), t)) + E(F(q(t), t) (D_x, G)(q(t), t)).$$

Moreover the following conditioned transport equations hold, as counterpart of 12), for $t_0 \leq t_1$.

$$89) \quad E(F(q(t_1), t_1) | q(t_0) = x_0) - F(x_0, t_0) = \\ = \int_{t_0}^{t_1} E((D_x, F)(q(t), t) | q(t_0) = x_0) dt,$$

$$90) \quad F(x_1, t_1) - E(F(q(t_0), t_0) | q(t_1) = x_1) = \\ = \int_{t_0}^{t_1} E((D_x, F)(q(t), t) | q(t_1) = x_1) dt.$$

89) and 90) follows easily from the definition 78) and the Markov property.

In some cases it is useful to compare two different processes, one, $q(t)$, evolving under the control v_{t+} , and the other $q'(t)$, evolving under v'_{t+} .

According to the general frame outlined before we assume that both processes have the same distribution density ρ_0 at time t_0 . Then we can easily give, under suitable regularity conditions, an explicit expression for the averages of functionals $F(q')$ of the process $q'(t)$, for

$t_0 \leq t \leq t_2$, in terms of the same functionals $F(q)$ for the other process $q(t)$, considered as a known reference process. In fact the Ito-Girsanov formula [22,11] gives

$$91) \quad E(F(q')) = \\ = E\left(F(q) \exp\left\{\frac{1}{2v} \int_{t_0}^{t_2} [\delta v_{t'}(q(t), t) \cdot dW(t) - \frac{1}{2} \delta v_{t'}^2(q(t), t)] dt\right\}\right),$$

where $\delta v_{t'}(x, t) = (v'_{t'} - v_{t'}) (x, t)$.

Finally we give the explicit expression of the variations of the transition probability density and the density corresponding to a change 92) in the control drift field, while the external Brownian noise and the initial density are kept fixed.

We follow the same method leading to 46) in the classical case. In fact we start from 87) and exploit the trick explained in 21), but using $D_{t'}$ in place of D . Then we use the transport equation 89) instead of 12). Also in ^{the} stochastic case we end up with an expression similar to 46), where now ρ and ρ' are not given by 40), as in the classical case, but are the solutions of 66) or 87) with initial condition 67). For the variation of the density in the stochastic case we still have an expression similar to 47), in fact 45) and 68) are structurally the same. The explicit expressions for the stochastic case are

$$\begin{aligned}
 92) \quad & p'(x_1, t_1; x_0, t_0) - p(x_1, t_1; x_0, t_0) = \\
 & = \int_{t_0}^{t_1} \nabla p'(x_1, t_1; x, t) \cdot \delta v_{\text{st}}(x, t) p(x, t; x_0, t_0) dx dt, \\
 & p'(x_1, t_1) - p(x_1, t_1) = \int_{t_0}^{t_1} \nabla p'(x_1, t_1; x, t) \cdot \delta v_{\text{st}}(x, t) p(x, t) dx dt.
 \end{aligned}$$

This typical procedure will be also exploited in the following (see 110)).

This ends our discussion about the kinematical properties of the stochastic control theory. The reader will notice that we have written most of Section 2 having in mind the generalization to the stochastic case. In fact the classical and the stochastic case are very similar and most of the structure is preserved, provided we take care of the fact that the Brownian disturbances produce the splitting of the velocity field v in v_{st} and v_c . Some physical consequences of this splitting with reference to the quantum uncertainty principle and Bohr complementarity are analyzed in [24,25].

In the next Sections we generalize the constrained and unconstrained forms of the variational principle to the stochastic case.

4 - THE STOCHASTIC ACTION FUNCTIONALS

The main problem for the formulation of stochastic variational principles, leading to a simulation of quantum mechanical behavior, is to find the right action. Our proposal is the following.

In analogy with the classical case 13) we introduce the Lagrangian field defined by

$$93) \quad \mathcal{L}(x, t) = \frac{1}{2} m \dot{v}_{(+)}(x, t) \cdot \dot{v}_{(-)}(x, t) - V(x) ,$$

where $\dot{v}_{(+)}$ is the control field and $\dot{v}_{(-)}$ is defined through 69). Notice that the Lagrangian field depends not only on $\dot{v}_{(+)}$, but also on the initial density ρ_0 . This is a peculiar aspect of the theory.

Clearly 93) is time reversal invariant. In fact for the time reversal transformation we have [18]

$$94) \quad t \rightarrow t' = -t , \quad q(t) \rightarrow q'(t') = q(t) , \quad x \rightarrow x' = x , \\ \dot{v}_{(\pm)}(x, t) \rightarrow \dot{v}'_{(\pm)}(x, t') = -\dot{v}_{(\mp)}(x, t) .$$

As $\nu \rightarrow 0$ 93) reduces to the classical case.

In analogy with 49) let us define the average stochastic action, spent by the controller in order to move the system from time t_0 to time t_1 , $t_1 \geq t_0$, for some initial distribution $\rho_0(\cdot)$.

$$95) \quad A(t_0, t_1; \rho_0; \dot{v}_{(+)}) = \\ = \int_{t_0}^{t_1} E(\mathcal{L}(q(t), t)) dt = \int_{t_0}^{t_1} \int \mathcal{L}(x, t) \rho(x, t) dx dt ,$$

with \mathcal{L} given by 93). Notice the additivity property

$$96) A(t_0, t_1; \rho_0; \psi_{t_1}) = A(t_0, t; \rho_0; \psi_{t_1}) + A(t, t_1; \rho_t; \psi_{t_1}),$$

where ρ_t is $\rho(\cdot, t)$ and $t_0 \leq t \leq t_1$.

Let us point out the following intuitive physical picture at the basis of 95).

Let $s_0 = t_0, s_1, \dots, s_N = t_1$ be an equipartition of the interval $[t_0, t_1]$, with $\Delta s = (t_1 - t_0)/N$. First of all let us consider a classical trajectory

$q(t)$; then we can assume the limit

$$97) A(t_0, t_1) = \frac{1}{2} m \lim_{\Delta s \rightarrow 0} \sum_{i=0}^{N-1} \frac{(q(s_{i+1}) - q(s_i))^2}{(\Delta s)^2} \Delta s$$

as an expression for the kinetic part of the action in the interval $[t_0, t_1]$ according to the definitions 14) and 49). For the stochastic case let us modify 97) in the form

$$98) A(t_0, t_1) = \frac{1}{2} m \lim_{\Delta s \rightarrow 0} \sum_{i=1}^{N-1} \frac{(q(s_{i+1}) - q(s_i))}{\Delta s} \cdot \frac{(q(s_i) - q(s_{i-1}))}{\Delta s} \Delta s.$$

Clearly 97) and 98) give the same limit in the classical case, because the change is only of second order in Δs . On the other hand the limit 98) does not exist in the stochastic case because $q(t)$ is nowhere differentiable. But we can take (conditional) averages on \mathcal{A} in 98) before and then take the limit. Then it is immediate to verify that we have

$$99) E(A(t_0, t_1)) = \frac{1}{2} m \int_{t_0}^{t_1} \int (\psi_{t_1}^* \cdot \psi_{t_1})(x, t) \rho(x, t) dx dt,$$

so that the kinetic part of 95) is correctly reproduced. On the other hand the limit in 97) would give divergent results also in the case some average is taken before. It is clear that the tendency of the Brownian motion to produce highly irregular trajectories for $q(t)$ favours 98) over 97) as far as the convergence is concerned. In fact the contributions in 98) come from scalar products of the displacement in each infinitesimal time interval multiplied by the displacement in the next interval. It is clear that only a physical explanation of the nature of the underlying Brownian motion could also produce a deeper justification of 98). For the moment 98) and the equivalent 95) are taken as basic assumption of our theory.

Notice that 93) can be written in equivalent form

$$93') \quad \mathcal{L}(x, t) = \frac{1}{2} m (v^2 - \mu^2)(x, t) - V(x) ,$$

by exploiting the current and osmotic velocities defined in 70), 72).

The minus sign before μ^2 in 93') is physically relevant, it means that the osmotic part of the action plays more a role similar to a potential rather than a kinetic contribution. This is in agreement with the Schrödinger variational formulation for stationary states of quantum mechanical systems, where μ^2 and V enter with the same sign in the energy functional, see for example [35] .

It is also useful to introduce conditioned expectations for 98). It is very simple to prove, exploiting Markov property and the expression 85), that

$$100) \quad E(\mathcal{A}(t_0, t_2) | q(t_0) = x_0) = \\ = \frac{1}{2} m \int_{t_0}^{t_2} \int (\dot{v}_{(\pm)}^2 + 2v \nabla \cdot \dot{v}_{(\pm)})(x, t) p(x, t; x_0, t_0) dx dt.$$

Analogous expressions hold for a forward conditioning at time t_2 .

Then we are motivated to introduce the following forward and backward Lagrangian fields

$$101) \quad \mathcal{L}_{(\pm)}(x, t) = \frac{1}{2} m \dot{v}_{(\pm)}^2(x, t) \pm m v (\nabla \cdot \dot{v}_{(\pm)})(x, t) - V(x),$$

for which one can easily prove

$$102) \quad E(\mathcal{L}(q(t), t)) = E(\mathcal{L}_{(\pm)}(q(t), t)).$$

We also define, for $t_0 \leq t \leq t_2$,

$$103) \quad I(x, t; t_2; \dot{v}_{(\pm)}) = - \int_t^{t_2} E(\mathcal{L}_{(\pm)}(q(t'), t') | q(t) = x) dt' = \\ = - \int_t^{t_2} \int \mathcal{L}_{(\pm)}(x', t') p(x', t'; x, t) dx' dt'.$$

Notice that I satisfies the transport equation

$$104) \quad (\mathcal{D}_{(\pm)}, I)(x, t) = \mathcal{L}_{(\pm)}(x, t),$$

with the boundary condition

$$105) \quad I(\cdot, t_2) = 0$$

Moreover

$$106) \quad A(t_0, t_2; p_0; \dot{v}_{(\pm)}) = - \int I(x_0, t_0; t_2; \dot{v}_{(\pm)}) p_0(x_0) dx_0.$$

103), 104), 105) and 106) are the counterpart of 14), 19)₁, 19)₂, 49) respectively.

In analogy with the classical case it is also useful to introduce action functionals with additional end point contributions. For some given smooth field $S_2(\cdot)$ on \mathcal{M} we then define

$$107) \quad J(x, t; t_2, S_2; v_{(t_2)}) = \\ = I(x, t; t_2; v_{(t_2)}) + E(S_2(q(t_2)) | q(t) = x) ,$$

$$108) \quad B(t_0, t_2; \rho_0, S_2; v_{(t_2)}) = \\ = A(t_0, t_2; \rho_0; v_{(t_2)}) - E(S_2(q(t_2))) = \\ = - \int J(x_0, t_0; t_2, S_2; v_{(t_2)}) \rho_0(x_0) dx_0 .$$

One easily checks the following

$$109) \quad (D_{(t_2)} J)(x, t) = \mathcal{L}'_{(t_2)}(x, t) , \\ J(\cdot, t_2) = S_2(\cdot) .$$

In order to introduce and prove variational principles it is necessary to obtain explicit expressions for the variations of the stochastic action functionals, introduced before, when the control field is changed from $v_{(t_2)}$ to some other $v'_{(t_2)} = v_{(t_2)} + \delta v_{(t_2)}$, while the initial distribution is kept fixed. We follow the same route as in 21). With obvious short hand notations, we have

$$110) \quad D'_{(t_2)}(J' - J) = \mathcal{L}'_{(t_2)} - \mathcal{L}_{(t_2)} + (D_{(t_2)} - D'_{(t_2)}) J = \\ = \left(\frac{1}{2} m (v'_{(t_2)} + v_{(t_2)}) + m \nu \nabla - \nabla J \right) \cdot \delta v_{(t_2)} .$$

Then the transport equation 89) and the boundary condition in 109) give

$$111) \quad (\delta J)(x_0, t_0) = - \int_{t_0}^{t_1} \int \left[\left(\frac{1}{2} m (\dot{v}_{tt} + \dot{v}_{t-}) + m v \nabla - \nabla J \right) \cdot \delta v_{tt} \right] (x, t) \rho'(x, t; x_0, t_0) dx dt.$$

If we multiply by $\varphi_0(x_0)$, integrate on dx_0 , and get rid of $\nabla \cdot \delta v_{tt}$ through a simple integration by part on dx , then we obtain

$$112) \quad (\delta B)(t_0, t_1) = \int_{t_0}^{t_1} \int \left(\frac{1}{2} m (\dot{v}_{tt} + \dot{v}_{t-}) - \nabla J \right) \cdot \delta v_{tt}(x, t) \rho'(x, t) dx dt$$

By following the same method, but exploiting I in 110) in place of J , we also get

$$113) \quad (\delta A)(t_0, t_1) = \int_{t_0}^{t_1} \int \left(\frac{1}{2} m (\dot{v}_{tt} + \dot{v}_{t-}) - \nabla I \right) \cdot \delta v_{tt}(x, t) \rho'(x, t) dx dt,$$

which is the counterpart of 50).

To the first order in δv_{tt} , we have

$$114) \quad (\delta B)(t_0, t_1) = \int_{t_0}^{t_1} \int (m v - \nabla J) \cdot \delta v_{tt}(x, t) \rho(x, t) dx dt,$$

$$115) \quad (\delta A)(t_0, t_1) = \int_{t_0}^{t_1} \int (m v - \nabla I) \cdot \delta v_{tt}(x, t) \rho(x, t) dx dt.$$

Let us also recall that the first order variation of the density at t_1 is

$$116) \quad (\delta \rho)(x_1, t_1) = \int_{t_0}^{t_1} \int \nabla \rho(x_1, t_1; x, t) \cdot \delta v_{tt}(x, t) \rho(x, t) dx dt,$$

as follows from 92)₂.

Then we can state the equivalence between the variational principle based on $\delta B = 0$ and the constrained variational principle based on $\delta A = 0$ with $\delta q(x_1, t_1) = 0, \forall x_1$. In fact, as in the classical case, the condition $\delta B = 0$ for any δv_m implies, as a consequence of 114), the Hamilton-Jacobi condition

$$117) \quad v(x, t) = (vS)(x, t) / m ,$$

where we have called S the value of J for the stationarizing field (physical control field). On the other hand if $\delta A = 0$ for any δv_m , satisfying $\delta q(\cdot, t_1)$, we can not conclude that $mv - \nabla I = 0$, but only

$$118) \quad mv - \nabla I = \nabla \int S_1(x_1) p(x_1, t_1; x, t) dx_1 ,$$

for some suitable function S_1 . But 118) is equivalent to 117) in view of 107), provided the boundary term S_1 in 107) is chosen in agreement with 118). Therefore also in the stochastic case there is the interpretation of the S function as a field of Lagrangian multipliers associated to the density constraint at the final time in the variational principle.

Notice that, while in the classical case the variational principles are based on conditions of the type $\delta I(x_0, t_0) = 0$ or $\delta J(x_0, t_0)$, here in the stochastic case they are based on the averaged conditions $\delta A \equiv E(\delta I(q(t_0), t_0)) = 0$, $\delta B \equiv E(\delta J(q(t_0), t_0)) = 0$. In this way the density enters in an essential way in the dynamical evolution as the next considerations show. In fact let us now investigate about

the consequences of 117). Since S coincides with J for the stationarizing field we have

$$119) \quad (\mathcal{D}_{(t)}, S)(x, t) = \frac{1}{2} m v_{(t)}^2 + m \nu \nabla \cdot v_{(t)},$$

$$120) \quad S(\cdot, t_2) = S_2(\cdot) .$$

But for $v_{(t)}$, we have

$$121) \quad v_{(t)} = v + u = \nabla S / m + 2\nu \nabla R$$

as a consequence of 117) and 70). If we substitute 121) in 119) then we find the following Hamilton-Jacobi equation with stochastic corrections

$$122) \quad (\partial_t S)(x, t) + \frac{1}{2m} (\nabla S)^2(x, t) + V(x) - 2m\nu^2 ((\nabla R)^2(x, t) + (\Delta R)(x, t)) = 0 .$$

The connection with quantum mechanics is established in the standard way [17, 18]. First of all we fix the diffusion constant so that

$$123) \quad 2\nu = \hbar / m .$$

Then the Ansatz

$$124) \quad \psi(x, t) = \sqrt{\rho(x, t)} \exp\left(\frac{i}{\hbar} S(x, t)\right)$$

reduces the two real equations 73), 122) to the Schrödinger equation

$$125) \quad i\hbar (\partial_t \psi)(x, t) = -\frac{\hbar^2}{2m} (\Delta \psi)(x, t) + V(x) \psi(x, t) .$$

If we do not impose 123) then for the wave function 124) we find some non linear equation [26,27] instead of 125).

Clearly all our considerations extend to the case where the action functional J is modified through boundary terms at the initial time t_0 and in the case where linear terms in v are introduced in the Lagrangian field 93).

Finally let us notice, on the basis of equations 124), 107), 103) and the connection between S and J , that the expression of the phase function S is given in terms of the physical control field $v_{cr} = (\nabla S + \hbar^{-1} \nabla R)/m$ as follows

$$126) \quad S(x, t) = E(S_x(q(t_2)) | q(t) = x) + \\ - \int_t^{t_2} E(L_{cr}(q(t'), t') | q(t) = x) dt', \quad t_0 \leq t \leq t_2,$$

where $q(t)$ is the process subject to the physical control field.

If the initial value S_0 of S is known, then we can write 126) in the equivalent form

$$127) \quad S(x, t) = E(S_0(q(t_0)) | q(t) = x) + \\ + \int_{t_0}^t E(L_{cr}(q(t'), t') | q(t) = x) dt', \quad t_0 \leq t \leq t_2,$$

as can be easily checked on the basis of the transport equations 89), 90) and the conditioning identities 85), 86).

As a consistency check of our theory it is also possible to connect the stochastic Lagrangian field 93) with the standard Lagrangian

$$128) \quad L(\psi(\cdot, t), \psi^*(\cdot, t), \dot{\psi}(\cdot, t), \dot{\psi}^*(\cdot, t)) \equiv \\ = \int \left[\frac{i\hbar}{2} (\psi^* \dot{\psi} - \dot{\psi} \psi^*) - \frac{\hbar^2}{2m} \nabla \psi \cdot \nabla \psi^* - V \psi \psi^* \right](x, t) dx,$$

giving rise to 124) through Euler-Lagrange independent variations of ψ and ψ^* .

In fact 128), with the Ansatz 124), reduces to

$$129) \quad L = - \int \left[\gamma_t S + \frac{\hbar^2}{2m} (\nabla R)^2 + \frac{1}{2m} (\nabla S)^2 + V \right](x, t) \rho(x, t) dx.$$

A simple calculation shows that 129) gives the equations 73), 122) of the Madelung fluid through Euler-Lagrange independent variations of S and ρ . On the other hand by exploiting the continuity equation 73) and 117) one easily finds

$$130) \quad L = - \frac{d}{dt} \int S(x, t) \rho(x, t) dx + E(\mathcal{L}(q(t), t)),$$

where L is defined in 128), 129), \mathcal{L} is given in 93) and $q(t)$ is the associated process. Since the total time derivative is irrelevant we have the equivalence between our stochastic variational principle and the standard variational principle leading to Schrödinger equation from 128).

6 - CONCLUSIONS AND OUTLOOK

We have seen that it is possible to choose the Lagrangian field in a stochastic variational problem so that the resulting programming equation corresponds to the Hamilton-Jacobi equation with the right quantum corrections. From this point of view stochastic quantization [17,28,29,18] can be based either on the smoothed form of the second principle of dynamics or on a stochastic variational problem. The advantage of this second method relies on the fact that the gradient form of the velocity fields follows from the variational principle and is not an independent assumption.

We have given two equivalent forms of the variational principle. For one version boundary terms are inserted in the action functional and unconstrained control drift variations are allowed. For the other version the action functional does not contain boundary terms, but the control variations are restricted so that the density is kept point wise fixed at the end points of the generic time interval of variations. This equivalence between the two principles allows the interpretation of the phase function \mathcal{S} as a field of Lagrangian multipliers associated to the density constraints. Therefore \mathcal{S} can be considered as a variable conjugated to ρ , according to the usual hydrodynamical picture.

While in classical mechanics and in standard stochastic control theory the variational principles are usually expressed in optimal form, where the functionals are not only made stationary with respect to

control variations but acquire optimum values (maxima or minima), we have not considered here this problem. We plan to deal with it in the future.

The extension of our considerations to the case where the configuration space is a general Riemannian manifold is straight forward and is considered in [19]. In this framework it is interesting to see the interplay between geometric features and the ability of the Brownian motion to feel second order effects, as shown for example in [23].

Of particular interest is the case where the underlying configuration space is not simply connected. Then the variational principles, even in the classical case, can have a more refined formulation, where boundary terms or density constraint may depend on the connectivity properties of the trajectories. This peculiar features may lead to simulation of the half integer spin effects in the case of rotating bodies [30,31,32] or to spin-statistics effects, according to the development of general ideas already pointed out for example in [33].

Variational principles, very similar to those described in this paper, were considered for the ground state process of Euclidean field theory [34], in the frame of a general program aiming at the connection between classical statistical mechanics and quantum field theory. It would be interesting to see whether similar interpretations hold also in our case. In particular the stochastic action functional could be connected to entropy properties of the trajectories appearing

in the Ito-Girsanov formula.

Finally we would like to remark that the kinetic metric enters in our theory in two different places. Firstly it gives the kinetic action (as in the classical case), but it also determines the underlying Brownian disturbance (see also [23]).

A natural extension of interesting remarks by Davidson [26] seems to suggest that the Brownian disturbance could be introduced without any relation with the kinetic metric on the configuration space.

Therefore it would be very interesting to include also in the controller domain the metric associated to the Brownian motion, providing a wide generalization of the stochastic variational principles introduced in this paper.

But it is clear that a major progress in the whole program of stochastic quantization would come from a better understanding of the physical origin of the underlying Brownian motion, so that in particular the form of stochastic action, assumed in this paper, could be explained in more fundamental terms.

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