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CALCULATIONS OF STATIONARY SOLUTIONS FOR THE NON LINEAR VISCOUS RESISTIVE MHD EQUATIONS IN SLAB GEOMETRY

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The reduced system [1;2] of the non linear resistive MHD equations is used in the 2-D one helicity approximation [3] in the numerical computations of stationary tearing modes. The critical magnetic Raynolds number S ($S = \tau_r / \tau_H$ where τ_R and τ_H are respectively the characteristic resistive and hydro magnetic times) and the corresponding linear solution are computed as a starting approximation for the full non linear equations. These equations are then treated numerically by an iterative procedure which is shown to be rapidly convergent.

A numerical application is given in the last part of this paper.

I-INTRODUCTION :

The computation of stationary solutions of the non linear tearing model is important in the theory of confinement in Tokamaks because of the probable role played by this mode in the disruptive instability [4]. It is well known that the tearing mode has an exponential growth [5] during the linear regime when the island size is smaller than the resistive layer ; when the island width is of the order of the resistive layer size, the mode enters the Rutherford's regime [6;7] characterized by a linear growth of the island ; finally, when the island size is greater than the resistive layer, the non linear effects will play an important role in the saturation of the mode ; the quasilinear theory of saturation has been studied by Roscoe B. white and Al. [8] : it predicts stationary islands with fluid vortex flow.

In section II we give a brief description of the helicoidal model for the non linear tearing mode.

Section III describes the computation of the critical magnetic Raynold's number in a plane plasma layer.

In section IV we describe the computational precedure for the non linear equations.

In section V we give a numerical application and the conclusion.

II THE MODEL :

We start from the reduced system [2] of the non linear resistive MHD equations written in the small plasma pressure limit for the couple of flux and stream functions Ψ and u .

$$\frac{\partial w}{\partial t} + \vec{v}\cdot\vec{\nabla} w - \nu_0 \Delta w = \frac{1}{\rho_0} \vec{B}\cdot\vec{\nabla}\Delta\psi \tag{1a}$$

$$\frac{\partial \psi}{\partial t} + \vec{v}\cdot\vec{\nabla}\psi = \eta_0 \Delta\psi + E_0 \tag{1b}$$

with $B_x = \partial\psi/\partial y \quad B_y = -\partial\psi/\partial x \qquad v_x = \partial\psi/\partial y \quad v_y = -\partial\psi/\partial x$

and $w = \Delta u$ is the vorticity function.

$\rho_0, \nu_0 = \mu_\perp/\rho_0$ are respectively the specific mass and the kinematic viscosity taken as constants for simplicity ; $E_0 = \eta_0(x) j_0(x)$ is the equilibrium electric field where η_0 and j_0 are respectively the resistivity and the current equilibrium profiles. Defining the perturbed flux Ψ_1 by $\Psi(x,y) = \Psi_0(x) + \Psi_1(x,y)$ where $\Psi_0(x)$ is the equilibrium flux ($\underset{\sim}{B}_0 = \underset{\sim}{\nabla}\Psi_0 \times \underset{\sim}{e}_z$), the system of eqs (1a), (1b) becomes:

$$\frac{\partial w}{\partial t} + \vec{v}\cdot\vec{\nabla} w - \nu_0 \Delta w = \frac{1}{\rho_0}\vec{B}\cdot\vec{\nabla}(\Delta\psi_1) - \frac{1}{\rho_0} B_{1x}\frac{dj_0}{dx} \tag{2a}$$

$$\frac{\partial \psi_1}{\partial t} = \eta_0 \Delta\psi_1 + \vec{B}\cdot\vec{\nabla} u \tag{2b}$$

where $\quad B_{1x} = \partial\psi_1/\partial y$

In order to exhibit the magnetic Raynolds and viscous Prandtl numbers, we introduce dimensionless variables ; we define the following characteristic lenght, times and velocities :

$$\tau_H = L\rho_0^{1/2}/\bar{B} \;;\quad \tau_R = L^2/\bar{\eta}_0 \;;\quad \tau_{visc} = L^2/\nu_0$$

$$V_A = \bar{B}/\rho_0^{1/2} \;;\quad V_R = \bar{\eta}_0/L \;;\quad V_{visc} = \nu_0/L$$

where V_A is the alfven velocity, τ_H the hydromagnetic time, τ_R the resitive time, $\bar{B}$ a characteristic magnetic field strengh, L the thickness of the plane plasma layer and $\bar{\eta}_0$ the resistivity at the plasma center.

We then normalize as follows :

$$t \to \tau_R t \; ; \; x \to Lx \; ; \; y \to Ly \; ; \; \vec{B} \to \bar{B}\vec{B} \; ; \; \psi \to \bar{B} L \psi$$

$$v \to v_R v \; ; \quad u \to L v_R u \; ; \quad W \to v_R W/L$$

with these normalization, equations (2a) and (2b) become :

$$\frac{\partial W}{\partial t} + \vec{v}\cdot\vec{\nabla} W - \gamma \Delta W = S^2 \left(\vec{B}\cdot\vec{\nabla}\Delta\psi_1 - B_{1x} \frac{dJ_0}{dx} \right) \tag{3a}$$

$$\frac{\partial \psi_1}{\partial t} = \eta_0 \Delta \psi_1 + \vec{B}\cdot\vec{\nabla} u \tag{3b}$$

where $\eta_0(x) = 1/J_0(x)$ with $\eta_0(0) = 1$

the magnetic Raynold's number S and the viscous Prandtl number γ are defined by

$$S = \tau_R / \tau_H \qquad ; \quad \gamma = \tau_R / \tau_{visc} \tag{4}$$

As is noticed in [9] the right hand side of equation (3a) contains the driving force of the tearing mode ; therefore the bifurcation parameter is S and the bifurcation problem can be posed as follows :

Find the periodic solutions in y direction of the following system

$$\vec{V}\cdot\vec{\nabla} W - \gamma \Delta W = S^2 \left(\vec{B}\cdot\vec{\nabla}\Delta\psi_1 - B_{1x} \frac{dJ_0}{dx} \right) \tag{5a}$$

$$\eta_0(x) \Delta \psi_1 + \vec{B}\cdot\vec{\nabla} u = 0 \tag{5b}$$

with the boundary conditions

$$u = W = \psi = 0 \qquad \text{at} \qquad x = \pm 1 \tag{6}$$

Non trivial solutions exist [9;10] for values of the magnetic Raynold's number S, greater than a critical value S_c.

III COMPUTATION OF THE CRITICAL RAYNOLDS NUMBER

We start from the linearized version of the equations (5a - 5b) and suppose that the stationary island is small compared to the shear lenght; then, far from the island, the nonviscous linear tearing "outer" solution holds ; and near the island, the "inner" solution of the viscous linear tearing is computed and matched to the outer solution. The matching condition gives the critical Raynolds number ; now the couple (S_c,U_L) is used as the starting approximation in the computation of the full non linear equations.

The linearized system writes

$$\varepsilon\left(\frac{d^2W}{dx^2}-k^2W\right)+k^2J_0\psi_0'^2u=-\frac{dJ_0}{dx}k\psi_1 \qquad (7a)$$

$$\frac{d^2\psi_1}{dx^2}-k^2\psi_1=J_0k\psi_0'u \qquad (7b)$$

where $$\varepsilon=\gamma/S^2 \quad \text{and} \quad W=\frac{d^2u}{dx^2}-k^2u \qquad (8)$$

k is the wave lenght of the mode

* Putting $\varepsilon = 0$ in the eqs 7a-7b we obtain the classical equation of the linear tearing :

$$\psi_0'(x)\left[\psi_1''-k^2\psi_1\right]=\psi_0'''\psi_1 \qquad (9)$$

from which the outer solution vanishing at $\pm\infty$ and the slope jump $\Delta'(0)=\left[\psi_1'/\psi_1\right]_-^+$ can be computed if the equilibrium profile $\psi_0(x)$ is given.

* The inner solution is then computed in the region of high gradients ($\psi''\gg k^2\psi$) assuming the " ψ_1 =cte" approximation and keeping $\varepsilon \neq 0$ in the equation 7a-7b.

The necessary matching to the outer solution will give the following relation between the critical Raynolds number and the slope jump :

$$\gamma/S^2 \simeq k^2\left(\frac{\Delta'(0)}{2\pi}\right)^6 \qquad (10)$$

IV COMPUTATIONAL PROCEDURE FOR THE NON LINEAR EQUATIONS :

We start from the eqs 5a-5b-6 written in the new form :

$$\vec{v}\cdot\vec{\nabla}W - \gamma\Delta W + S^2\vec{B}\cdot\vec{\nabla}(J_0\vec{B}\cdot\vec{\nabla}u) = -S^2\frac{dJ_0}{dx}\frac{\partial\psi_1}{\partial y} \tag{11a}$$

$$\vec{v}\cdot\vec{\nabla}\psi_1 - \eta_0\Delta\psi_1 = \vec{B}_0\cdot\vec{\nabla}u \tag{11b}$$

with the boundary conditions :

$$\begin{array}{l} u = W = \psi_1 = 0 \quad \text{at} \quad x = \pm 1 \\ u, \psi, W \quad \text{periodic in } y \end{array} \tag{11c}$$

and a fixed island size (or equivalently a given value of $\psi_1(o)$)

The computational procedure applied to the system (11a)-(11b) is summarized below :

Starting with the critical set $S_c = S^{(n)}$, $U_c = U^{(n)}$, $\psi_c = \psi^{(n)}$ one compute the Fourier components $\delta U_l(x)$, $\delta\psi_l(x)$ of the perturbations $\delta U(x,y) = \sum_{l=1}^{L}\delta u_l(x)$ sin ly and $\delta\psi(x,y) = \sum_{l=0}^{L}\delta\psi_l(x)$ cos ly around $U_c(x,y)$ and $\psi_c(x,y)$ from the system :

$$\left\{\begin{array}{l} -\Delta^2\delta u_l - \Lambda_c(l-1)^2x^2\delta u_l + 2(l-1)\Lambda_c x\delta\psi_l = \delta\Lambda(x^2u_c - 2x\psi_c)_{l=2} + (S_u)_l \\ -\Delta^2\delta\psi_l - (l-1)x\,\delta u_l = (S_\psi)_l \end{array}\right. \tag{12a}$$

$$(l = 0, 1, 2, \ldots L)$$

where $S_u = -\frac{1}{\gamma}\vec{v}\cdot\vec{\nabla}\Delta^2 u + \Lambda\left[(\vec{B}\cdot\vec{\nabla})^2 - (\vec{B}_0\cdot\vec{\nabla})^2\right]u$

and $S_\psi = -\vec{v}\cdot\vec{\nabla}\psi \quad ; \quad \Lambda_c = K^2S_c^2/\gamma$ (12b)

Now the left hand side operator of the system (12a) is singular because the critical set $(\Lambda_c; U_c; \psi_c)$ is a proper solution of the homogenous equations. Hence for a non trivial solution of (12a) to exist, it is necessary that the right hand side be orthogonal to the solution of the homogeneous adjoint system. [10]. In our case the adjoint solution is simply :

$$\psi_c^* = -2\Lambda_c \psi_c \text{ and } U_c^* = U_c \tag{13a}$$

which gives the free parameter $\delta\Lambda^{(n)}$ at each n iteration by :

$$\delta\Lambda^{(n)} = \frac{\int_{-1}^{+1} u_c (S_u)_{l=2}^{(n)} dx - 2\Lambda_c \int_{-1}^{+1} \psi_c (S_\psi)_{l=2}^{(n)} dx}{\int_{-1}^{+1} 2x u_c \psi_c dx - \int_{-1}^{+1} x^2 u_c^2 dx} \tag{13b}$$

and an improved eigenvalue $\Lambda_{(n+1)}$ by

$$\Lambda_{(n+1)} = \Lambda_c + \delta\Lambda_{(n)} \tag{13c}$$

To achieve the convergence of the above procedure, we must impose to the solution $(\delta\psi, \delta u)$ to stay in the subset orthogonal to (ψ^*_c, u^*_c), that is the new solution is to be computed from :

$$\vec{Z}_{new} = \vec{Z}_{(n+1)} - (\vec{Z}_{(n+1)}, \vec{Z}_c^*) \frac{\vec{Z}_c}{\|Z_c\|^2} \tag{14}$$

where $Z = (\delta\psi ; \delta U)$

(15)

and $Z_c = (\psi_c ; U_c)$

The above procedure is then continued until convergence is reached (convergence of the non-linear steps is achieved after 10 iterations as seen on the figures)

V-NUMERICAL APPLICATION :

The following data are used :

- Equilibrium current profile :

  $J_0\ (x) = 1/ch^2\ x$
- The island size is fixed to

  $\Psi_1\ (o) = 0.01$
- The wave-lenght of the mode is fixed to :

  $k^{-1} = 8/7$

- The $\Delta'(o)$ for the choosen current profile is equal to 0.5 and $\gamma = 0.13$

- The numerical value of the ratio $K^{1}S^{2}/\gamma$ as computed from (10) is :

  $\Lambda_c = 6\ 10^{+6}$
- The numerical results obtained for $S_c = 10^3$, are shown on the following figures.

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FIGURE CAPTIONS

- Fig. 1 shows the initial and converged profiles of the stream function u.

- Fig. 2 shows the converged profile of the flux function ψ.

- Fig. 3 and 4 show how the kinetic and magnetic energies converge in the linear and non linear stages versus the iteration number.

- Fig. 5 gives the converged eigenvalues in the linear stage ($\Lambda_c = \frac{K^2 S_c^2}{\nu} = 6.10^6$) and the non linear stage ($\Lambda_{NL} = 6.7\ 10^6$). We observe that $\Lambda_{NL} > \Lambda_c$ as is predicted mathematically, and the non linear convergence is attained after only 10 iterations.

- Fig. 6 gives the slab representation of the magnetic island with the stream lines (vortex)

- Enlargements of Fig. 6 near "X" point and "O" point are shown respectively on the figures 7 and 8.

- Fig. 9 and 10 give the 3-D representations of the converged flux and stream funtions respectively.

CONCLUSION

As a conclusion, the method described in this paper has confirmed the existence of stationary solutions of the non linear resistive MHD equations and thus. proved to be a powerfull tool in the investigation of the so important problem of the saturation of the tearing modes in plasma tokomaks.

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GRAPHIQUE NO 8

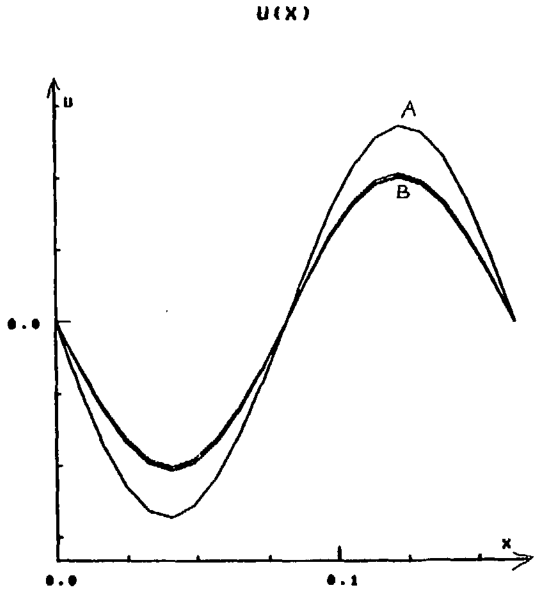

Fig. 1 INITIAL (A) AND CONVERGED (B) PROFILES
OF THE STREAM FUNCTION

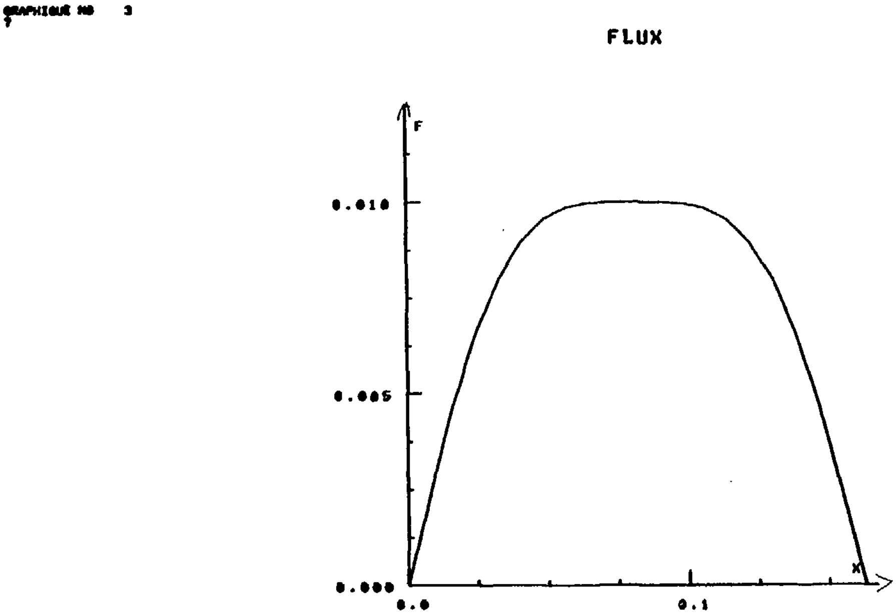

Fig. 2 CONVERGED PROFILE OF THE FLUX FUNCTION

GRAPHIQUE NO 5

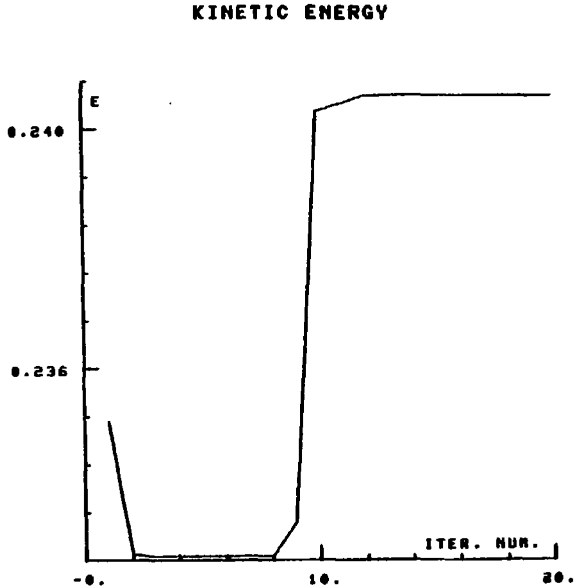

Fig. 3 CONVERGENCE OF KINETIC ENERGY
VERSUS THE ITERATION NUMBER

GRAPHIQUE NO 6

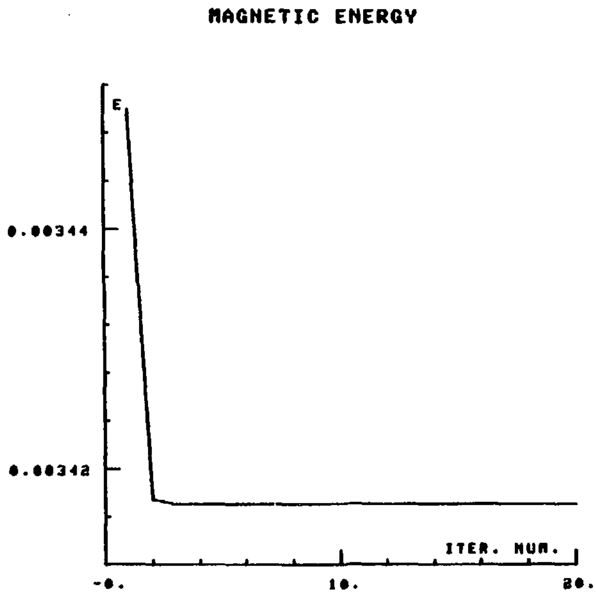

Fig. 4 CONVERGENCE OF THE MAGNETIC ENERGY VERSUS THE ITERATION NUMBER

GRAPHIQUE NO 7
7

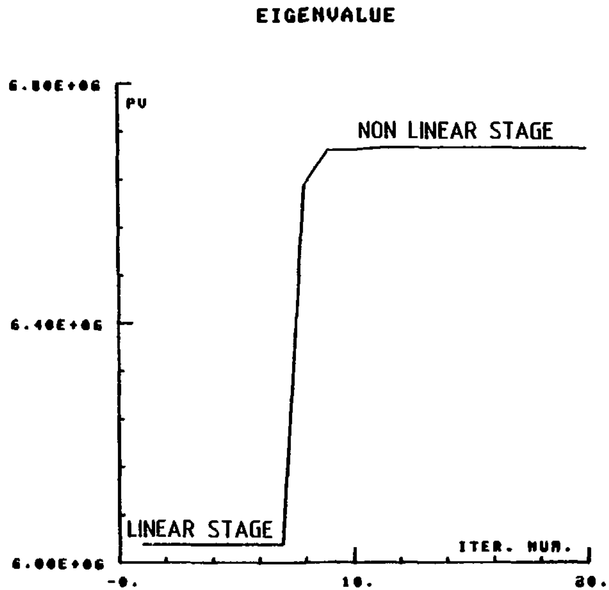

Fig. 5 CONVERGENCE OF THE MAGNETIC RAYNOLD'S NUMBER K^2S^2/γ VERSUS THE ITERATION NUMBER

GRAPHIQUE NO

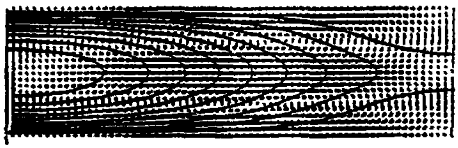

Fig. 6 FLUX LINES AND STREAM LINES (FLOW)

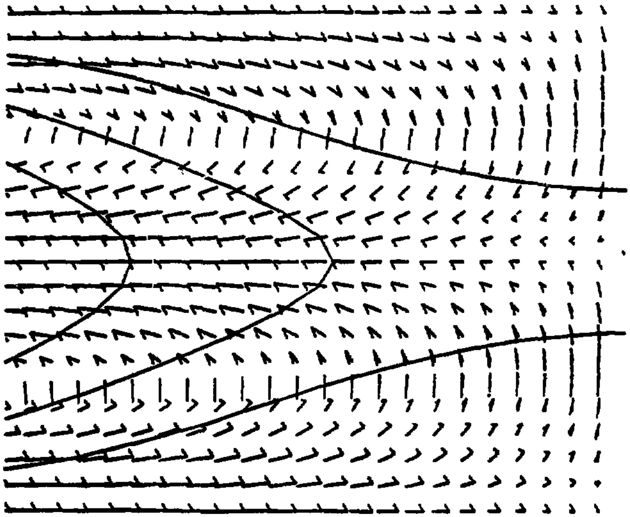

Fig. 7 FLUX LINES AND FLOW NEAR X POINT (right)

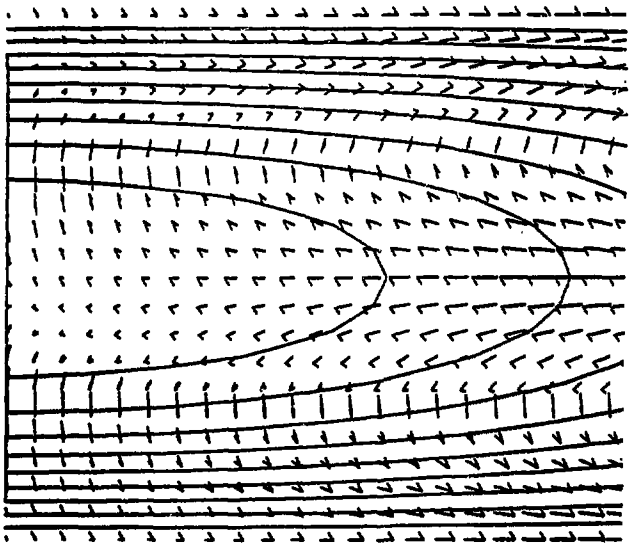

Fig. 8 FLUX LINES AND FLOW NEAR O POINT (left)

GRAPHIQUE NO 10
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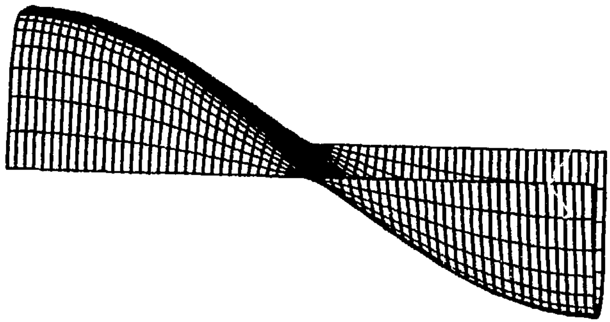

Fig. 9 3-D REPRESENTATION OF THE FLUX FUNCTION

GRAPHIQUE NO 11
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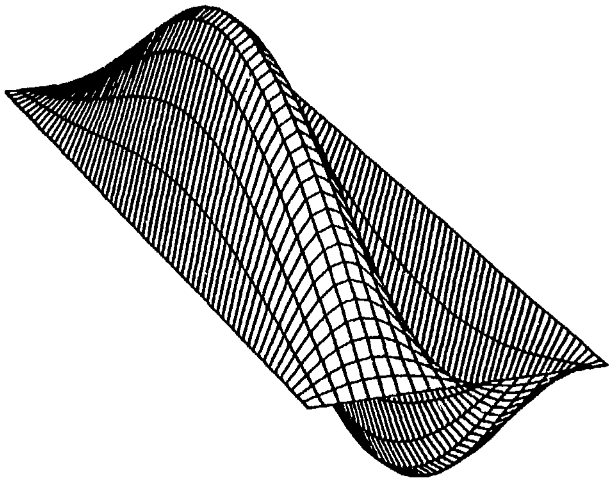

Fig. 10 3-D REPRESENTATION OF THE STREAM FUNCTION